## Working example: lines on surfaces.

VARIETY OF LINES ON SURFACES OF GIVEN DEGREE. We are going to analyze the set of lines laying on a surface $S \subset \mathbb{P}_3$ of a given degree $d$.

Exercice 13.1. Carry out the complete analysis for $d = 2$.

To this aim consider the space $\mathbb{P}_N = \mathbb{P}(S^d V^*)$, of surfaces of degree $d$ in $\mathbb{P}_3 = \mathbb{P}(V)$, and identify the set of all lines in $\mathbb{P}(V)$ with the Plücker quadric $Q_P \subset \mathbb{P}_5 = \mathbb{P}(\Lambda^2 V)$. Let $\Gamma \stackrel{\text{def}}{=} \{ (S, \ell) \in \mathbb{P}_N \times Q_P \mid \ell \subset S \} \subset \mathbb{P}_N \times Q_P$ be the incidence graph.

Claim 13.2. *$\Gamma$ is closed submanifold of $\mathbb{P}_N \times Q_P$.*

Proof. 2-dimensional subspace spanned by $u, w \in V$ coincides with the image of the contraction map $V^* \longrightarrow V$, which sends $\xi \in V^*$ to $\langle\, \xi\,,\, u \wedge w \,\rangle$. So, the line $\ell = (uw)$ lies on a surface $S$ given by $F = 0$ iff $F(\langle\, \xi\,,\, u \wedge w \,\rangle) \cong 0$ identically in $\xi \in V^*$. In coordinates, let $e_\nu$ form a base of $V$, $\xi_\nu$ be the coordinates of $\xi$ w. r. t. the dual base of $V^*$, and $u \wedge w = \sum\limits_{\mu \neq \nu} p_{\mu\nu}\, e_\mu \wedge e_\nu$, where $p_{\mu\nu} = -p_{\nu\mu}$ are the associated Plücker coordinates on $\mathbb{P}_5 \supset Q$. Then $\langle\, \xi\,,\, u \wedge w \,\rangle = \sum\limits_i \xi_i \cdot (\sum_\nu p_{i\nu} e_\nu)$. Substitute this into $F$, expand the result through the monomials in $\xi$, and write down that all coefficients of this expansion vanish — this gives a system of polynomial equations on the coefficients of $F$ and $p_{ij}$ describing $\Gamma \subset \mathbb{P}_N \times Q_P \subset \mathbb{P}_N \times \mathbb{P}_5$. □

Claim 13.3. *Projection $\Gamma \xrightarrow{\pi_2} Q_P$ is surjective; all its fibers are projective spaces of dimension $d(d+1)(d+5)/6 - 1$.*

Proof. Let a line $\ell \subset \mathbb{P}(V)$ be given by $x_0 = x_1 = 0$. Then $S \supset \ell$ iff $S$ has an equation $0 = x_2 \cdot F_2(x) + x_3 \cdot F_3(x)$, where $F_2, F_3 \in S^{d-1} V^*$ are arbitrary homogeneous polynomials. These equations form a vector space $W$, which coincides with the image of the linear operator $S^{d-1}V^* \oplus S^{d-1}V^* \xrightarrow{(f,g) \mapsto x_2 f + x_3 g} S^d V^*$ whose kernel consists of all $(f, g)$ such that $x_2 f = -x_3 g$ that is possible iff $f = x_3 h$ and $g = -x_2 h$ for some $h \in S^{d-2} V^*$. Hence, the kernel is isomorphic to $S^{d-2}V^*$ and $\dim W = 2 \dim(S^{d-1}V^*) - \dim(S^{d-1}V^*) = \Big( 2\, d(d+1)(d+2) - (d-1)d(d+1) \Big)/6 = d(d+1)(d+5)/6$. □

Corollary 13.4. *$\Gamma$ is an irreducible projective manifold of dimension $d(d+1)(d+5)/6 + 3$.*

Proof. This follows at once from Corollary 12.33 and Theorem 12.29. □

Claim 13.5. *A generic[1] surface $S_d \subset \mathbb{P}_3$ of degree $d \geqslant 4$ does not contain lines.*

Proof. By Corollary 12.13, the image of the projection $\Gamma \xrightarrow{\pi_1} \mathbb{P}_N$, that is the set of all surfaces containing some lines, is closed irreducible submanifold of $\mathbb{P}_N = \mathbb{P}(S^d V^*)$. By Theorem 12.29, its dimension equals $\dim \Gamma$ minus the minimal dimension of non-empty fibres of $\pi_1$. In particular, the image is proper, as soon as $\dim \Gamma < N$, i. e. if $d(d+1)(d+5)/6 + 3 < (d+1)(d+2)(d+3)/6$, which holds for all $d \geqslant 4$. □

Claim 13.6. *Each cubic surface $S_3 \subset \mathbb{P}_3$ contains lines; generically, this line set is finite.*

Proof. Taking in the previous proof $d = 3$, we get $\dim \Gamma = N = 19$ and $\pi_1$ is surjective, if it has a non-empty 0-dimensional fiber. Let us find, say, all the lines on a cubic $C$ with affine equation $xyz = 1$. This affine piece does not contain the lines at all, because $x = x_0 + \alpha\, t$, $y = y_0 + \beta\, t$, $z = z_0 + \gamma\, t$ lies on $C$ iff $\alpha\beta\gamma = 0$, $\alpha\beta z_0 + \beta\gamma x_0 + \gamma\alpha y_0 = 0$, and $\alpha y_0 z_0 + \beta x_0 z_0 + \gamma x_0 y_0 = 0$, but $x_0 y_0 z_0 = 1$, which leads to contradiction when we go from the left to the right: for example, $\alpha = 0 \Rightarrow \beta = 0$ or $\gamma = 0 \Rightarrow \beta = \gamma = 0$. To describe $C$ at infinity, put $x = x_1/x_0$, $y = x_2/x_0$, $z = x_3/x_0$ and rewrite its equation as $x_1 x_2 x_3 = x_0^3$. Thus, $C \cap \{x_0 = 0\}$ consists of 3 lines: $x_i = x_0 = 0$, $i = 1, 2, 3$. □

Exercice 13.7*. Find all lines on the (smooth) Fermat cubic $C_F$, given by $\sum x_i^3 = 0$.

HINT. $C_F$ is preserved by the permutations of the coordinates; up to such a permutation, a pair of linear equations for $\ell \subset C_F$ is reduced, by Gauss, to $x_0 = \alpha x_2 + \beta x_3$, $x_1 = \gamma x_2 + \delta x_3$; now substitute, show that $\alpha\beta\gamma\delta = 0$, and find them...

LINES ON A SMOOTH CUBIC. Now, let $S \subset \mathbb{P}_3$ be a smooth cubic surface with an equation $F(x) = 0$.

Claim 13.8. *Reducible plane section of $S$ splits either into a line and a smooth conic or into a triple of distinct lines.*

Proof. We have to show that a plane section $\pi \cap S$ can not contain a double line component. If there is a double line $\ell \subset \pi \cap S$, take the coordinates where $\pi$ is given by $x_2 = 0$ and $\ell$ by $x_2 = x_3 = 0$. Then $F(x) = x_2 Q(x) + x_3^2 L(x) = 0$ for some linear $L$ and quadratic $Q$. Let $a$ be an intersection point of $\ell$ with the quadric $Q(x) = 0$. Then $x_2(a) = x_3(a) = Q(a) = 0$ and all partial derivatives $\partial F/\partial x_i$ vanish at $a$, i. e. $S$ is singular at $a$. □

Corollary 13.9. *A point of $S$ can belong at most to 3 lines laying on $S$ and these lines should be coplanar.*

Proof. Indeed, all lines passing through $p \in S$ and laying on $S$ belong to $S \cap T_p S$. □

Claim 13.10. *Given $\ell \subset S$, there are precisely 5 distinct planes $\pi_1, \pi_2, \ldots, \pi_5$ containing $\ell$ and intersecting $S$ in a triple of lines; moreover, if $\pi_i \cap S = \ell \cup \ell_i \cup \ell_i'$, then $\ell_i \cap \ell_j = \ell_i \cap \ell_j' = \ell_i' \cap \ell_j' = \varnothing$ $\forall\, i \neq j$ (in particular, $S$ contains some skew lines) and any line on $S$ skew to $\ell$ must intersect for each $i$ precisely one of $\ell_i$, $\ell_i'$.*

[1] i. e. any from some dense Zariski open subset in the space of all hypersurfaces

Proof. Fix a base $\{e_0, e_1, e_2, e_3\}$ for $V$ such that the line $\ell = (e_0e_1)$ (given by equations $x_2 = x_3 = 0$) lays on $S$. Then the equation $F(x) = 0$, defining $S$, can be written in this base as:

$$L_{00}(x_2, x_3) \cdot x_0^2 + 2\, L_{01}(x_2, x_3) \cdot x_0x_1 + L_{11}(x_2, x_3) \cdot x_1^2 + 2\, Q_0(x_2, x_3) \cdot x_0 + 2\, Q_1(x_2, x_3) \cdot x_1 + R(x_2, x_3) = 0 \qquad (42)$$

where $L_{ij}, Q_\nu, R \in k[x_2, x_3]$ are homogeneous of degrees $1, 2, 3$ respectively. Let us parameterize the plain pencil through $\ell$ by the points $e_\vartheta = \vartheta_2 e_2 + \vartheta_3 e_3 \in (e_2e_3)$ and write $(t_0 : t_1 : t_2)$ for homogeneous coordinates in the plane $\pi_\vartheta = (e_0e_1e_\vartheta)$ w. r. t. these 3 base points. The equation for the plane conic $(\pi_\vartheta \cap S) \setminus \ell$ is obtained from (42) by the substitution $x = (t_0 : t_1 : \vartheta_2 t_3 : \vartheta_3 t_3)$ and cancelling the common factor $t_3$. The resulting conic has the Gram matrix

$$G = \begin{pmatrix} L_{00}(\vartheta) & L_{01}(\vartheta) & Q_0(\vartheta) \\ L_{01}(\vartheta) & L_{11}(\vartheta) & Q_1(\vartheta) \\ Q_0(\vartheta) & Q_1(\vartheta) & R(\vartheta) \end{pmatrix}$$

whose determinant $D(\vartheta_2, \vartheta_3) = L_{00}(\vartheta)L_{11}(\vartheta)R(\vartheta) + 2\, L_{01}(\vartheta)Q_0(\vartheta)Q_1(\vartheta) - L_{11}(\vartheta)Q_0^2(\vartheta) - L_{00}(\vartheta)Q_1^2(\vartheta) - L_{01}(\vartheta)^2 R(\vartheta) \in k[\vartheta_2, \vartheta_3]$ is homogeneous of degree 5 in $\vartheta = (\vartheta_2 : \vartheta_3)$, i. e. has 5 roots counted with multiplicities. We have to show that all these roots are simple. Each root corresponds to the splitting of the conic into a pair of lines, whose intersection point lies either on $\ell$ or outside $\ell$.

In the first case, we can fix a base in such a way that two lines in question are $\ell' = (e_0e_2)$ and $\ell'' = \big(e_0\,(e_1 + e_2)\big)$ given by equations $x_3 = x_1 = 0$ and $x_3 = (x_1 - x_2) = 0$; this corresponds to the root $\vartheta = (1 : 0)$ and its multiplicity equals the highest power of $\vartheta_3$ which divides $D(\vartheta_2, \vartheta_3)$. Since $\ell, \ell', \ell'' \subset S$, the equation (42) has a form $x_1x_2(x_1 - x_2) + x_3 \cdot q(x)$ with some quadratic $q(x)$, i. e. the only elements of $G$ possibly not divisible by $\vartheta_3$ are $L_{11} \equiv x_2 \pmod{\vartheta_3}$ and $Q_1 \equiv -x_2^2/2 \pmod{\vartheta_3}$. So, $D(\vartheta_2, \vartheta_3) \equiv -L_{00}Q_1^2 \pmod{\vartheta_3^2}$ and this term is of order 1 in $t_3$ as soon $x_1x_2^2$ and $x_0^2x_2$ do come in (42) with non zero coefficients. But the first is the only monomial that gives non zero contribution into $\partial F/\partial x_1$ computed at $e_2 \in S$ and the second — in $\partial F/\partial x_2$ at $e_0 \in S$; so, they do.

In the second case, we can fix a base in order to have $\ell' = (e_0e_2)$, $\ell'' = (e_1e_2)$ (given by $x_3 = x_1 = 0$ and $x_3 = x_0 = 0$), and the same root $\vartheta = (1 : 0)$. Now, (42) takes $x_0x_1x_2 + x_3 \cdot q(x)$ non zero modulo $\vartheta_3$ entry of $G$ is only $L_{01} \equiv x_2/2 \pmod{\vartheta_3}$. So, $D(\vartheta_2, \vartheta_3) \equiv -L_{01}^2 R \pmod{\vartheta_3^2}$, which is of order 1 in $t_3$ if $x_2^2x_3$ and $x_0x_1x_2$ really do appear in (42). The second does, because otherwise $F$ is divisible by $x_3$. The first is the only monomial that gives non zero contribution into $\partial F/\partial x_3$ computed at $e_2 \in S$.

The rest assertions follow immediately from Corollary 13.9, Claim 13.8 and remark that any line in $\mathbb{P}_3$ intersects any plane. □

Claim 13.11. *Any 4 mutually skew lines on $S$ do not lay on a common quadric and always there exist either 1 or 2 (but no more!) lines on $S$ intersecting all these 4 lines.*

Proof. Any line passing through 4 distinct points of $S$ has to lay on $S$. If 4 lines on $S$ belong to a line family ruling a quadric[2] then the second ruling line family of this quadric, that is the quadric itself, lays on $S$, i. e. $S$ is reducible. Now it remains to apply Exercice 2.20. □

Configuration of 27 lines. Take 2 skew lines $a, b \subset S$ and construct 5 pairs of lines $\ell_i, \ell_i'$ predicted by Claim 13.10 applied to $\ell = a$. Let us write $\ell_i$ for those which do meet $b$ and $\ell_i'$ for those which do not. Then there are 5 more lines $\ell_i''$ coupled with $\ell_i$ by Claim 13.10 applied to $\ell = b$. Each $\ell_i''$ meets $b$ but neither $a$ nor $\ell_j$ with $j \neq i$; so, it intersects all $\ell_j'$ with $j \neq i$. Any line $c \subset S$, different from 17 just constructed, is skew to $a$, $b$ but meets either $\ell_i$ or $\ell_i'$ for each $i$. By Claim 13.11, all lines meeting $\geqslant 4$ of $\ell_i$'s are exhausted by $a$, $b$. If $c$ meets $\leqslant 2$ out of $\ell_i$'s, then, up to a permutation of indices, it meets $\ell_1'$, $\ell_2'$, $\ell_3'$ and, say, either $\ell_4'$ or $\ell_5$; but in the both cases $a$ and $\ell_5''$ are the only such the lines. So, $c$ has to intersect precisely 3 out of $\ell_i$.

Claim 13.12. *Remaining lines $c \subset S$ are in 1–1 correspondence with 15 triples $\{i, j, k\} \subset \{1, 2, 3, 4, 5\}$.*

Proof. There is at most one line $c$ intersecting a given triple of $\ell_i$'s — this is the second possible line besides $a$ meeting all these $\ell_i$'s and the rest $\ell_j'$'s (all 5 are mutually skew). On the other hand, by Claim 13.10, for each $i$ there are precisely 10 lines on $S$ intersecting $\ell_i$: 4 of them are $a$, $b$, $\ell_i'$, $\ell_i''$ and other 6 have to intersect exactly 2 of the rest four $\ell_j$'s. So, we have 1–1 correspondence between these 6 lines and $6 = \binom{4}{2}$ choices of pairs of $\ell$'s. □

Corollary 13.13. *Each smooth cubic surface $S \subset \mathbb{P}_3$ contains precisely 27 lines; their incidence combinatorics is the same for all $S$.* □

Exercice 13.14. Let $\mathfrak{G} \subset \mathfrak{S}_{27}$ be a group of all permutations of the 27 lines preserving all the incidence relations between them; find the order of $\mathfrak{G}$. (ANSWER: $|\mathfrak{G}| = 51\,840 = 2^7 \cdot 3^4 \cdot 5$)

Exercice 13.15*. Consider the field of 4 elements: $\mathbb{F}_4 \stackrel{\text{def}}{=} \mathbb{F}_2[\omega]/(\omega^2 + \omega + 1)$, where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Like for $\mathbb{R} \subset \mathbb{C}$, the extension $\mathbb{F}_2 \subset \mathbb{F}_4$ has a conjugation automorphism: $z \longmapsto \overline{z} \stackrel{\text{def}}{=} z^2$, which lives $\mathbb{F}_2$ fixed and permutes two roots of $\omega^2 + \omega + 1$. Show that a factor group of the *unitary* $4 \times 4$ - matrices with entries in $\mathbb{F}_4$ modulo the scalar matrices is canonically embedded as a (normal) subgroup of index 2 into the group $\mathfrak{G}$ from Exercice 13.14.

Hint. This group preserves the Fermat cubic $C_F$ (see Exercice 13.7) whose equation over $\mathbb{F}_4$ turns to Hermitian form $\sum x_i\overline{x}_i$.

[2] comp. with Corollary 2.19