Lecture 8

THE KONTSEVICH INTEGRALS

The Kontsevich integrals $\{Z_m\}$ constitute a beautiful and sophisticated tool whose main purpose is to prove the existence of Vassiliev invariants by providing a proof of the Vassiliev–Kontsevich Theorem 7.1. These integrals actually form an infinite series in $m = 0, 1, 2, \dots$ and are assigned to any concrete knot $K : \mathbb{S}^1 \to \mathbb{R}^3$ from the class of "strictly Morse" knots. Their values are elements of the algebra Δ_m of chord diagrams and their domains of definition are subsets of Euclidean space $\mathbb{R}^m$.

In this lecture, we shall describe the Kontsevich integrals (in the form originally defined by Kontsevich) for a specific trefoil knot K_{TR}, and calculate the integral $Z_2(K_{TR})$ in detail. We then calculate the integral of a specific unknot H (having the shape of a hump) and learn that $Z_2(H) \neq 0 = Z_2(\bigcirc)$. This means that the original Kontsevich integrals *are not* isotopy invariants.

After that, we redefine the Kontsevich integrals, learn that their new version *is* isotopy invariant, and explain in what sense they provide a proof of the existence of Vassiliev invariants.

8.1. The original Kontsevich integral of a trefoil knot

Consider the trefoil knot

$$K_{TR} : \mathbb{S}^1 \to \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$$

shown in Fig.8.1. Note that it is a *strictly Morse smooth knot*, which means that it has finite number c of extrema ($c = 4$), which are either maxima or minima and are located at different levels of the t-axis.

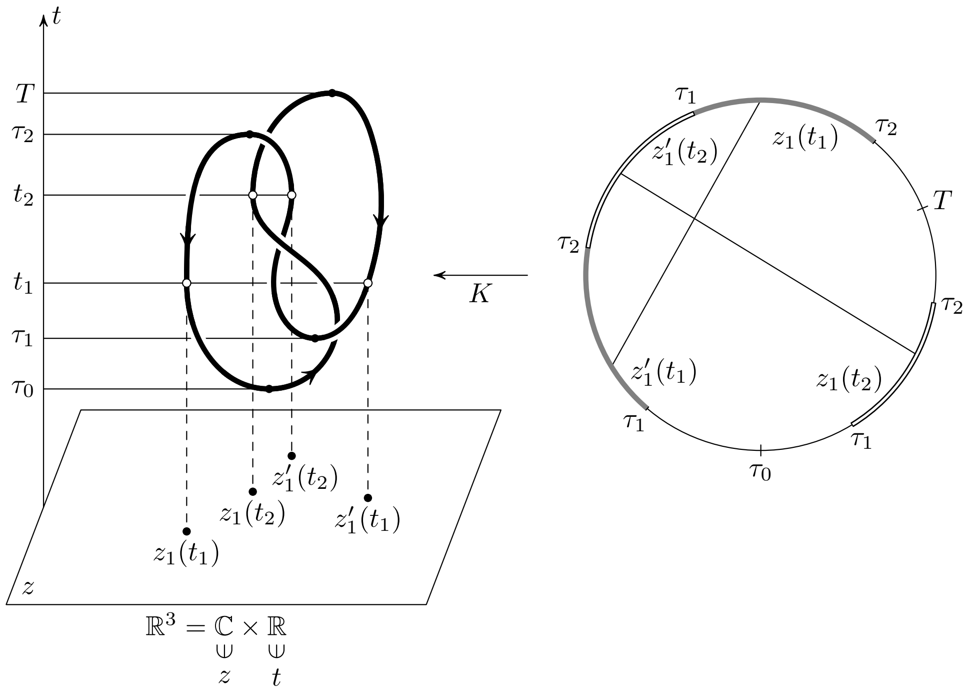


Figure 8.1. A strictly Morse trefoil

The *Kontsevich integral* $Z(K_{TR})$ is defined as the series

$$Z(K_{TR}) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} Z_m(K_{TR}),$$

where

$$Z_m(K_{TR}) = \int\limits_{\tau_0 \leq t_1 < \cdots < t_m \leq T} \sum_{P=\{(z_j, z_j')\}} (-1)^{\downarrow_j} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}.$$

The domain of integration of the m-th integral is a subset of $\mathbb{R}^m$; for $m = 2$, it is the triangle shown in Fig. 8.2 below.

The sum inside the integral is taken over all choices of pairings, a *pairing* being a pair of functions (z_j, z_j') each of which is defined on a strand of the knot – to each point (t, z) of that strand each of the functions assigns the complex number z, which is the projection of the point (t, z) on the horizontal plane $\mathbb{C}$; thus the pairing is determined by a choice of two stands of the knot located between the same pair of extrema. In the case $m = 2$, there are two nontrivial pairings, the first one (z_1, z_1') is shown in Fig. 8.1.

The symbol $\downarrow_j$ in the expression $(-1)^{\downarrow_j})$ stands for the number of downward-oriented stands on which the pairing (z_j, z_j') is defined. Thus, for $m = 2$, for the first pairing, we have $\downarrow_j = 2$.

The symbol D_P stands for the element of the algebra Δ determined by the chord diagram obtained by joining by chords the inverse images by K_{TR} of the points at the same level on the chosen strands of the knot. Thus for $m = 2$ for the first pairing, we have $D_P = \otimes$, where $\otimes$ is the chord diagram with two intersecting chords, as can be seen in Fig. 8.1. In the case when there is only one pairing at some levels of the knot (this happens near the absolute minimum and the absolute maximum), we set $D_P = 0$ (because of the one-term relation).

Finally, the last term under the integral sign is a complex-valued differential m-form that is to be integrated (the other terms are "constants" and can be taken out of the integral). Note that this integral converges despite the fact that the denominators of the differential 1-forms become equal to zero near the extrema – this is a consequence of the one-term relation.

8.2. Calculation of the integral for $m = 2$

Let $m = 2$. Let us partition the domain of integration into three triangles, two rectangles and one square as shown in Fig. 8.2. The partitions corresponding to the two small triangles contribute nothing to the integral (there is only one pairing, so D_P vanishes by the one-term relation). The reader will check (Exercise 8.2) that for all the pairings P in the two rectangles and the square, D_P is the 2-chord diagram with nonintersecting chords, so that these domains contribute nothing (once again by the one-term relation). There are, however, two nontrivial pairings corresponding to the middle triangle – they are shown in Fig. 8.2.

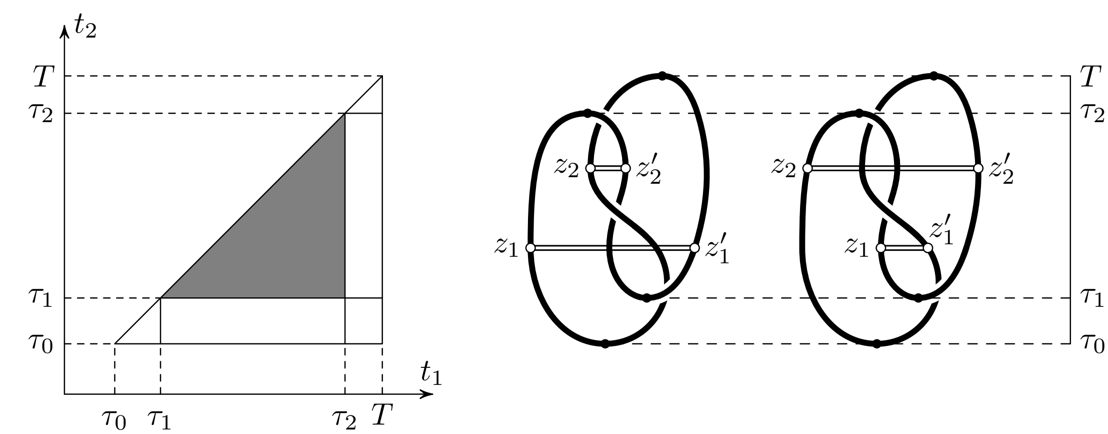


Figure 8.2. The nontrivial pairings for $Z_2(K_{TR})$

As noted above, there are only two nontrivial pairings – the ones shown in Fig. 8.2. It can be seen from Fig. 8.1 that $\downarrow_1 = 2$. A similar study of the second pairing (Exercise 8.3) shows that $\downarrow_2 = 0$. Therefore, we can write

$$
\begin{aligned}
Z_2(K_{TR}) &= \int_{\tau_0 \le t_1 < t_2 \le T} \sum_{(P=\{(z_j, z_j')\})} (-1)^{\downarrow_j} \otimes \frac{dz_1 - dz_1'}{z_1 - z_1'} \wedge \frac{dz_2 - dz_2'}{z_2 - z_2'} \\
&= \left[\int_{\tau_0 \le t_1 < t_2 \le T} (-1)^2 \, \omega_2 + \int_{\tau_0 \le t_1 < t_2 \le T} (-1)^0 \, \omega_2 \right] \otimes \\
&= \zeta \otimes,
\end{aligned}
$$

where $\otimes$ is the chord diagram with two intersecting chords,

$$\omega_2 = \frac{dz_1 - dz_1'}{z_1 - z_1'} \wedge \frac{dz_2 - dz_2'}{z_2 - z_2'}, \text{ and } \zeta = 2 \cdot \int_{\tau_0 \le t_1 < t_2 \le T} \omega_2.$$

Thus we see that the (original) Kontsevich integral for $m = 2$ is equal to the chord diagram $\zeta\otimes$, where ζ is a complex number. Altough the calculation of the exact value of ζ is a hopelessly difficult task, it is not difficult to show that $\zeta \neq 0$.

When $m = 1$, it is easy to show (Exercise 8.4) that the corresponding integral is zero (more precisely, the zero element of the graded algebra Δ of chord diagrams).

For $m > 2$, the knot K_{RT}, having only four strands between its extrema, cannot produce more that two different pairings, so that the integrals with $m > 2$ are of no interest.

8.3. Kontsevich integral of the hump

Consider the (trivial) knot H shown in Fig. 8.3. It is a smooth strictly Morse knot – let us find its m-th Kontsevich integral. Arguing as in the previous section, we note that the domain of integrations of this integral is the same as that for the trefoil K_{TR}, that it can be subdivided into squares, rectangles, and triangles, just as in Fig. 8.2, we can prove (Exercise 8.5) that that there are two nontrivial pairings for $Z_2(H)$, one of which is shown in Fig. 8.3.

It follows that $Z_2(K_{TR})$ is nonzero, which means that the original Kontsevich integral is not an isotopy invariant, contrary to Kontsevich's claim. This fact (and the counterexample H) was noticed by Bar Natan in 2005, but this did not put an end to the Kontsevich approach.

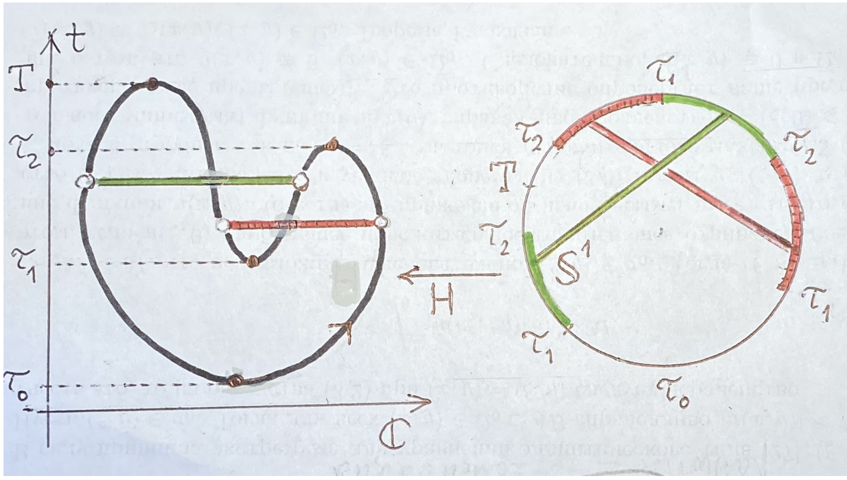
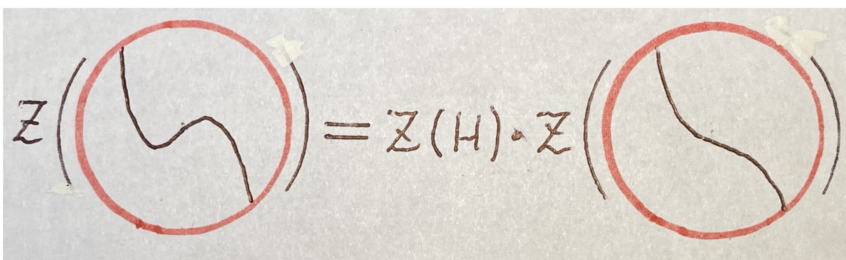


Figure 8.3. Kontsevich integral of the hump

Bar Natan showed that the original Kontsevich integral Z is invariant under isotopic deformations of Morse knots that do not change the number of maxima and established the following formula in the case when the Morse knot acquires one additional maximum:



which should be understood as follows: Z is applied to two Morse knots that are identical outside the circles and are as shown in the picture inside the circles, the product "·" being the usual product in the graded algebra Δ of chord diagrams.

We now define a genuine knot invariant I, that we call the (*modified*) *Kontsevich integral* by setting

$$I(K) := \frac{Z(K)}{Z(H)^{c/2}}.$$

8.4. Results

We now state the main results concerning the (modified) Kontsevich integral. For the proofs, see the CD book.

I. *The integral $I_m(K)$ is defined (converges) for any strictly Morse knot K.*

II. *For the unknot $\bigcirc$ presented as a circle in a vertical plane,*

$$I_m(\bigcirc) = 0 \quad (\forall\, m).$$

III. *The integral $I_m(\,\cdot\,)$ is an isotopy invariant of strictly Morse smooth knots for all m.*

IV. *The integral I_m is universal in the sense that for any m and any strictly Morse smooth knot K there exists a linear map $l : \Delta_m \to \Delta_m$ such that*

$$\mathcal{V}_m(K) = l \circ (I_m(K))^*,$$

where $(I_m(K))^*$ is the linear space dual to the space $I_m(K)$.

Note that the last result implies the surjectivity of the map α_n (see Lecture 7) and thereby concludes the proof of the Vassiliev–Kontsevich theorem.

8.5. Exercises.

8.1. Verify that for the three rectangles, the two small triangles, and the square in the domain of integration (Fig. 8.2), the corresponding chord diagrams all consist of two nonintersecting chords.

8.2. Check that for the second pairing (see Fig 8.2), we have $\downarrow_2 = 0$.

8.3. Show that for $m = 1$ the integral $Z_m(K_RT)$ is zero.

8.4. Represent the eight knot K_8 as a strictly Morse smooth knot and compute the integral $Z_2(K_8)$ (you need not calculate the actual values of the integrals of the 2-forms).

8.5. Prove that there are two nontrivial pairings for $Z_2(H)$.

8.6. Construct an example of a smooth strictly Morse unknot H_1 such that $Z_3(H_1)$ is nonzero.