Lecture 1

Knots and links, Reidemeister moves

In this lecture, we shall introduce knots and links – the main protagonists of this course. Intuitively, you can think of a knot as a string in three-dimensional space whose extremities have been identified, of a link, as several such strings; these strings can be deformed, i.e., moved about in space, stretched and compressed, but they cannot be cut or glued. Two knots (or links) are considered equivalent if one can be deformed so as to have the same shape as the other (i.e., be isometric to the other).

There are several ways to define knots, links, and the corresponding equivalence relation. In the first part of this course, we will use elementary geometric definitions of these notions – they appear in the next section.

The main goal of the present lecture is to give some examples of knots and links, introduce the notions of knot diagram, link diagram, and their Reidemeister moves, and prove the Reidemeister Theorem, which transforms three-dimensional topological knot theory into a branch of two-dimensional combinatorial geometry.

1.1. Main definitions

A (*nonoriented*) *knot* is defined as a closed broken line without self-intersections in Euclidean space $\mathbb{R}^3$. A (*nonoriented*) *link* is a set of pairwise nonintertsecting closed broken lines without self-intersections in $\mathbb{R}^3$; the set may be empty or consist of one element, so that a knot is a particular case of a link. An *oriented* knot or link is a knot or link supplied with an orientation of its line(s), i.e., with a chosen direction for going around its line(s).

Some well known knots and links are shown in the figure below. (The lines representing the knots and links appear to be smooth curves rather than broken lines – this is because the edges of the broken lines and the angle between successive edges are tiny and not distinguished by the human eye :-).

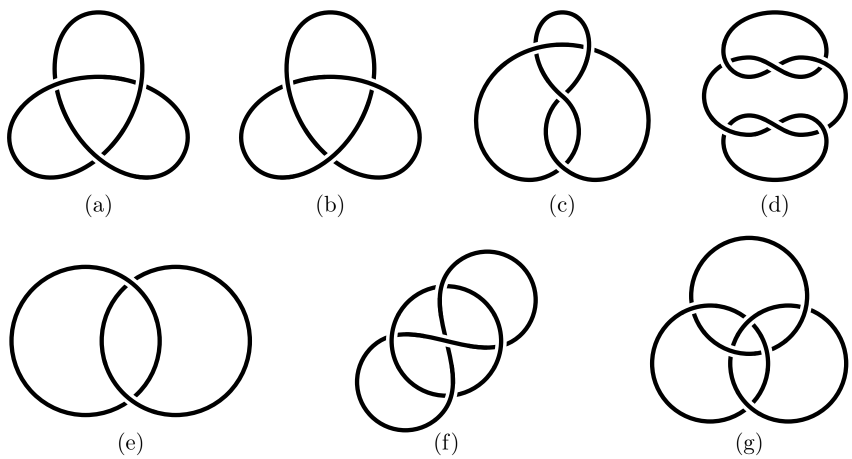


Figure 1.1. Examples of knots and links

The knot (a) is the *right trefoil*, (b), the *left trefoil* (it is the mirror image of (a)), (c) is the *eight knot*, (d) is the *granny knot*; the link (e) is called the *Hopf link*, (f) is the *Whitehead link*, and (g) is known as the *Borromeo rings*.

Two knots (or links) K, K' are called *equivalent*) if there exists a finite sequence of Δ-*moves* taking K to K', a Δ-move being one of the transformations shown in Figure 1.2; note that such a transformation may be performed only if triangle ABC does not intersect any other part of the line(s).

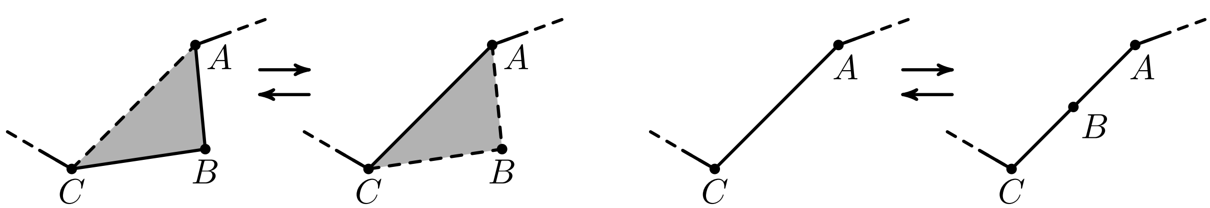


Figure 1.2. Δ-moves

The next figure shows how a knot's shape can be transformed by a successsion of Δ-moves.

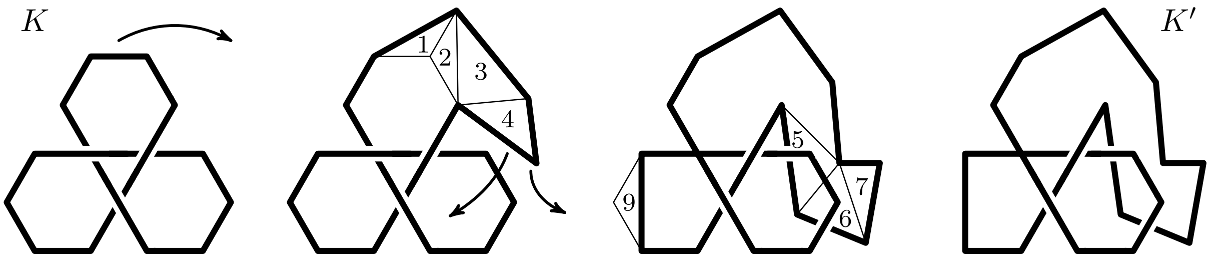


Figure 1.3. Modifying a knot by Δ-moves

As explained above, you can think of a knot as a thin elastic string in three-dimensional space that can be deformed (i.e., stretched, compressed, and moved about), and consider two knots to be equivalent if one of them can be deformed to exactly the same shape as the other's (i.e., made to be isometric to it).

Using the term knot (or link), we often use this term to stand for the entire equivalence class of knots containing the given concrete knot. When two knots are in the same equivalence class, we often say that they have the same *knot type*. Thus when we say "the knot shown in Figure 1.3 is the left trefoil", we mean that it is equivalent to the trefoil (say the one shown in Figure 1.1.(b)).

One of the basic problems of knot theory, often called the *knot classification problem*, is to determine whether any two given concrete knots are equivalent. An important particular case of that problem is the *unknotting problem*: given any concrete knot, determine if it is the unknot (also called trivial knot), i.e., the round circle. In this course, we shall be particularly interested in these two problems.

Remark 1.1. In the examples in Fig. 1.1, there are two trefoil knots – the "right" and the "left" one – (a) and (b). They are obviously mirror symmetric, but are *not equivalent* (the proof, which uses a powerful invariant – the Jones polynomial – will be given later in the course). You will find out, when you do Exercise 1.3, that the “eight knot” (Fig. 1.1 (c)) *is*, on the contrary, equivalent to its mirror image. Later in this lecture, we will discuss this and similar facts (the “chirality” of knots), as well as the “invertibilty” of knots (whether their equivalence class changes or doesn’t change when their oriention is reversed).

Remark 1.2. Most books on knot theory give a definition of knot equivalence different from ours. Namely, they say that two knots K and K' are *ambient isotopic* if there exists an orientation-preserving homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K) = K'$. This definition is equivalent to ours in the sense that the two definitions yield the same equivalence classes of knots and links. The proof of this fact is quite difficult, and we omit it. From now on, we will use the following terminology: instead of saying that two knots (links) are equivalent, we shall say they are *isotopic*, omitting the adjective “ambient” for brevity.

Remark 1.3. This remark is for those readers who are familiar with the notions of homotopy and of isotopy, and it is purely terminological – it explains why the adjective “ambient” appears in the definition mentioned in Remark 1.2 – so that the present remark can be skipped by the other readers. The point here is that if we declare two knots K and K' to be “equivalent” when there is an isotopy taking K to K', then it is easy to show that all knots are equivalent to the unknot – see Figure 1.4.

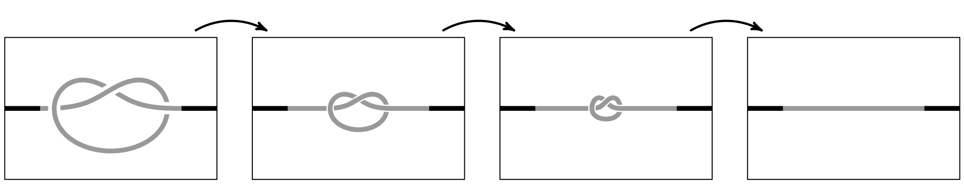


Figure 1.4. All knots are (non-ambient) isotopic to the unknot

1.2. Reidemeister moves

All the knots and links in Figures 1.1-1.3 are pictured as *knot diagrams* or *link diagrams*, i.e., as projections in general position of the knot or link on the horizontal plane showing, at each double point, which of its branches passes above the other one; "in general position" means that no vertex is projected to another vertex, there is only a finite number of double points, and they are all transversal self-intersections. Thus the double points become *crossing points*, at which one branch is an *overpass*, the other, an *underpass*. The projection (with double points instead of crossing points) is called the *shadow* of the link or knot diagram.

The advantage of picturing knots and links as being (almost!) planar is that we pass from a three-dimensional problem (which is not easy to visualize) to a planar one, which is easier to work with, especially after the so-called *Reidemeister moves* $\Omega_1, \Omega_2, \Omega_3$ are introduced: they are defined as shown in Fig.1.5.

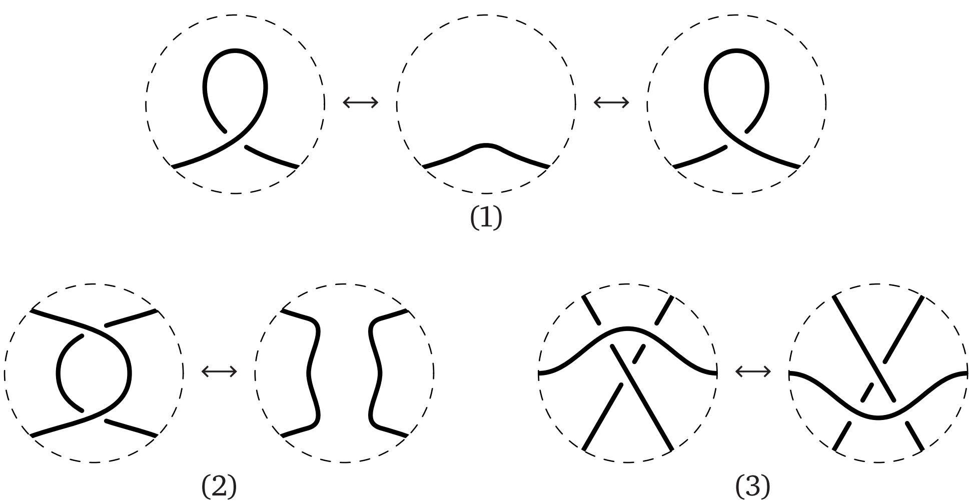


Figure 1.5. Reidemeister moves

These figures should be understood as follows. The pictures show only the part of the link located inside the disk bounded by a hashed line; each move does not modify the part of the link lying outside of the disk, but changes the part of the link that lies inside the disk as shown in the picture. Thus Ω_1 creates/kills a little loop, Ω_2 creates/kills a double overpass, Ω_3 shifts a branch of the knot over a crossing point.

We shall also need the following definition: two knot diagrams are called *planar isotopic* if their shadows can be obtained from each other by a finite sequence of moves shown in Fig. 1.6.

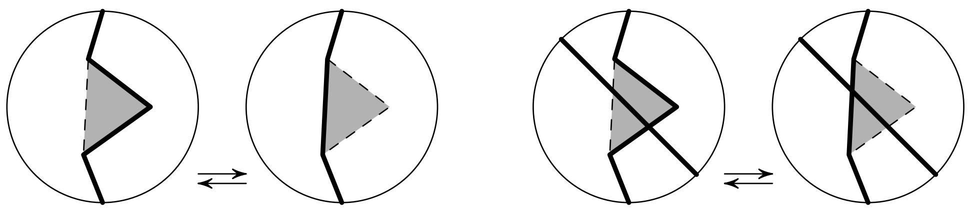


Figure 1.6. Planar isotopy moves

Reidemeister Theorem 1.1. *Two link diagrams, in particular knot diagrams, L and L' are isotopic if and only L can be taken to L' by a finite sequence of Reidemeister moves and planar isotopy moves.*

Proof. The "if" part of the statement is obvious. The proof of the "only if" part is basically a general position argument – it suffices to show that any Δ-move can be replaced by Reidemeister moves.

Suppose we are given a Δ-move $[AB] \mapsto [AC] \cup [BC]$ (Fig.1.7). Without loss of generality, we can assume that the shadows of the edges $[AD]$ and $[BD]$ of our link issuing from A and B do not go inside triangle ABC. Indeed, if (say) $[AD]$ goes into the triangle, we chose a point A' on AC near A and perform an Ω_1 move as shown in Fig.1.7, obtaining the new triangle$A'BC$ such that the edge issuing from A' does not go into the new modified triangle.

Now let us note that the shadows of the branches of our link that intersect triangle ABC either lie entirely above it or entirely below it (otherwise these branches would pierce the triangle, which is forbidden by the definition of Δ-moves).

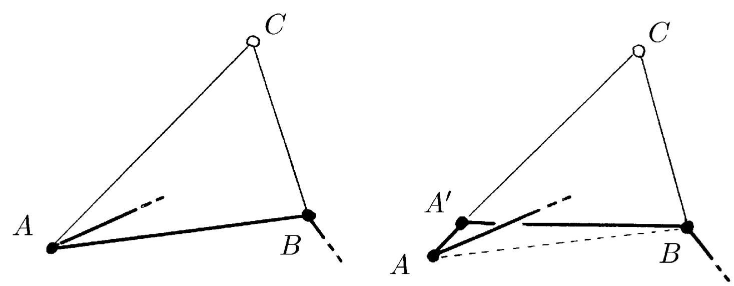


Figure 1.7. Modifying triangle ABC

Let us partition triangle ABC into small triangles (whose edges do not contain vertices of the shadow of L) of the following four types (Fig.1.8). Triangles of type I contain only one crossing of L with both edges of L intersecting two sides of such a triangle. Triangles of type II contain only one vertex of the shadow of L. Triangles of type III contain only a part of one edge of the shadow of L. Triangles of type IV contain nothing (the shadow of L does not intersect them).

Such a partition can be constructed as follows. First, for each crossing and each vertex of the shadow of L, we construct nonintersecting little triangles of type I and II, respectively, and then partition the remaining part of triangle ABC into triangles of type III and IV.

Now, instead of the given Δ-move, we progressively move from $[AB]$ to $[AC] \cup [CB]$, performing Reidemeister moves associated to each of the little triangles. Namely, for each triangle of type I we do an Ω_3 move, for each triangle of type II, an Ω_2 move or a planar isotopy, for each triangle of type III,an Ω_2 move or a planar isotopy, for each triangle of type IV, a planar isotopy. This concludes the proof. □

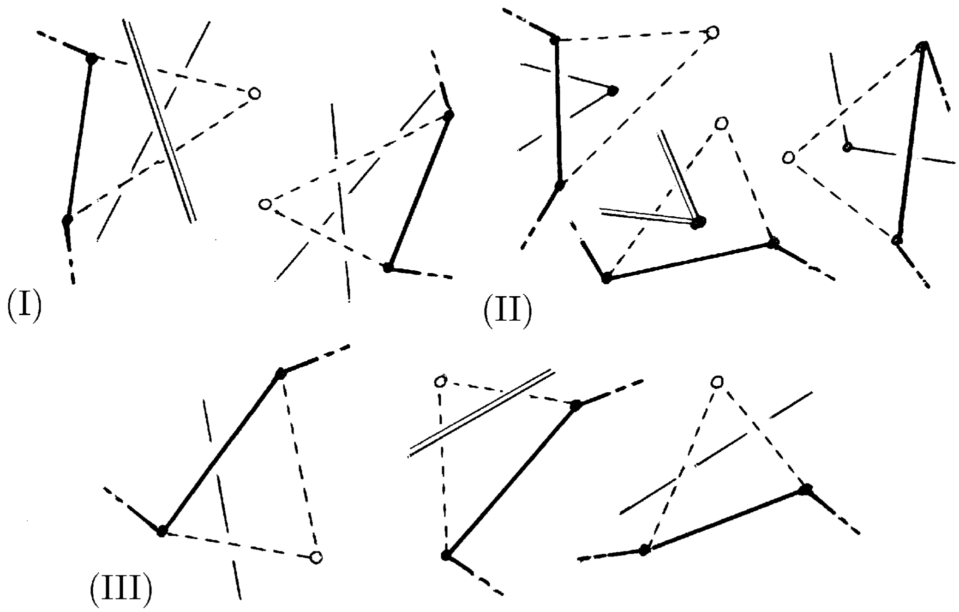


Figure 1.8. Triangles of types I, II, III, and IV

Remark 1.4. The fundamental importance of the theorem is due to the fact that it reduces a difficult three-dimensional topological problem to a (simpler) problem in two-dimensional geometric combinatorics. Actually, some textbooks in knot theory define links as link diagrams up to Reidemeister moves, thus transforming 3D knot theory into a branch of 2D geometric combinatorics, thereby hiding its three-dimensional nature. In this course, we prefer to stay in three dimensions.

1.3. Torus knots

If p, q are coprime positive integers, the *torus knot* $T(p, q)$ is defined as the closed curve lying on the standard torus and winding p times around the meridian of the torus and q times around its parallel.

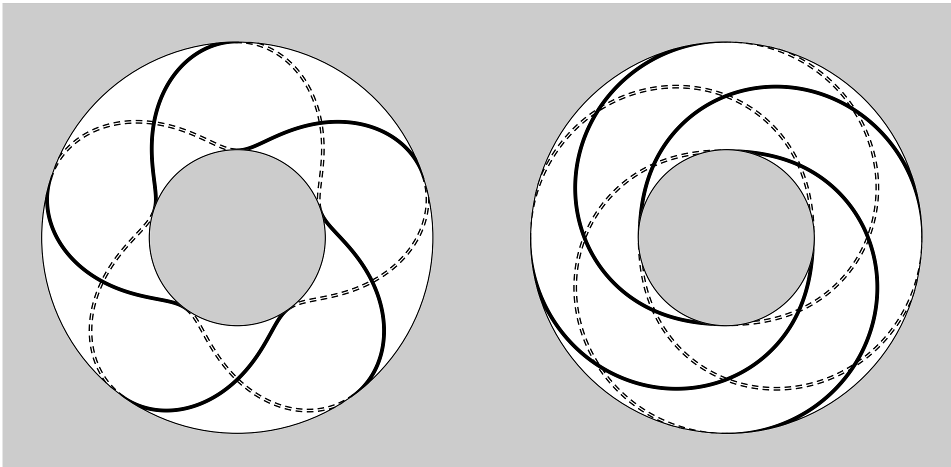


Figure 1.6. The torus knots $T(5,2)$ and $T(4,3)$

Fig. 1.7 shows two torus knots. The trefoil is obviously a torus knot, namely $T(3,2)$. The mirror image of a torus knot is also a torus knot, not isotopic to the given one. Torus knots are classified (up to mirror symmetry) by pairs of coprime natural numbers.

One can also study torus links – their definition is the object of Exercise 1.8.

1.4. Invertbility and chirality

For an orientable knot K, let us denote by $\overleftarrow{K}$ the knot obtained by reversing the orientation of K and by K^* the mirror image of K. A knot K is called *invertible* if $\overleftarrow{K} = K$, *plus-amphicheiral* if $K^* = K$, and *minus-amphicheiral* if $K^* = \overleftarrow{K}$.

In what follows, we use Rolfsen's notation for knots (e.g. 3_1 for the left trefoil and 4_1 for the eight knot). The reader can look at the other knots indicated below by googling "Rolfsen knot tables".

The following five logically possible combinations of invertibility and chirality are actually realized by specific knots:

(1) all four knots $K, \overleftarrow{K}, K^*, \overleftarrow{K}^*$ are the same – by the eight knot;

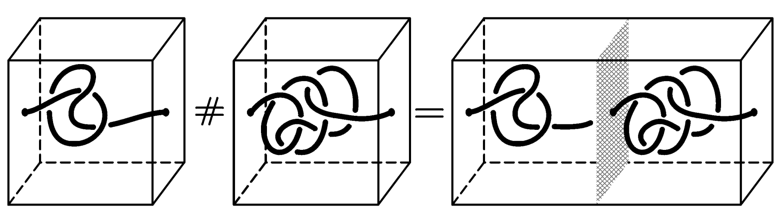
(2) all four knots $K, \overleftarrow{K}, K^*, \overleftarrow{K}^*$ are different – by the knot 9_{32};
(3) the knot K is invertible and differs from its mirror image – by the trefoil knot 3_1;
(4) the knot K is noninvertible and plus-amphicheiral – by the knot 12_{427};
(5) the knot K is noninvertible and minus-amphicheiral – by the knot 8_{17}.

We will be able to prove these facts only much later in the course by using different knot invariants.

1.5. The arithmetic of knots

In this lecture, we introduce a composition operation (called connected sum) in the set of (isotopy equivalence classes of) knots and study the algebraic structure thereby obtained. In order to do that, we shall put knots into boxes. By definition, a *boxed knot* is a non-self-intersecting polygonal curve in a cube (or a in rectangular parallelipipedon) joining the centers of two of its opposite faces. Boxed knots are oriented (from left to right in the figures). Two boxed knots are called *equivalent* (or *ambient isotopic*) if there exists a finite sequence of Δ-moves (see Sec. 1.2) performed inside the cube and taking one curve to the other.

The *composition* or *connected sum* of two knots K_1 and K_2, denoted by $K_1 \# K_2$, is obtained by joining the boxes containing them as shown in Fig. 1.6.



(a) (б)

Figure 1.6. Connected sum of two knots

Our immediate aim is to investigate the algebraic structure of the set $\mathcal{K}$ (of equivalence classes of) knots w.r.t. the connected sum operation #. It turns out that this structure is surprisingly similar to that of the set of natural numbers w.r.t. multiplication, with the unknot $\bigcirc$ playing the role of the unit 1.

The connected sum operation is obviously well defined on equivalence classes of boxed knots and possesses the two following important properties:

I. *The connected sum operation is associative and commutative:*

$$K_1\#K_2 = K_2\#K_1, \quad \big(K_1\#K_2\big)\#K_3 = K_1\#\big(K_2\#K_3\big).$$

II. *There are no inverse elements under the connected sum operation, i.e.,* $K\#K' = \bigcirc \implies K = K' = \bigcirc$.

Associativity is obvious, the proof of commutativity is shown in Fig.1.7 for a concrete example, but the construction is clearly general.

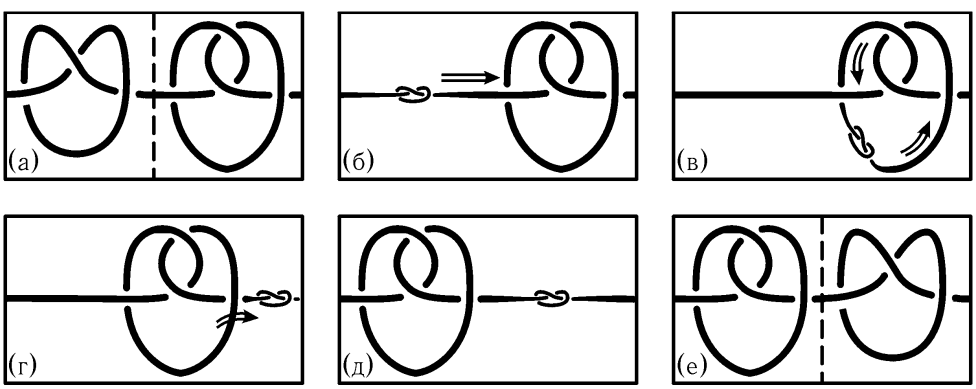


Figure 1.7. Commutativity of the connected sum operation

The proof of assertion II is similar to the following "proof" of the "equality" $1 = 0$,

$$1 = 1 + 0 + 0 + 0 + \dots = 1 + (1-1) + (1-1) + (1-1) + \dots$$
$$= 1 + (-1+1) + (-1+1) + (-1+1) + \dots$$

$$= (1-1) + (1-1) + (1-1) + \cdots = 0 + 0 + 0 + \cdots = 0,$$

The proof of assertion II is sketched in Fig. 3.3.

From the intuitive point of view, assertion II means that if you have a nontrivial knot tied at one end of a rope, then it is impossible to tie another knot at the other end so that when you pull the two ends apart, the two knots will cancel each other. In particular, if you tie a right trefoil at one end of a rope and a left trefoil at the other end, you get the granny knot, which is not trivial.

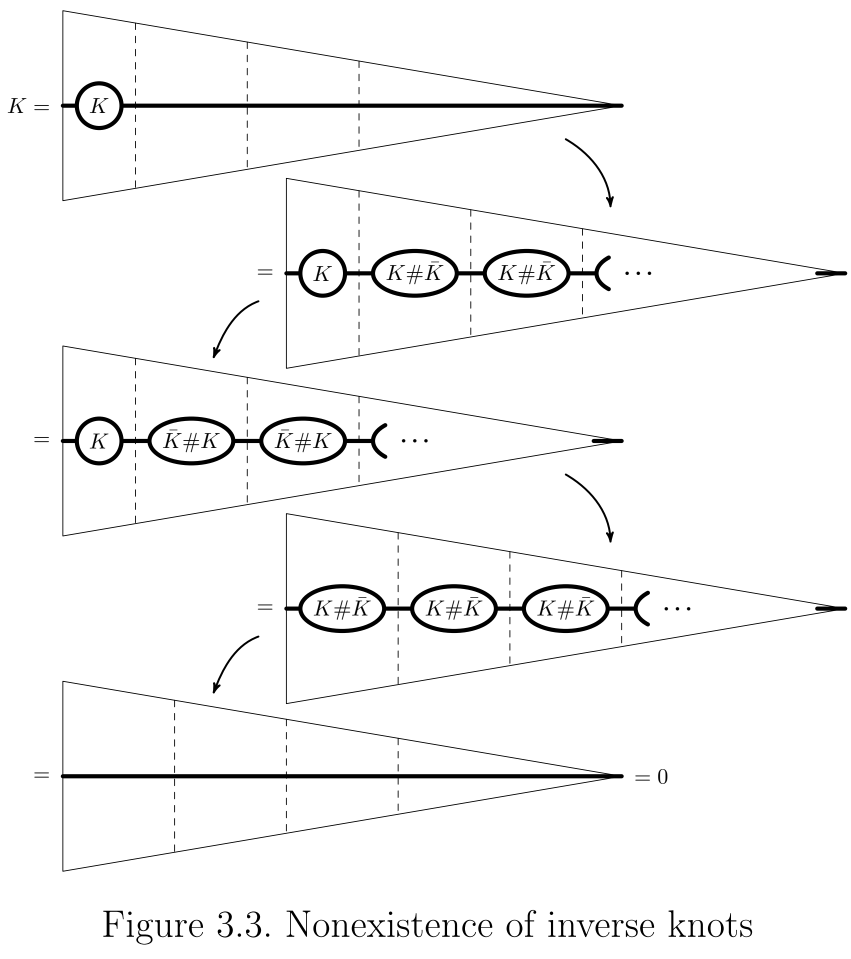


Figure 3.3. Nonexistence of inverse knots

Unlike the above "proof" of one equaling zero, the proof of assertion II can be made rigorous by using the definition of knot equivalence indicated in Remark 1.2 (Lecture 1), but we omit the details.

Before we continue the study of the algebraic structure of $(\mathcal{K}, \#)$, let us compare knots regarded as closed polygonal curves (i.e., as defined in Sec. 1.1) to the theory of boxed knots. Fortunately, the two notions turn out to yield the same theory – this follows from the next statement.

Proposition 1.1 *There is a canonical bijection between the equivalence classes of boxed knots and the isotopy equivalence classes of oriented knots.*

Sketch of the proof. To any concrete boxed map we assign an oriented knot by joining the endpoints of the boxed knot by a polygonal non-self-intersecting curve lying outside the box in the vertical plane containing the endpoints. It is obvious that this assigment is well defined on equivalence classes, and it is also obvious that it is surjective. The rigorous proof of injectivity involves some delicate results on the topology of $\mathbb{R}^3$ and is omitted.

The canonical bijection carries over to ordinary knots all the structures that we have considered for boxed knots, in particular the connected sum operation. The latter can be defined directly for ordinary knots – how this is done is shown in Fig.1.9.

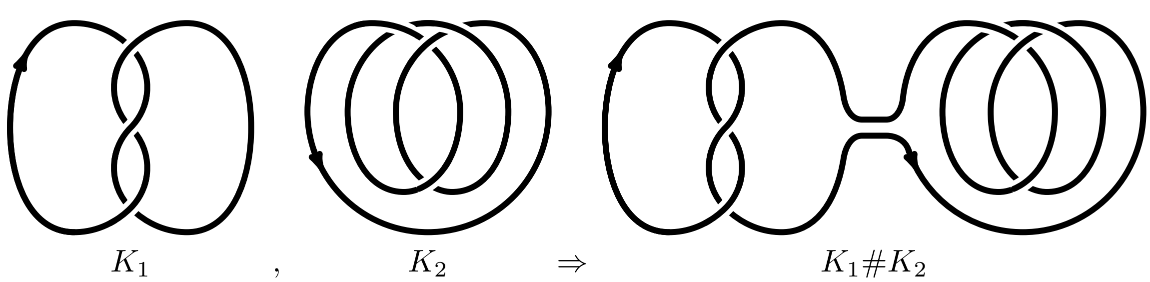


Figure 1.9. Connected sum of ordinary oriented knots

We have avoided that definition because it is rather difficult to prove that it does not depend on the choice of representatives in the equivalence classes of the given knots.

A nontrivial boxed knot is called *prime* if it cannot be presented as the sum of two nontrivial knots, i.e.,

$$K = K_1 \# K_2 \implies K_1 = \bigcirc \quad \text{or} \quad K_2 = \bigcirc.$$

If a knot is not prime, we say that it is *composite*. An example of a composite knot is shown in Fig. 1.10. The reader is invited to decompose this knot into (three) prime knots (Exercise 1.9)

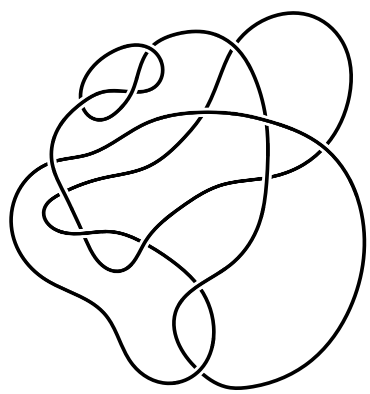


Figure 1.10. A composite knot

The main result of this section is the following

Theorem 1.2. *The set of equivalence classes of knots w.r.t. the connected sum operation is a commutative semigroup* $(\mathcal{K}, \#)$ *with unique* (*up to order*) *decomposition into prime knots, i.e.,*

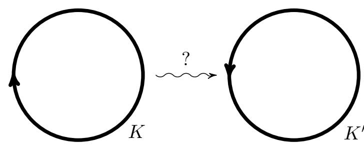
$$K \in \mathcal{K},\ K \neq \bigcirc \implies \exists! \{P_1, \ldots, P_n\} : K = P_1 \# \ldots \# P_n,$$

where the P_i *are prime knots.*

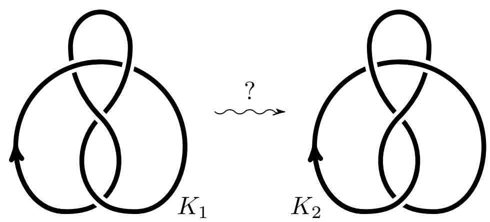
The proof of the existence and uniqueness involves some delicate 3D topology and the fundamental group, and so is omitted.

1.5. Exercises

1.1 Using Reidemeister moves, show that the two diagrams of the unknot with opposite orientations are equivalent.

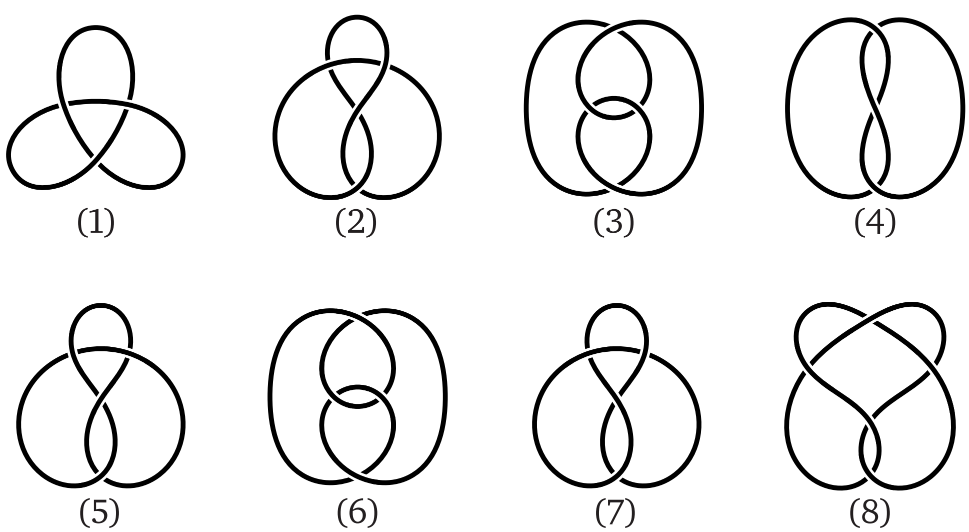


1.2. Using Reidemeister moves, show that the two knot diagrams K_1 and K_2 represent the same knot.



1.3. Using Reidemeister moves, show that reversing the orientation transforms the knot diagram of the right trefoil into an ambient isotopic knot.

1.4. Which of these knots represent the right trefoil? the figure eight knot? the unknot?



1.5. Using Reidemeister moves, show that reversing the orientation transforms the knot diagram of the eight knot into a diagram of the same eight knot.

1.6. Find a knot diagram with five crossings which is a torus knot. What are the values of p and q for that knot?

1.7. Find a knot diagram with 17 crossings which is a torus knot. What are the values of p and q for that knot?

1.8. For any pair of integers (k, l) give a reasonable definition of torus link of type (k, l) (similar to the definition of a torus knot of type (p, q)) and find the number of its components.

1.9. Represent the knot in Fig. 1.10 as the connected sum of three prime knots.