What is a category of metric spaces? Answering the question

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1. Word metric $d(-,-)$ on a finitely generated group G with respect to a (finite) generating set $X \subset G$ is defined as follows: $d(1_G, g)$ equals to the *least* number of letters in a word $w = x_{i_1}^{\mp 1} x_{i_2}^{\mp 1} \ldots x_{i_k}^{\mp 1}$ representing element g in a group, and it's left invariant, so $d(g, g') = d(1_G, g^{-1}g')$.

   Slightly different way to say the same is that d is induced from geodesic metric on Cayley graph w.r.t. generating set X via embedding $G \longrightarrow \mathrm{Cay}(G, X)$ sending the group to vertices of Cayley graph.

   (a) Recall those traits of a function between metric spaces $f : X \longrightarrow Y$ and explore a bit their behaviour under composition in different directions and combinations:

   i. Cobounded (quasi-dense): f is surjective up to a finite error, i. e. there is a positive D such that $d_Y(y, f(X)) \leq D$ for any point $y \in Y$; equivalently, every D-ball contains a point in image.

   ii. Uniform embedding: f preserves divergence, i. e. for every sequence of pairs $\{x_i, x_i'\} \in X \times X$ with $\lim d_X(x_i, x_i') = \infty$, $\lim d_Y(f(x_i), f(x_i'))$ is also ∞.

   iii. Quasi-Lipschitz: dilation of metric by f is at most affine, i. e. there exist positive real numbers C, D such that $d_Y(f(x), f(x')) \leq C \cdot d_X(x, x') + D$. (NB: C, D are *numbers*, not functions of elements)

   iv. Bornologous: dilation of metric by f depends only on *value* of metric, i. e. there exists a function $scale : \mathbb{R} \longrightarrow \mathbb{R}$ such that $d_Y(f(x), f(x')) \leq scale(d_X(x, x'))$.

   v. Metrically proper: preimage of every ball is contained in a ball. (NB: no uniformity conditions, it's just usual properness for topologies induced by metric)

   vi. Noncollapsing: contraction of metric by f is at most affine, i. e. there exist positive real numbers C, D such that $d_X(x, x') \leq C \cdot d_Y(f(x), f(x')) + D$.

   vii. Effectively proper: contraction of metric by f depends only on *value* of metric, i. e. there exists a function $scale : \mathbb{R} \longrightarrow \mathbb{R}$ such that preimage of every r-ball is contained in a $scale(r)$-ball.

   viii. Quasi-isometric embedding: (iii) + (vi), i. e. f is "isometry up to affine error". (sidenote: affine functions on $\mathbb{R}$ form a group! and affine functions on $\mathbb{R}_{\geq 0}$ form a monoid, which is good enough)

   (b) Prove that when X is a length space (...look up a definition elsewhere or cook it up yourself) then (iii) and (iv) are equivalent, and both are equivalent to a following "local boundedness" condition: there exist positive real R, R' such that image of every R-ball in X is contained in an R'-ball in Y. One can also consider a similar condition that will be slightly stronger if X is not a length space: $d_X(x, x') < R \implies d_Y(f(x), f(x')) < R'$)

   (c) Define a *quasi-homogeneous* metric space. Prove that (i)+(ii) is equivalent to (i)+(viii) for that class. If you have failed, prove it for finitely generated groups with word metric.

2. Call two maps q, r from a set A to a metric space Y *close*, if $d_Y(q(a), r(a)) \leq D$ for some positive D. Call maps $f : X \to Y$, $g : Y \to X$ between metric spaces *quasi-inverse*, if gf and fg are close to identity maps on X and Y respectively.

   (a) Suppose that f satisfies (i)+(viii). Construct a quasi-inverse map $g : Y \to X$ with the same properties. Such pair f, g is usually called a *quasi-isometry*.

   (b) Construct an example of f, X, Y such that f is (i)+(ii), but has no quasi-inverse of same type.

   (c) Find reasonable conditions on X (and maybe Y) such that (i)+(ii) maps admit quasi-inverses.

3. Map is called *coarse* (if it preserves coarse structure defined by metric, look it up in wiki)

   ...if it is (iv)+(v).

   (a) Note an obvious fact that maps $f, g : X \to Y$ are close if there is a coarse map $X \times [0, 1] \to Y$ restricting to f and g on the ends of cylinder. Rethink your actions during the quest for quasi-inverses in previous section. Find a funny analogy with homotopy. (Maybe that analogy is useful.)

   (b) Coarse map admits a quasi-inverse $\iff$ it is (i)+(iv)+(vii).

   (c) (*) Let B, C be locally compact, compactly generated topological groups (finitely generated discrete group is an example of such; Lie groups as well). Prove if there is a coarse equivalence $B\ toC$, then it is actually a quasi-isometry.

4. (*) Let G, H be finitely generated (or countable, if you're feeling brave; or locally compact, if your eagerness knows no bounds) groups. Suppose there are proper cocompact commuting actions of G and H on a locally compact Hausdorff X (no metric here!). Prove that existence of such an action is equivalent to G and H being coarsely equivalent. As we can deduce from previous exercise, they are even quasi-isometric! (Hint: look at $f \in H^G$ where function space has compact-open topology. Find a suitable subspace where action will be cocompact.)

5. Prove Hopf-Rinow theorem (or find a proof in literature): metric as a function on $X \times X$ is proper for complete locally compact length space X. Conversely, properness of metric on a length space X implies that X is locally compact, complete (and geodesic).

6. "Lemma about geometric action": suppose G isometrically acts on a *connected* metric space X properly (action map $G \times X \to X$ is proper for discrete topology on G) and cocompactly (there's a compact subset of X such that its orbit covers X). Prove (or assume) that X is locally compact and complete. Prove that G is finitely generated. (Hint: take some nbhd V of certain compact subset in X and look at elements which do not translate V inside its complement). Let's call those actions geometric, because it's appropriate: if M is a Riemannian manifold, or a finite CW complex with decent metric on it making it into a compact metric space, then $\pi_1(M)$ acts geometrically on universal cover on M.

7. Prove Milnor-Svarc theorem: suppose G acts geometrically on a length space X. Then G is quasi-isometric to G.