Lecture 5

Braids

Braids, like knots and links, are curves in $\mathbb{R}^3$ with a natural composition operation. They have the advantage of being a group under the composition operation (unlike knots, all braids have inverses). The braid group is denoted by B_n. The subscript n is a natural number, so that there is an infinite series of braid groups

$$B_1 \subset B_2 \subset \dots \subset B_n \subset \dots$$

At first, we shall define and study these groups geometrically, then look at them as a purely algebraic object via their presentation obtained by Emil Artin (who actually invented braids in the 1920ies). Then we shall see that braids are related to knots and links via a geometric construction called "closure" and study the consequences of this relationship.

We note at once that braids play an important role in many fields of mathematics and theoretical physics, exemplified by such important terms as "braid cohomology", "braided vector space", "braided monoidal category", "braided Hopf algebra", and this is not due to their relationship to knots. But here we are interested in braids mainly because of their relationship to knots and links.

5.1. Geometric braids

A *braid in n strands* is a set consisting of n pairwise nonintersecting polygonal curves (called *strands*) in $\mathbb{R}^3$ joining n points aligned on a horizontal line L to n points with the same x, y coordinates aligned on a horizontal line L' parallel to L and located lower than L; the strands satisfy the following condition: when we move downward along any strand from a point on L to a point on L', the tangent vectors of this motion cannot point upward (i.e., the z-coordinate of these vectors is always negative).

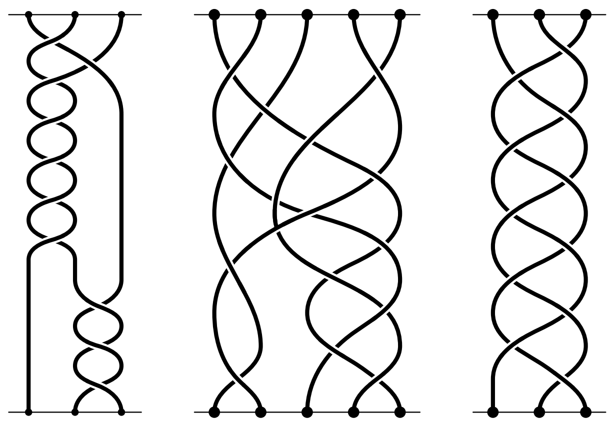

Figure 5.1. Examples of braids

Two braids are called *equivalent* (or *isotopic*) if there is a sequence of braids in which avery braid is obtained from the previous one by a Δ-move.

Two braids b and b' with the same number of strands have a natural composition operation, consisting in identifying the n lower endpoints of b with the upper endpoints of b' (see Fig. 6.2 for the case $n = 4$).

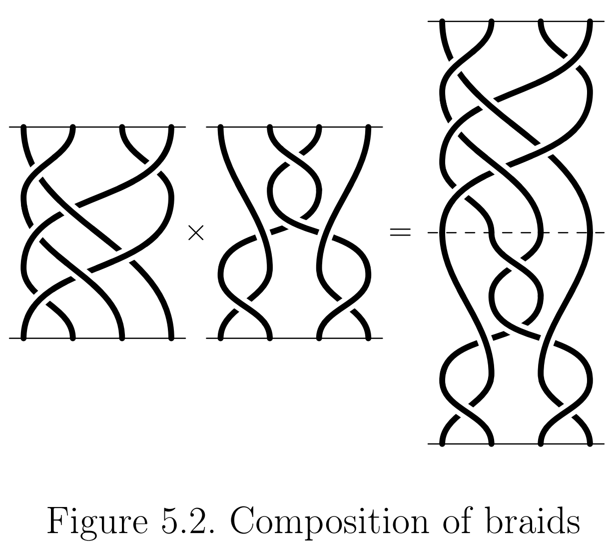

Figure 5.2. Composition of braids

It is easy to see that the composition operation is well defined on equivalence classes of braids. In what follows, we will use the term braid on n stands (or simply n-braid or braid) both for concrete geometric braids and for equivalence classes – the reader will understand what is meant from the context.

5.2. The geometric braid group B_n

Theorem 5.1. *For any $n \geq 1$, the set (of equivalence classes) of braids forms a group, denoted by B_n.*

Proof. The neutral element is the braid all of whose strands are straight vertical lines. The composition operation is obviously associative. Any braid has an inverse, namely its mirror image w.r.t. the horizontal plane containing its lower endpoints (this is clear from Fig.5.2).

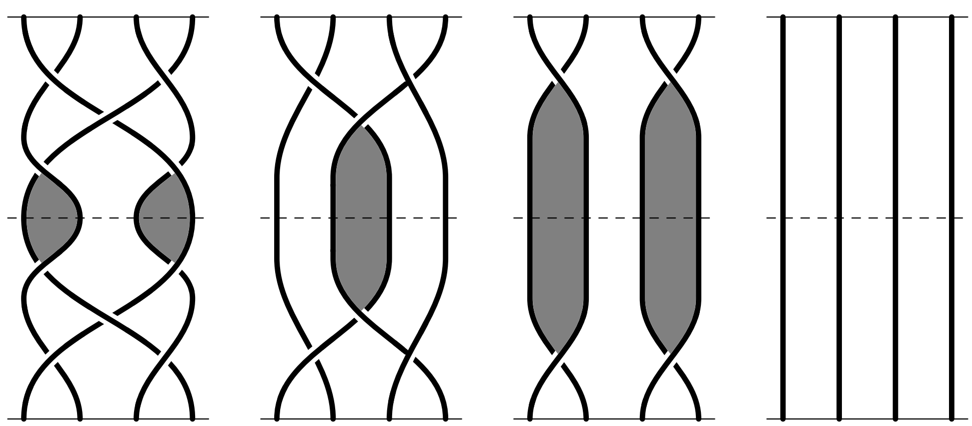

Figure 5.3. Inverse braid

This completes the proof of the theorem.

It is easy to see that $B_1 = 0$ and $B_2 \cong \mathbb{Z}$. Is the group B_n Abelian? The answer is “no”, provided $n \geq 3$; finding an example is easy (Exercise 5.1).

Is the group B_n finitely generated? The answer is yes, B_n has $n-1$ *canonical generators*

$$b_1, b_2, \ldots, b_{n-1}.$$

They are shown on the left-hand side of Fig. 5.4.

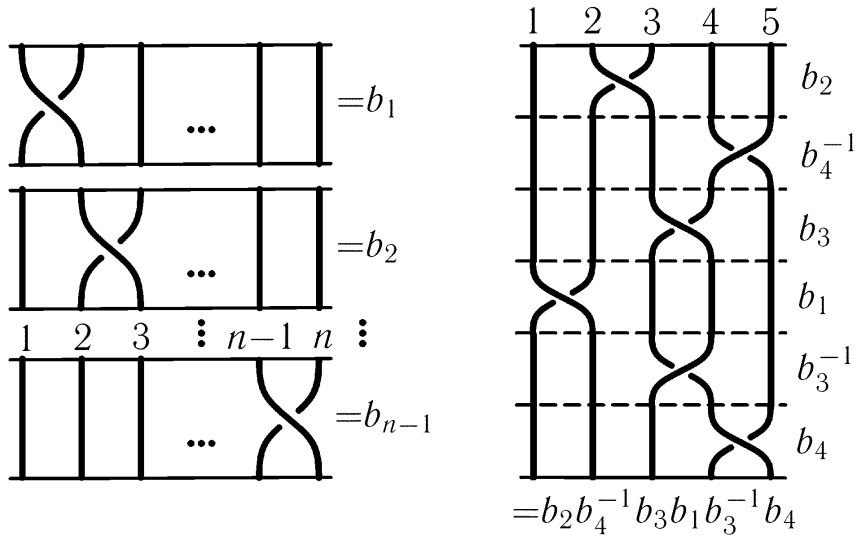

Figure 5.4. Braid generators

The right-hand side of the figure shows how a 5-braid can be expressed as the product of canonical generators. This constuction is general.

5.3. Digression on group presentations

Readers familiar with the notion of group presentation can skip this section and go on to the next one.

Roughly speaking, a group presentation of a group G is a method for defining the group by listing its generators and the relations that these generators must satisfy.

Thus the free group in n generators F_n is represented in the form

$$F_n \leftrightarrow \langle g_1, g_2, \ldots g_n \mid \ \rangle,$$

(following tradition, we do not explicitely indicate the *trivial relations*

$$g_i g_i^{-1} = g_i^{-1} g_i = e, \quad g_j e = e g_j = g_j,$$

which are satisfied by any group, so only the generators are indicated in the presentation of free groups). Thus the free group F_n in n generators is defined as the set of equivalence classes of words in the alphabet $g_1, g_1^{-1}, g_2, g_2^{-1} \dots g_n, g_n^{-1}$, where two words are considered equivalent if one can be transformed into the other by means of the trivial relations. For example, in F_3 we have $g_1 = g_2 g_2^{-1} g_1 g_3^{-1} g_3$ because

$$g_1 = e g_1 = (g_2 g_2^{-1}) g_1 = (g_2 g_2^{-1} g_1) e = (g_2 g_2^{-1} g_1)(g_3^{-1} g_3).$$

The group of residues modulo n is presented in the form

$$\mathbb{Z}/n\mathbb{Z} \leftrightarrow \langle g \,|\, g^n = e \rangle,$$

while the direct sum of two copies of the integers is presented as

$$\mathbb{Z} \oplus \mathbb{Z} \leftrightarrow \langle g, h \,|\, g h g^{-1} h^{-1} = e \rangle.$$

The formal definition is as follows: a *group presentation* in the alphabet

$$\mathcal{A} = \{ g_1, g_1^{-1}, g_2, g_2^{-1}, \dots, g_n, g_n^{-1} \}$$

of a group G is an expression of the form

$$G \leftrightarrow \langle g_1, g_2, \dots, g_n \ : R_1 = R_2 = \dots R_k = e \rangle,$$

where the R_j are words in the alphabet $\mathcal{A}$ if $G \cong F_n / \langle\langle R_1, \dots R_k \rangle\rangle$, where $\langle\langle R_1, \dots R_k \rangle\rangle$ denotes the minimal normal subgroup containing the elements $R_1, \dots R_k$.

Although the presentation of groups provides us with a convenient way of performing calculations with elements of the group (for examples, see below), it doesn't always help to identify the presented group (we will discuss this in Sec. 5.5 below).

The presentation of a group $G \leftrightarrow \langle g_i, R_j \rangle$ allows to perform calculations within the group by replacing R_i by the neutral element e (and vice versa) and using the trivial relations. Here is an example of a calculation in $\mathbb{Z} \oplus \mathbb{Z}$:

$$ghg^{-1}h^{-1} = e \implies (ghg^{-1}h^{-1})(hg) = e(hg) \implies$$

$$\implies ghg^{-1}(h^{-1}h)g = e(gh) \implies ghg^{-1}g = hg,$$

which means that the two elements hg and gh in $\mathbb{Z} \oplus \mathbb{Z}$ are equal, i.e., represent the same element of the group $\mathbb{Z} \oplus \mathbb{Z}$. Read these calculations carefully and identify the specific relations used at each step.

5.4. Artin presentation of the braid group

The braid group B_n was discovered by Emil Artin in 1925 as a geometric object, but he soon obtained its purely algebraic interpretation by writing out its presentation.

Theorem 5.2. *The geometric braid group B_n has the following presentation:*

$$\Big\langle b_1, \dots, b_{n-1} \,|\, b_i b_{i+1} b_i = b_{i+} b_i b_{i+1},\ i < n-1, \\ b_i b_j = b_j b_i,\ |i-j| \geq 2 \Big\rangle.$$

About the proof. It is easy to show that the generators b_i of B_n satisfy the relations indicated in the presentation – one must simply take a good look at Fig. 5.5. The fact that these relations suffice to determine B_n is not at all obvious: we omit its proof.

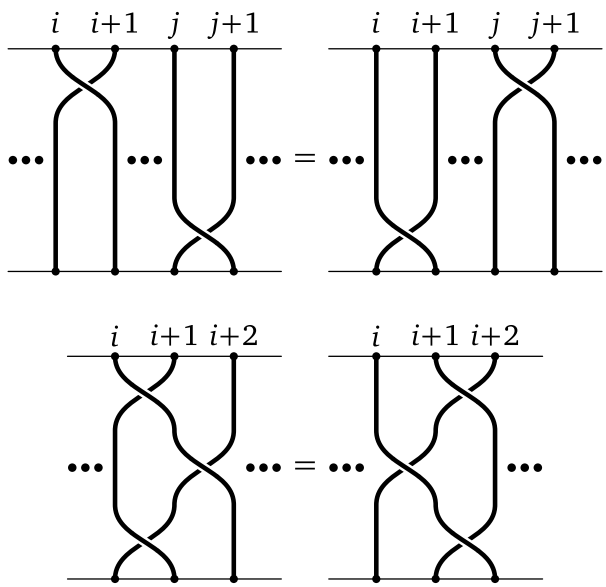

Figure 5.5. Geometry of the braid relations

Further algebraic study of the braid group is outside of the scope of this course. Here we only mention that that the braid group has important applications in topology, complex analysis, theoretical physics, but it interests us because there is a simple construction, called closure (see Sec. 5.6 below), that assigns a knot or link to any braid.

But before going on to this, we digress on algorithmically undecidable problems in group theory and knot theory.

5.5. Digression on undecidable problems

In this section, I will digress about decidable and undecidable problems in group theory (in particular in the braid group) and in knot theory. These topics will not be studied in the course, there will be no proofs, but in studying knots, links, and braids, it is necessary to know what problems in the theory are solvable in principle and what problems are undecidable (cannot be solved in principle). Also, the undecidability of many fundamental problems of group theory, which are not related to knot theory in any way, are part of the basic mathematical culture that students must acquire sooner or later – the sooner the better, in my opinion.

We begin with some bad news. Let G be a group presented as $\langle g_1, \dots, g_n \,|\, R_1 = \dots = R_k = e\rangle$ Then the *word problem in G* is as follows: Does there exist an algorithm which, given two words in the alphabet $\{g_1, g_1^{-1} \dots g_n, g_n^{-1}\}$, tells us whether these two words represent the same element of G?

The first bad news is

Fact 1. *There exist groups for which the word problem is undecidable, and most* (*in a certain natural sense*) *groups have this property.*

Fortunately for us, here is some good news:

Fact 2. *The word problem in the braid group B_n is decidable for any n.*

The *conjugation problem in G* is as follows: Does there exist an algorithm which, given two words w_1, w_2 in the alphabet $\{g_1, g_1^{-1} \dots g_n, g_n^{-1}\}$, tells us whether these two words are conjugate in G, i.e., whether there exists a word w such that $ww_1w^{-1} = w_2$?

The bad news here is

Fact 3. *There exist groups for which the conjugation problem is undecidable, and most* (*in a certain natural sense*) *groups have this property.*

But we have the following good news:

Fact 4. *The conjugation problem in the braid group B_n is decidable for any n.*

However,

Fact 5. *There exists no algorithm which, given a group presentation, correctly answers the following questions: Is the corresponding group trivial? Is it Abelian? Is it cyclic? Is it amenable? as well a many other quations of that type.*

We conclude this digression by looking at algorithmic decidability in knot theory. The main problem of knot theory, namely,

Does there exist an algorithm which, given two knot diagrams, correctly tells us whether they represent the same knot?

has a positive solution. In particular, this means that there exists an algorithm which, given a knot diagram, correctly tells us whether it represents the unknot.

Unfortunately, the proofs of Fact 5 are practiaclly useless "existence theorems" – the existence of the required algorithm is rigorously proved, but the actual algorithm is not sufficiently well described to implement it as a progam for humans and/or computers.

5.6. Closure of a braid

The *closure* operation for braids is defined as shown in Fig. 5.6. In the figure, we see that the closure of a braid can be an oriented knot (as in Fig.5.6(a)) or an oriented link (as in Fig.5.6(b)). The closure of a braid b is denoted by $\mathrm{cl}(b)$.

In fact, any oriented link can be obtained as the closure of an appropriate braid, as the following theorem, due to James Alexander, tells us.

Theorem 5.3. *For any oriented link L, there exists a braid b such that $cl(b) = L$.*

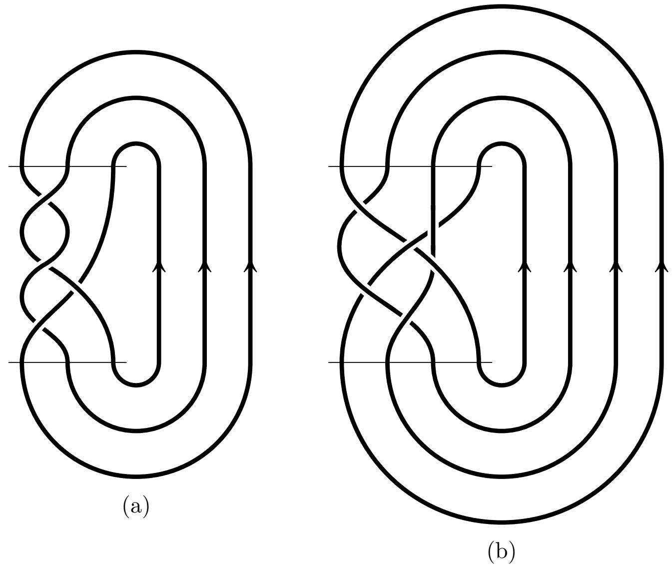

Figure 5.6. Closures of braids

Proof. We first illustrate the main ideas of the proof by Fig. 5.7, which shows how to find a braid whose closure is the eight knot 4_1. This is done in two steps. In the first one, we isotope the given knot into a *circular* one, i.e., an oriented knot whose tangent vector is always directed to the left when we look at it from a fixed point, called the *center*. In the second step, we "unroll" the circular knot into a closed braid.

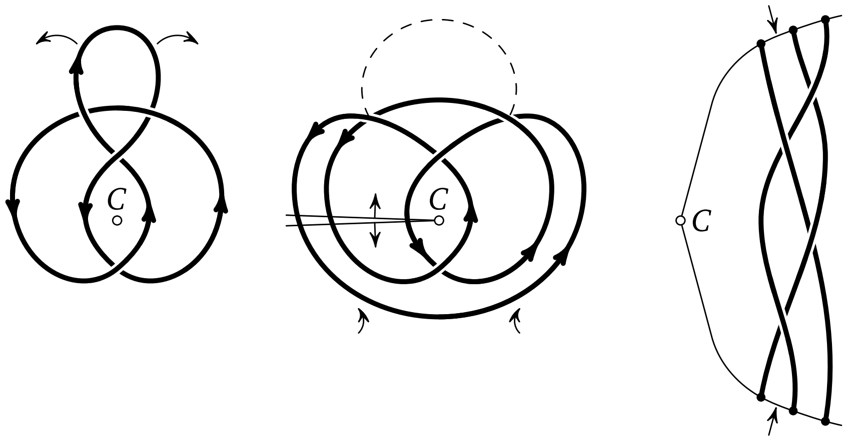

Figure 5.7. Finding a braid whose closure is the eight knot

In the general case of an arbitrary knot K, it suffices to prove that we can isotope it into a circular one, because the unrolling procedure is the same as in the particular case of the eight knot. To perform this isotopy, we first choose the center point C somewhere in the middle of the diagram and paint in red all the edges of the knot that are oriented in the wrong direction (i.e., point to the right, instead of the left, if observed from C). Let $[A, B]$ be such an edge. Without loss of generality, we can assume that no more than one edge of K crosses $[A, B]$, because if there is more than one, we simply subdivide $[A, B]$ into smaller edges, each crossed by only one other edge of K.

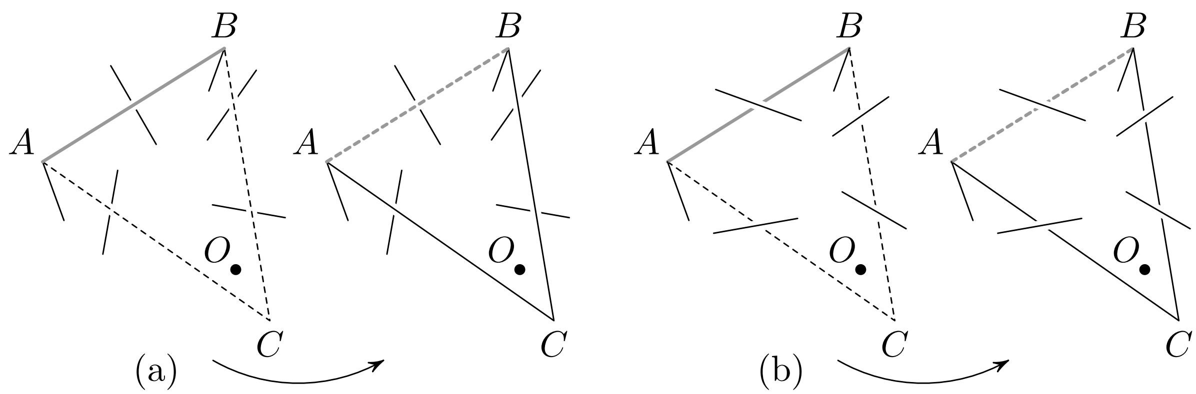

Figure 5.8. Proof of the Alexander theorem

We consider two cases. In the first, the edge that crosses $[A, B]$ forms an underpass. We then choose a point O "behind and above" C (see Fig. 5.8 (a)) and perform the Δ-move replacing $[A, B]$ by $[A, O] \cup [O, B]$; if O is high enough, this will indeed be a legal Δ-move.

In the second case, when the edge that crosses $[A, B]$ forms an overpass, we choose the point O "behind and below" C and perform the same Δ-move. Note that no new red edges arise in these constructions, so that we can successively get rid of all the red edges, obtaining a circular knot. This proves the theorem for arbitrary knots.

The proof for arbitrary links is similar and left to the reader (Exercise 5.14).

5.6. Exercises

5.1. Show that the group B_3 is not Abelian.

5.2. Express the braids shown in Fig. 7.1 in terms of the canonical braid generators.

5.3. Are the two 4-strand braids on the left hand side of Fig. 7.2 isotopic?

5.4. Prove that the group presented as $\langle\, g|\ \rangle$ (the free group in one generator) is isomorphic to $\mathbb{Z}$ (the additive group of integers).

5.5. Prove that a group whose generators pairwise commute (i.e., $ghg^{-1}h^{-1} = e$ for any pair of generators g, h) is Abelian.

5.6. Find a presentation of the permutation group of n objects S_n.

5.7. Prove that the group with two commuting generators is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

5.8. Construct an epimorphism of the braid group B_n onto the permutation group on $n-1$ elements.

5.9. Find a braid whose closure is the 7_1 knot.

5.10. Find a braid whose closure is the 5_2 knot.

5.11*. A braid is called *pure* if the endpoints of each strand have the same x, y coordinates. Prove that pure braids on n strands form a subgroup of B_n denoted PB_n and find a minimal set of generators for PB_n.

5.12. Show that the the conjugate of a given braid by another braid has the same closure as the given braid, i.e., for all braids $b, b_0 \in B_n$ we have $\mathrm{cl}(b^{-1}\, b_0\, b) = \mathrm{cl}(b)$.

5.13. If a braid $b \in B_n$ can be expressed in terms of the generators $b_1, \ldots, b_{n-2}$, then $\mathrm{cl}(b\, b_{n-1}^{\pm 1}) = \mathrm{cl}(b)$.

5.14. Prove the Alexander theorem for links.

5.15. Find a presentation of the braid group B_3 (only one relation suffices).