CHAPTER 2

Von Neumann algebras and traces

2.1. Topologies in the algebra of bounded linear operators in a Hilbert space

By $\mathcal{H}$ we will usually denote a Hilbert space with the inner product $(\cdot,\cdot)$. We shall only consider complex Hilbert spaces; then $(\cdot,\cdot)$ is a Hermitian inner product.

Recall that a complex linear space $\mathcal{H}$ with an Hermitian positive definite inner product $(\cdot,\cdot)$ is called a *pre-Hilbert space*. The inner product induces a norm and a metric defined by the formulas

$$||x|| = (x,x)^{1/2}, \quad \rho(x,y) = ||x-y||, \quad x, y \in \mathcal{H}.$$

Then $\mathcal{H}$ is a *Hilbert space* if it is complete with respect to the metric. An important property of Hilbert spaces $\mathcal{H}$ is that for every closed subspace $L \subset \mathcal{H}$ the space $\mathcal{H}$ becomes an orthogonal direct sum $\mathcal{H} = L \oplus L^{\perp}$, where $L^{\perp}$ is the orthogonal complement of L:

$$L^{\perp} = \{x | x \perp y \text{ for every } y \in L\}.$$

Later on $\mathcal{H}$ will always denote a Hilbert space. The metric ρ defines topology on $\mathcal{H}$. A linear operator $A : \mathcal{H} \to \mathcal{H}$ is continuous if and only if it is bounded, that is, the operator norm of A is finite:

$$||A|| = \sup_{x \in \mathcal{H} \setminus \{0\}} \frac{||Ax||}{||x||} < \infty.$$

Denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded linear operators in $\mathcal{H}$. This is a $*$-algebra with the involution $A \mapsto A^*$ where the adjoint operator is defined by the usual identity

$$(Ax, y) = (x, A^*y), \quad x, y \in \mathcal{H}.$$

It is easy to see that the involution is an (antilinear) isometry with respect to the operator norm: $||A^*|| = ||A||$.

For any Hilbert space $\mathcal{H}$, the space $\mathcal{B}(\mathcal{H})$ is a Banach space with respect to the operator norm. It is even a Banach algebra, which means that $||AB|| \le ||A|| ||B||$ for $A, B \in \mathcal{B}(\mathcal{H})$. It is an involutive Banach algebra which means that the given involution is an antilinear isometric antiisomorphism, that is, $(AB)^* = B^*A^*$ for A, $B \in \mathcal{B}(\mathcal{H})$. Moreover $||A^*A|| = ||A||^2$ for every $A \in \mathcal{B}(\mathcal{H})$ which means that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.

Exercise 1. Prove that $||A^*A|| = ||A||^2$, $A \in \mathcal{B}(\mathcal{H})$.

Definition 1. *C^*-algebra* is an involutive Banach algebra such that $||a^*a|| = ||a||^2$ for every $a \in \mathcal{H}$. Every norm-closed $*$-subalgebra in $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. It can be proved that every C^*-algebra is isometrically $*$-isomorphic to a norm-closed $*$-subalgebra in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (but neither $\mathcal{H}$ nor the $*$-subalgebra are uniquely defined).

Definition 2. The *strong operator topology* on $\mathcal{B}(\mathcal{H})$ is defined by the following set of basic neighborhoods of any operator $A \in \mathcal{B}(\mathcal{H})$;

$$U(x_1, x_2, \ldots, x_N; \varepsilon; A) := \{B | B \in \mathcal{B}(\mathcal{H}), \|(B - A)x_i\| < \varepsilon, \quad i = 1, \ldots, N\}$$

where $\{x_1, x_2, \ldots, x_N\}$ is an arbitrary finite subset in $\mathcal{H}$, $\varepsilon > 0$.

Of course by dividing all $x_1, x_2, \ldots, x_N$ by ϵ we can replace ε by 1 without changing the neighborhood. So we can always assume $\varepsilon = 1$.

Definition 3. The *weak operator topology* on $\mathcal{B}(\mathcal{H})$ is defined by the following set of basic neighborhoods of any operator $A \in \mathcal{B}(\mathcal{H})$:

$$\begin{aligned} U(x_1, x_2, \ldots, x_N; y_1, y_2, \ldots, y_N; \varepsilon; A) : \ = \ & \{B | B \in \mathcal{B}(\mathcal{H}), |((B - A)x_i, y_i)| < \varepsilon, \\ & i = 1, \ldots, N\} \end{aligned}$$

where $\{x_1, x_2, \ldots, x_N\}$ and $\{y_1, y_2, \ldots, y_N\}$ are two finite subsets in $\mathcal{H}$ (with the same number N of elements; some of the elements x_i, y_j may coincide), and $\varepsilon > 0$.

We will also use the convergence of nets in different topologies. Let us recall necessary definitions. A *directed set* is a partially ordered set $(\Gamma, \geq)$ such that for every $\gamma_1, \gamma_2 \in \Gamma$ there exists $\gamma \in \Gamma$ such that $\gamma \geq \gamma_1$ and $\gamma \geq \gamma_2$. A *net* in a set X is a map $x : \Gamma \to X$ of a directed set Γ to X. If X is a topological space then we shall say that the net $x : \Gamma \to X$ converges to $\bar{x} \in X$ (and write $x_\gamma \to \bar{x}$ or $\lim_\Gamma x_\gamma = \bar{x}$, where $x_\gamma = x(\gamma)$, $\gamma \in \Gamma$) if for every neighborhood U of $\bar{x}$ there exists $\gamma_0 \in \Gamma$ such that $\gamma \geq \gamma_0$ implies $x_\gamma \in U$. In a particular case $\Gamma = \mathbb{N} = \{1, 2, 3, \ldots\}$ (with the standard order) a net becomes a sequence and we get the usual notion of convergence. If $\{A_\gamma | \gamma \in \Gamma\}$ is a net in $\mathcal{B}(\mathcal{H})$ then we shall write $A_\gamma \xrightarrow{s} A$ or $s\text{-}\lim_\Gamma A_\gamma = A$ if A is the limit of this net $\{A_\gamma | \gamma \in \Gamma\}$ in the strong operator topology. Similarly $A_\gamma \xrightarrow{w} A$ or $w\text{-}\lim_\Gamma A_\gamma = A$ will mean that the net converges to A in the weak operator topology.

Obviously $A_\gamma \xrightarrow{s} A$ if and only if $\|(A_\gamma - A)x\| \to 0$ for every $x \in \mathcal{H}$. Similarly $A_\gamma \xrightarrow{w} A$ if and only if $((A_\gamma - A)x, y) \to 0$ for every $x, y \in \mathcal{H}$.

Exercise 2. Find an example of a sequence of operators that converges strongly, but not in the norm-topology.

Exercise 3. Find an example of a sequence of operators that converges weakly, but not strongly.

2.2. Spectral Theorem for bounded self-adjoint operators.

Theorem 1. *Let $A = A^* \in \mathcal{B}(\mathcal{H})$. Then there exist a measure space. $(X, d\mu)$, a bounded measurable function $a : X \to \mathbb{R}$ and a unitary operator $V : \mathcal{H} \to L^2(X, d\mu)$ such that $A = V^{-1} M_a V$ where M_a is the operator of multiplication by a in $L^2(X, d\mu)$, i.e. $(M_a f)(x) = a(x) f(x)$ for every $f \in L^2(X, d\mu)$.*

This means that A is unitarily equivalent to a multiplication operator by a bounded real-valued function. The same is true for a unitary operator in $\mathcal{H}$ except that the function defining the corresponding multiplication operator will be a measurable complex-valued function $u : X \to \mathbb{C}$ such that $|u(x)| = 1$ for every $x \in X$. Similarly if $A \in \mathcal{B}(\mathcal{H})$ is normal i.e. $AA^* = AA^*$ then it is unitarily equivalent to a multiplication operator by a bounded measurable complex-valued function.

An important modification concerns commuting families of self-adjoint operators and says that they can be treated simultaneously:

THEOREM 2. (Spectral Theorem for commuting families of self-adjoint operators.) *Let $\{A_i | i \in I\}$ be any set of self-adjoint operators in $\mathcal{H}$ such that $A_i A_j = A_j A_i$ for every $i, j \in I$. Then there exists a measure space $(X, d\mu)$, a family of bounded measurable functions $a_i : X \to \mathbb{R}$, $i \in I$, and a unitary operator $V : \mathcal{H} \to L^2(X, d\mu)$, such that $A_i = V^{-1} M_{a_i} V$ for every $i \in I$.*

The spectral representation of a self-adjoint operator as a multiplication operator is not unique. However it allows to develop uniquely defined functional calculus.

DEFINITION 1. Let $A = A^* \in \mathcal{B}(\mathcal{H})$ and let $f : [-||A||, ||A||] \to \mathbb{C}$ be a Borel function. The operator $f(A) \in \mathcal{B}(\mathcal{H})$ is defined then as follows: take a spectral representation of A, that is, a unitary operator $U : \mathcal{H} \to L^2(X, d\mu)$ such that $A = U^{-1} M_a U$, where $a : X \to \mathbb{R}$ is a measurable bounded function; then define $f \circ a : X \to \mathbb{C}$ by $(f \circ a)(x) = f(a(x))$ and put $f(A) = V^{-1} M_{f \circ a} V$.

Note that $||A|| = ||M_a||$ implies the inequality $|a(x)| \le ||A||$ for almost all x. Hence the function $f \circ a$ is almost everywhere defined which is sufficient for the operator $M_{f \circ a}$ to be well defined. Therefore $f(A)$ can be defined in this way.

THEOREM 3. *The operator $f(A)$ from the previous definition does not depend on the choice of the spectral representation of A.*

PROOF. If f is a polynomial, then it is easy to see that the given definition of $f(A)$ gives the same result as the substitution of A to the polynomial f, hence the result does not depend on the choice of the spectral representation. Now if f is continuous then we can find a sequence of polynomials $p_k \in \mathbb{C}[\lambda]$, $k = 1, 2, \dots$ such that

$$\sup_{\lambda \in [-||A||, ||A||]} |f(\lambda) - p_k(\lambda)| \to 0 \text{ as } k \to +\infty.$$

It follows that
$||f \circ a - p_k(a)||_{L^\infty} \to 0$ as $k \to \infty$, where $||h||_{L^\infty} = \operatorname{ess\,sup}_{x \in X} |h(x)|$. Hence
$||f(A) - p_k(A)|| \to 0$ as $k \to \infty$. This proves the required statement for continuous functions f.

Now let Δ be an interval which may be open, closed or contain one of his ends, and also may be semi-infinite (i.e. it may have one of the forms $(-\infty, \lambda)$, $(-\infty, \lambda]$, $[\lambda, \mu]$, $(\lambda, \mu]$, (λ, μ), $[\lambda, \mu)$, (λ, ∞), $[\lambda, \infty)$). Let us consider the characteristic function χ_Δ of this interval i.e. $\chi_\Delta(\lambda) = 1$ if $\lambda \in \Delta$ and 0 otherwise. Let us put $E(\Delta) = \chi_\Delta(A) = V^{-1} M_{\chi_\Delta} V$ and prove that $E(\Delta)$ does not depend on the choice of the spectral representation. To do this, first note that replacing Δ by $\Delta_0 = \Delta \cap [-||A||, ||A||]$ does not change $E(\Delta)$. So we may suppose that $\Delta \subset [-||A||, ||A||]$. Now take a sequence of continuous functions $f_k : [-||A||, ||A||] \to \mathbb{R}$, $k = 1, 2, \dots$, such that $|f_k(\lambda)| \le 1$ for all $\lambda \in \mathbb{R}$, and $f_k \to \chi_\Delta$ almost everywhere. Then using the Lebesgue dominated convergence theorem we easily obtain that $s\text{-}\lim_{k \to \infty} M_{f_k \circ a} = M_{\chi_\Delta \circ a}$, hence $E(\Delta) = s\text{-}\lim_{k \to \infty} f_k(A)$ which proves the desired independence of $E(\Delta)$ of the spectral representation. In particular the operator $E_\lambda = E((-\infty, \lambda])$ is independent on the spectral representation.

Consider the function $\lambda \mapsto (E_\lambda y, z)$ for fixed $y, z \in \mathcal{H}$. If $y = z$ then it is increasing, equals 0 on $(-\infty, -||A||)$ and constant on $(||A||, +\infty)$. In general it has bounded variation because it can be represented as a linear combination of functions of the same form with $y = z$ by the polarization identity

$$\begin{aligned}(E_\lambda y, z) &= \tfrac{1}{4}(E_\lambda(y+z), y+z) - \tfrac{1}{4}(E_\lambda(y-z), y-z) \\ &+ \tfrac{1}{4}(E_\lambda(y+iz), y+iz) - \tfrac{1}{4}(E_\lambda(y-iz), y-iz)\end{aligned}$$

(which is true, of course, if we replace E_λ by any self-adjoint operator).

Now it is easy to prove that if $f : [-||A||, ||A||] \to \mathbb{C}$ is any Borel function then for every $y, z \in \mathcal{H}$

$$(f(A)y, z) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda y, z)$$

where the integral on the right-hand side is the Lebesgue-Stiltjes integral (integral of f with respect to the complex-valued measure defined by the function of bounded variation $\lambda \mapsto (E_\lambda y, z)$ in such a way that the measure of the interval $\Delta \subset \mathbb{R}$ is equal $(E(\Delta)y, z))$. To do this it is sufficient to prove the corresponding equality in $L^2(X, d\mu)$ for $A = M_a$ where it reduces to standard measure-theoretic considerations. Another proof may be obtained if we check the equality for the functions f which are characteristic functions of Borel subsets in $[-||A||, ||A||]$, by showing that the subsets where it is true, form a σ-algebra, and then representing each bounded Borel function f as a uniform limit of finite linear combinations of characteristic functions of Borel subsets. $\square$

REMARK 1. In particular the operator A itself can be represented by the formula

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda$$

which we proved in a weak sense but which is actually true in norm topology as is easily seen from the spectral representation.

REMARK 2. Similarly functional calculus of several commuting self-adjoint operators can be constructed.

DEFINITION 2. Let $A \in \mathcal{B}(\mathcal{H})$. We shall write $A \geq 0$ if $A = A^*$ and $(Af, f) \geq 0$ for every $f \in \mathcal{H}$.

EXERCISE 1. Prove that the condition $A = A^*$ may be omitted: if $A \in \mathcal{B}(\mathcal{H})$ and $(Af, f) \in \mathbb{R}$ for every $f \in \mathcal{H}$ then $A^* = A$.

Hint. Use the polarization equality.

DEFINITION 3. Let $A, B \in \mathcal{B}(\mathcal{H})$. We shall write $A \geq B$ if $A - B \geq 0$. (This is usually used only for self-adjoint A and B).

THEOREM 4. *Let $A \in \mathcal{B}(\mathcal{H})$ and $A \geq 0$. Then there exists $B \in \mathcal{B}(\mathcal{H})$ such that $B \geq 0$ and $B^2 = A$. Such an operator B is unique (and will be denoted $\sqrt{A}$ or $A^{1/2}$). It commutes with all operators commuting with A.*

PROOF. The existence of B may be deduced from the spectral theorem (though some proofs of the spectral theorem use independently proved existence of the square root of the operator $A \geq 0$). Namely if $A = U^{-1} M_a U$ is a spectral representation of A in $L^2(X, d\mu)$ then $A \geq 0$ equivalent to the inequality $a(x) \geq 0$ almost everywhere on X. Hence we can take $B = U^{-1} M_{\sqrt{a}} U$. Representing the function $\lambda \mapsto \sqrt{\lambda}$ on $[0, ||A||]$ as a uniform limit of polynomials $p_k(\lambda)$, $k = 1, 2, \ldots$, we see that

$$\lim_{k \to \infty} ||B - p_k(A)|| = 0.$$

Hence B commutes with every operator, commuting with A.

Now suppose that $B_1 \in \mathcal{B}(\mathcal{H})$, $B_1 \geq 0$ and $B_1^2 = A$. We have to prove that $B_1 = B$. Obviously $B_1 A = AB_1 = B_1^3$. Hence $B_1 B = BB_1$. Now the equality $0 = B_1^2 - B^2 = (B_1 - B)(B_1 + B)$ implies that $B_1 = B$ (it is obvious e.g. if B and B_1 are both represented in $L^2(X, d\mu)$ as multiplication operators by non-negative functions; such a representation is possible since B_1 and B commute). □

Exercise 2. (a) Prove existence of $\sqrt{A}$ without any use of the spectral theorem.

(b) Prove uniqueness of $\sqrt{A}$ without any use of the spectral representation of two commuting self-adjoint operators.

Exercise 3. Suppose that $A, B \in \mathcal{B}(\mathcal{H})$, $A \geq 0$, $B \geq 0$.

(a) Prove that if $A \geq B$ then $\sqrt{A} \geq \sqrt{B}$

(b) Give an example that shows that the converse is false.

Missing proofs and more details about the results of this section can be found e.g. in [**BeSh**], Appendix 1, and [**ReSi**], Ch.VII.

2.3. Polar decomposition

Definition. A *partial isometry* in $\mathcal{H}$ is an operator $U \in \mathcal{B}(\mathcal{H})$ such that it maps $(\mathrm{Ker}U)^\perp$ to $\mathrm{Im}U$ isometrically i.e. $\|Ux\| = \|x\|$ for every $x \in (\mathrm{Ker}U)^\perp$.

Due to the polarization equality we can say equivalently that

$$(Ux, Uy) = (x, y), \qquad x, y \in (\mathrm{Ker\ U})^\perp$$

or that $U^*U = E \in \mathrm{Proj}(\mathcal{H})$ (i.e. is an orthogonal projection in $\mathcal{H}$) such that $\mathrm{Ker}E = \mathrm{Ker}U$. Now it becomes obvious that U^* is then a partial isometry too, and $\mathrm{Ker}U^* = (\mathrm{Im}U)^\perp$. Hence $\mathrm{Im}U^* = (\mathrm{Ker}U)^\perp$ so U is a partial isometry if and only if $E = U^*U$ and $F = UU^*$ are orthogonal projections. Then $\mathrm{Im}E = \mathrm{Im}U^* = (\mathrm{Ker}U^*)^\perp$.

Theorem. *For every $A \in \mathcal{B}(\mathcal{H})$ there exist $S = S^* \in \mathcal{B}(\mathcal{H})$, $S \geq 0$, and a partial isometry U with* $\mathrm{Ker}\, U = (\mathrm{Im}\, S)^\perp = \mathrm{Ker}\, S$, *such that $A = US$. Such a decomposition of A is unique and is called polar decomposition. We shall denote $S = |A|$. Obviously* $\mathrm{Ker}\, S = \mathrm{Ker}\, A$.

Proof. Let us begin with the proof of uniqueness. If $A = US$ with all the conditions of the theorem satisfied, then $A^*A = SU^*US = S^2$, hence $S = \sqrt{A^*A}$ and U is obviously uniquely defined by the equality $A = US$.

To prove existence we can just take $S = \sqrt{A^*A}$ and then define U by the equalities $Ux = 0$ if $x \in (\mathrm{Im}S)^\perp = \mathrm{Ker}S = \mathrm{Ker}A$; $U(Sx) = Ax$, $x \in (\mathrm{Ker}S)^\perp$. This correctly defines U on $\mathrm{Im}S + (\mathrm{Im}S)^\perp$. It remains to note that $(Sx, Sx) = (S^*Sx, x) = (S^2x, x) = (A^*Ax, x) = (Ax, Ax)$, so $\|Sx\| = \|Ax\|$ hence U is an isometry on $\mathrm{Im}S$ and can be extended continuously to $\overline{\mathrm{Im}S}$ which gives the desired result. □

2.4. Monotone convergence of operators

Theorem. *Let Γ be a directed set and $\{A_\gamma | \gamma \in \Gamma\}$ an increasing net of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ i.e. $A_\gamma^* = A_\gamma$ for all γ, and $\gamma \leq \gamma'$ implies $A_\gamma \leq A_{\gamma'}$. Assume that it is bounded above i.e. there exists $B = B^* \in \mathcal{B}(\mathcal{H})$ such that $A_\gamma \leq B$*

for all $\gamma \in \Gamma$. Then there exists $A = s\text{-}\lim_{\Gamma} A_\gamma$. Moreover $A = \sup_{\gamma\in\Gamma} A_\gamma$ (i.e. A is the least upper bound of the set $\{A_\gamma | \gamma \in \Gamma\}$) which means that if $C \in \mathcal{B}(\mathcal{H})$ and $C \geq A_\gamma$ for every $\gamma \in \Gamma$ then $C \geq A$.

PROOF. First note that we can replace Γ by a cofinal set $\{\gamma | \gamma \in \Gamma, \gamma \geq \gamma_0\}$. Then we may suppose that $A_\gamma \geq A_{\gamma_0}$ for all $\gamma \in \Gamma$. By translating (subtracting A_{γ_0}) and scaling (dividing by a positive constant) we can further assume that $0 \leq A_\gamma \leq I$ for all γ.

Now for any $x \in \mathcal{H}$ the net $\{(A_\gamma x, x) | \gamma \in \Gamma\}$ is an increasing net in $\mathbb{R}$ which is bounded above by $\|x\|^2$. Hence there exists $\Phi(x,x) = \lim_{\Gamma}(A_\gamma x, x) = \sup_{\gamma\in\Gamma}\{(A_\gamma x, x)\}$. By polarization for every $x, y \in \mathcal{H}$ there exists a limit $\Phi(x,y) = \lim_{\Gamma}(Ax, y)$. The map $\{x, y\} \mapsto \Phi(x,y)$ is linear with respect to x and antilinear with respect to y. It is also Hermitian positive and satisfies the estimate $|\Phi(x,y)| \leq \|x\|\|y\|$ easily deduced from the Cauchy-Schwarz inequality $|\Phi(x,y)| \leq \Phi(x,x)\Phi(y,y)$. By the Riesz representation theorem, any antilinear continuous functional $\ell : \mathcal{H} \to \mathbb{C}$ has the form $(y_0, \cdot)$ with $y_0 = y_0(l)$, therefore we see that there exists a unique self-adjoint operator A such that $\Phi(x,y) = (Ax, y)$, $x, y \in \mathcal{H}$.

Clearly $w\text{-}\lim_{\Gamma} A_\gamma = A$ and $A_\gamma \leq A$ for every $\gamma \in \Gamma$. If $C \geq A_\gamma$ for every γ, then obviously $(Cx, x) \geq (A_\gamma x, x)$, hence $(Cx, x) \geq \sup_{\gamma\in\Gamma}(A_\gamma x, x) = (Ax, x)$ for every $x \in \mathcal{H}$. Therefore $A = \sup_{\gamma\in\Gamma} A_\gamma$. It remains to prove that $A = s\text{-}\lim_{\Gamma} A_\gamma$.

If $x \in \mathcal{H}$, then

$$\|(A - A_\gamma)x\|^2 \leq \|(A - A_\gamma)^{1/2}\|^2 \|(A - A_\gamma)^{1/2} x\|^2 \leq ((A - A_\gamma)x, x)$$

because $0 \leq A - A_\gamma \leq A \leq I$ implies $\|(A - A_\gamma)^{\frac{1}{2}}\| \leq 1$. Hence $\lim_{\Gamma}\|(A - A_\gamma)x\| = 0$ which means exactly that $A = s\text{-}\lim_{\Gamma} A_\gamma$. □

2.5. Basics on von Neumann Algebras

Let $\mathcal{H}$ be a Hilbert space. For any subset $M \subset \mathcal{B}(\mathcal{H})$ define its *commutant* as

$$M' = \{A \in \mathcal{B}(\mathcal{H}) | AB = BA \quad \text{for every} \quad B \in M\}.$$

Then M' is a subalgebra in $\mathcal{B}(\mathcal{H})$, $I \in M'$.

LEMMA. *M' is weakly closed. (It follows that M' is closed also in the strong operator topology and in the norm topology.)*

PROOF. Suppose that $A_\gamma \in \mathcal{B}(\mathcal{H})$, $A = w\text{-}\lim_{\Gamma} A_\gamma$, $B \in \mathcal{B}(\mathcal{H})$ and $A_\gamma B = BA_\gamma$ for all γ. We have then $(A_\gamma Bx, y) = (BA_\gamma x, y) = (A_\gamma x, B^* y)$ for every $x, y \in \mathcal{H}$. We can go to the limit in the first and last term, obtaining in this way that $(ABx, y) = (Ax, B^*y) = (BAx, y)$, hence $AB = BA$ and the statement of the Lemma follows. □

The following theorem is usually called the von Neumann Double Commutant Theorem.

THEOREM 1. *Let $\mathcal{M}$ be a $*$-subalgebra in $\mathcal{B}(\mathcal{H})$ and $I \in \mathcal{M}$. Then the following conditions are equivalent:*

(i) *$\mathcal{M}'' = \mathcal{M}$;*

(ii) *$\mathcal{M}$ is weakly closed (i.e. closed in the weak operator topology);*
(iii) *$\mathcal{M}$ is strongly closed (i.e. closed in the strong operator topology).*

Definition 1. A *von Neumann algebra* is a $*$-subalgebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ such that $I \in \mathcal{M}$ and $\mathcal{M}$ satisfies one of the three equivalent conditions of the theorem above.

Proof of Theorem 1. Due to Lemma (i)$\Rightarrow$(ii). The implication (ii)$\Rightarrow$(iii) is obvious. So it is sufficient to prove the implication (iii)$\Rightarrow$(i).

Suppose $A \in \mathcal{M}''$; $\varepsilon > 0$; $x_1, \dots, x_N \in \mathcal{H}$. We want to find $B \in \mathcal{M}$ such that $\|(A - B)x_i\| < \varepsilon$, $i = 1, \dots N$ (this will mean that $\mathcal{M}''$ lies in the strong closure of $\mathcal{M}$, hence it coincides with $\mathcal{M}$).

Let us consider $\widetilde{A} = A \otimes I_N \in \mathcal{M}'' \otimes \mathcal{L}(\mathbb{C}^N)$ and take

$$\widetilde{x} = \{x_1, x_2, \dots, x_N\} \in \mathcal{H} \oplus \dots \oplus \mathcal{H} \quad (N \text{ times}).$$

We will show that there exists $B \in \mathcal{M}$ such that if $\widetilde{B} = B \otimes I_N$ then $\|(\widetilde{A} - \widetilde{B})\widetilde{x}\| < \varepsilon$. Obviously $\widetilde{B} \in \widetilde{\mathcal{M}} = \mathcal{M} \otimes I_N$ and the $*$-subalgebra $\widetilde{\mathcal{M}} \subset \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ is strongly closed. It is easy to see that

$$(\widetilde{\mathcal{M}})'' = \widetilde{\mathcal{M}}'' = \mathcal{M}'' \otimes I_N.$$

Therefore $\widetilde{A} \in \widetilde{\mathcal{M}}''$. Let us consider the orthogonal projection $\widetilde{P}$ in $\mathcal{H} \otimes \mathbb{C}^N$ with the image $\operatorname{Im} \widetilde{P} = \overline{\widetilde{\mathcal{M}}\widetilde{x}}$. Then $\widetilde{P} \in \widetilde{\mathcal{M}}'$ and $\widetilde{P}$ commutes with $\widetilde{A}$ which means that $\overline{\widetilde{\mathcal{M}}x}$ is $\widetilde{A}$ invariant. In particular $\widetilde{A}x \in \overline{\widetilde{\mathcal{M}}x}$, hence we can find $B \in \widetilde{\mathcal{M}}$ such that $\|(\widetilde{A} - \widetilde{B})\tilde{x}\| < \varepsilon$ as required. $\square$

Corollary 1. *A subset $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if it has the form $\mathcal{M} = M'$ for a $*$-subset $M \subset \mathcal{B}(\mathcal{H})$, that is, a subset M such that $B \in M$ implies $B^* \in M$.*

Proof. It is easy to see that $M''' = M'$ for any subset $M \subset \mathcal{B}(\mathcal{H})$ (see Lemma 1 in 1.7). Now if M is a $*$-subset then M' is a $*$-subalgebra, hence we can apply the Theorem. $\square$

Definition 2. Let M be a subset of $\mathcal{B}(\mathcal{H})$. The *von Neumann algebra generated by M* is the smallest von Neumann algebra in $\mathcal{B}(\mathcal{H})$ containing M.

Define $M^* = \{B^* | B \in M\}$. Then the von Neumann algebra $\mathcal{M}$ generated by M is equal to $(M \cup M^*)''$. Indeed if $\mathcal{M}_1$ is a von Neumann subalgebra in $\mathcal{B}(\mathcal{H})$ and $M \subset \mathcal{M}_1$ then $M \cup M^* \subset \mathcal{M}_1$ and $(M \cup M^*)' \supset \mathcal{M}_1'$ hence $(M \cup M^*)'' \subset \mathcal{M}_1'' = \mathcal{M}_1$ and $\mathcal{M} = (M \cup M^*)''$ is the minimal von Neumann algebra containing M.

Definition 3. (a) The *center* of the von Neumann algebra $\mathcal{M}$ is

$$Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}' = \{B \in \mathcal{M} | BA = AB \text{ for every } A \in \mathcal{M}\}.$$

It is a commutative von Neumann algebra.

(b) A von Neumann algebra $\mathcal{M}$ is called a *factor* if $Z(\mathcal{M}) = \{\lambda I | \lambda \in \mathbb{C}\}$.
(c) Denote $\operatorname{Proj}(\mathcal{M}) = \{P \in \mathcal{M} | P^2 = P = P^*\}$ i.e. $\operatorname{Proj}(\mathcal{M})$ is the set of all orthogonal projections in $\mathcal{M}$.

Exercise 1. Prove the Cancellation Lemma (see 1.10) for factors in an arbitrary Hilbert space $\mathcal{H}$: if $\mathcal{M}$ is a factor in $\mathcal{H}$, $A \in \mathcal{M}, A' \in \mathcal{M}'$ and $AA' = 0$, then either $A = 0$ or $A' = 0$.

THEOREM 2. *Let $\mathcal{M}$ be a von Neumann subalgebra in $\mathcal{B}(\mathcal{H})$, $A = A^* \in \mathcal{M}$ and let $E(\Delta)$ be the spectral projection of A corresponding to an interval $\Delta \subset \mathbb{R}$. Then $E(\Delta) \in \mathcal{M}$.*

PROOF. We have seen in 2.2 that $E(\Delta) = s\text{-}\lim_{k\to\infty} P_k(A)$ where $P_k \in \mathbb{R}[x]$ are polynomials. Since $\mathcal{M}$ is strongly closed, we obtain $E(\Delta) \in \mathcal{M}$. □

COROLLARY 2. *If $\mathcal{M}$ is a von Neumann algebra then $\mathcal{M}$ is generated by* $\mathrm{Proj}(\mathcal{M})$. *Moreover, if* $\mathrm{span}(\mathrm{Proj}(\mathcal{M}))$ *is the set of finite linear combinations of elements from* $\mathrm{Proj}(\mathcal{M})$*, then $\mathcal{M}$ is the closure of* $\mathrm{span}(\mathrm{Proj}(\mathcal{M}))$ *in the norm topology.*

PROOF. Obviously it is sufficient to prove that if $A = A^* \in \mathcal{M}$ then A is in the norm closure of $\mathrm{span}(\mathrm{Proj}(\mathcal{M}))$. But this follows from the representation.

$$A = \int \lambda \, dE_\lambda$$

where the integral converges in norm topology. □

EXERCISE 2. Prove that if $A = A^* \in \mathcal{M}$, where $\mathcal{M}$ is a von Neumann algebra, and $f : [-||A||, ||A||] \to \mathbb{C}$ is a bounded Borel function, then $f(A) \in \mathcal{M}$.

THEOREM 3. *Every von Neumann algebra $\mathcal{M}$ in $\mathcal{H}$ is spanned by the set $U(\mathcal{M})$ of all unitary operators $U \in \mathcal{M}$.*

PROOF. Actually we shall prove that every element $A \in \mathcal{M}$ can be represented as a linear combination of four unitary operators from $\mathcal{M}$. First we may write $A = A_1 + iA_2$ with $A_i^* = A_i \in \mathcal{M}$, so it is sufficient to prove that every $A = A^* \in \mathcal{M}$ can be represented as a linear combination of two unitary operators in $\mathcal{M}$. By scaling we may suppose that $||A|| \le 1$. Now take $U_1 = A + i\sqrt{I - A^2}$, $U_2 = A - i\sqrt{I - A^2}$. It is easy to see that if $B \in \mathcal{M}$ and $B \ge 0$ then $\sqrt{B} \in \mathcal{M}$ because $\sqrt{B}$ is a norm-limit of polynomials $p_k(B)$. Now obviously $U_1, U_2 \in \mathcal{M}$ and $A = \frac{1}{2}(U_1 + U_2)$. On the other hand U_1, U_2 are unitary because $U_1^* = U_2$, $U_2^* = U_1$ and $U_1U_2 = U_2U_1 = I$. □

COROLLARY 3. *Let $\mathcal{M}$ be a von Neumann algebra in $\mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$. Then $A \in \mathcal{M}$ if and only if $U^{-1}AU = A$ for every unitary operator $U \in \mathcal{M}'$.*

PROOF. Let $A \in \mathcal{B}(\mathcal{M})$ and $U^{-1}AU = A$ for every unitary operator $U \in \mathcal{M}'$. We have to prove that $A \in \mathcal{M}$ (the converse is obvious). Clearly $BA = AB$ for all unitary $B \in \mathcal{M}'$, hence for all $B \in \mathcal{M}'$ due to the Theorem. It follows that $A \in \mathcal{M}'' = \mathcal{M}$. □

This important corollary provides a way to check that all operators appearing in unitarily invariant constructions from operators in $\mathcal{M}$, again belong to $\mathcal{M}$. We shall give an example of this situation now.

COROLLARY 4. *Let $\mathcal{M}$ be a von Neumann algebra in $\mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$, and let $A = US$ be the polar decomposition of A. Then $A \in \mathcal{M}$ if and only if $U \in \mathcal{M}$ and $S \in \mathcal{M}$.*

PROOF. Let us prove that $A \in \mathcal{M}$ implies $U \in \mathcal{M}$ and $S \in \mathcal{M}$ (the converse is obvious). Choose a unitary $V \in \mathcal{M}'$. Then $V^{-1}AV = (V^{-1}UV)(V^{-1}SV)$ gives a polar decomposition of $V^{-1}AV$ (e.g. it is obvious that $V^{-1}UV$ is a partial isometry and $V^{-1}SV \ge 0$). But $V^{-1}AV = A$, hence $V^{-1}UV = U, V^{-1}SV = S$, due to the uniqueness of this decomposition. Since V is an arbitrary unitary operator in $\mathcal{M}'$ we obtain $U \in \mathcal{M}'' = \mathcal{M}$ and $S \in \mathcal{M}'' = \mathcal{M}$ as required. □

We mentioned before that norm-closed $*$-subalgebras in $\mathcal{B}(\mathcal{H})$ can be characterized without mentioning their action on $\mathcal{H}$ as C^*-algebras. A similar characterization of von Neumann algebras is also possible due to S. Sakai: a C^*-algebra $\mathcal{M}$ is $*$-isomorphic with a von Neumann algebra if and only if $\mathcal{M}$ is the dual of a Banach space $\mathcal{M}_*$ (called the predual of $\mathcal{M}$), i.e. if $\mathcal{M}$ is isomorphic (as a Banach space) to the space of all linear continuous functions $f : \mathcal{M}_* \to \mathbb{C}$. (See e.g. [**Tak**], Theorem 3.5 in Ch.III.)

2.6. Trace

Let $\mathcal{M}$ be a von Neumann algebra in $\mathcal{H}$. Denote $\mathcal{M}^+ = \{A \in \mathcal{M} | A \geq 0\}$. We shall also use notation $\mathbb{R}^+ = \{\lambda \in \mathbb{R}, \lambda \geq 0\}$.

Definition. A *trace* on $\mathcal{M}$ is a map $\tau : \mathcal{M}^+ \to [0, \infty]$ satisfying the following conditions:

(i) $\tau(c_1 A_1 + c_2 A_2) = c_1 \tau(A_1) + c_2 \tau(A_2),\ c_i \in \mathbb{R}^+,\ A_i \in \mathcal{M}^+,\ i = 1, 2$ (τ is *linear*).
(ii) $\tau(A^* A) = \tau(A A^*),\ A \in \mathcal{M}$ (τ is *tracial*).
(iii) $\tau(A) = 0,\ A \in \mathcal{M}^+$ implies $A = 0$ (τ is *faithful*).
(iv) Let $A_\gamma \nearrow A,\ A_\gamma \in \mathcal{M}^+$ (i.e. $\{A_\gamma | \gamma \in \Gamma\}$ is an increasing net on elements in $\mathcal{M}^+$ that converges to A). Then $\tau(A_\gamma) \nearrow \tau(A)$ (τ is *normal*).
(v) For every $A \in \mathcal{M}^+$, $\tau(A) = \sup\{\tau(B) | B \in \mathcal{M}^+, B \leq A, \tau(B) < +\infty\}$ (τ is *semifinite*).

Some comments to this definition are necessary.

1° In (i) it is assumed that $c + \infty = \infty$ for every $c \in [0, \infty]$, $c \cdot \infty = \infty$ provided $c > 0$, but $0 \cdot \infty = 0$.
2° A map $\tau : \mathcal{M}^+ \to [0, \infty]$ satisfying (i) is called a *weight*. (Further conditions (iii), (iv) and (v) might be imposed on a weight and then it will be called faithful, normal and semifinite respectively).
3° Usually conditions (iii)–(v) are not automatically imposed on a trace: other terminology exists where trace is a map $\tau : \mathcal{M}^+ \to [0, \infty]$ satisfying (i) and (ii); then (iii), (iv) and (v) will mean that we have a faithful, normal and semifinite trace respectively. We will use the terminology introduced in the definition for the sake of brevity and also because all important traces satisfy (iii)–(v) as well.

Lemma 1 (Relative unitary invariance of the trace). *Let $\mathcal{M}$ be a von Neumann algebra in $\mathcal{H}$, τ a trace on $\mathcal{M}$. Then $\tau(U A U^{-1}) = \tau(A)$ for any $A \in \mathcal{M}^+$ and any unitary $U \in \mathcal{M}$.*

Proof. We have $A = \sqrt{A}\sqrt{A} = \sqrt{A} U^* U \sqrt{A} = B^* B$, where $B = U\sqrt{A}$. On the other hand $B B^* = U \sqrt{A}\sqrt{A} U^* = U A U^{-1}$ so the desired equality follows from the property (ii) in the definition of trace. □

Let $\mathrm{Gr}_{\mathcal{M}}(\mathcal{H}) = \{L \subset \mathcal{H} | L = \mathrm{Im}\, P, P = P^2 = P^* \in \mathcal{M}\}$ for any von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$. So $\mathrm{Gr}_{\mathcal{M}}(\mathcal{H})$ is the set of all "admissible" subspaces in $\mathcal{H}$. If $L \in \mathrm{Gr}_{\mathcal{M}}(\mathcal{H})$ then we shall say that L is *affiliated to* $\mathcal{M}$, and we will also denote this relation as $L \eta \mathcal{M}$.

The affiliation relation $L \eta \mathcal{M}$ is equivalent to L being $\mathcal{M}'$-invariant. (This equivalence is useful in particular because the later sometimes may be fulfilled for a non-closed subspace L.) Obviously there is a one-one correspondence

$$\mathrm{Proj}(\mathcal{M}) \cong \mathrm{Gr}_{\mathcal{M}}(\mathcal{H})$$
$$P \mapsto \mathrm{Im}\, P$$

DEFINITION. Given a trace $\tau : \mathcal{M} \to [0, \infty]$ define the corresponding *dimension function*

$$\dim_\tau : \mathrm{Gr}_{\mathcal{M}}(\mathcal{H}) \to [0, \infty]$$

by

$$\dim_\tau \mathrm{Im}\, P = \tau(P), \;\; P \in \mathrm{Proj}(\mathcal{M}).$$

We shall denote it by $\dim_{\mathcal{M}}$ if the trace was fixed before. It is also often called *relative dimension.*

Let us list some important properties of the relative dimension.

LEMMA 2. (i) $\dim_{\mathcal{M}} L = 0$ *if and only if* $L = \{0\}$*;*
(ii) *If* $L_1, L_2 \in \mathrm{Gr}_{\mathcal{M}}(\mathcal{H})$ *and* $L_1 \subset L_2$*, then* $\dim_{\mathcal{M}} L_1 \leq \dim_{\mathcal{M}} L_2$*;*
(iii) *If* $L_1, L_2 \in \mathrm{Gr}_{\mathcal{M}}(\mathcal{H})$ *and* $L_1 \perp L_2$*, then*

$$dim_{\mathcal{M}}(L_1 \oplus L_2) = \dim_{\mathcal{M}} L_1 + \dim_{\mathcal{M}} L_2;$$

(iv) *If* $\{L_i | i \in I\}$ *are mutually orthogonal closed subspaces in* $\mathcal{H}$*,* $L_i \in \mathrm{Gr}_{\mathcal{M}}(\mathcal{H})$ *for every* $i \in I$ *and* $L = \sup_i L_i$ *is the closure of the linear span of* $\cup_i L_i$ *(i.e. the smallest closed subspace in* $\mathcal{H}$ *including all* L_i*,* $i \in I$*), then* $L \in \mathrm{Gr}_{\mathcal{M}}(\mathcal{H})$ *and*

$$\dim_{\mathcal{M}} L = \sum_{i \in I} \dim_{\mathcal{M}} L_i,$$

where the sum can be understood e.g. as the least upper bound of the finite sums;
(v) *If* $L \in \mathrm{Gr}_{\mathcal{M}}(\mathcal{H})$ *and* $U \in \mathcal{M}$ *is unitary then* $\dim_{\mathcal{M}} U(L) = \dim_{\mathcal{M}} L$*.*

PROOF. (i) is obvious due to the faithfulness of τ.

(ii) follows if we notice that for $L_i = \mathrm{Im}\, P_i$, $i = 1, 2$, where $P_i \in \mathrm{Proj}(\mathcal{M})$, the inclusion $L_1 \subset L_2$ is equivalent to the inequality $P_1 \leq P_2$, hence $\tau(P_2) = \tau(P_1) + \tau(P_2 - P_1) \geq \tau(P_1)$.

To prove (iii) notice that if $L_i = \mathrm{Im}\, P_i$, $i = 1, 2$, and $L_1 \perp L_2$ then $L_1 \oplus L_2 = \mathrm{Im}(P_1 + P_2)$ and $P_1 + P_2 \in \mathrm{Proj}(\mathcal{M})$.

To check (iv) consider the set Γ of all finite subsets in I and define a partial order $\geq$ on Γ by inclusion: $\gamma' \geq \gamma''$ if $\gamma' \supset \gamma''$. Then Γ is a directed set and if $P_i \in \mathrm{Proj}(\mathcal{M})$, $\mathrm{Im}\, P_i = L_i$, then defining $P_\gamma = \sum_{i \in \gamma} P_i$ we obtain an increasing net in $\mathcal{M}^+$. It is easy to see that $\lim_\Gamma P_\gamma = P$ where $P \in \mathrm{Proj}(\mathcal{M})$, $\mathrm{Im}\, P = L = \sup_i L_i$. Hence (iv) follows from the normality of the trace.

To prove (v) notice that if $P \in \mathrm{Proj}(\mathcal{M})$ and $\mathrm{Im}\, P = L$, then $UPU^{-1} \in \mathrm{Proj}(\mathcal{M})$ and $\mathrm{Im}(UPU^{-1}) = U(L)$ so the necessary property will follow from the relative unitary invariance of the trace. □

2.7. Classification of factors

F. Murray and J. von Neumann classified factors (and von Neumann algebras) according to properties of traces on these algebras. We shall briefly mention the classification of factors without proofs.

It can be proved that if $\mathcal{M}$ is a factor then there is at most one trace on $\mathcal{M}$ (up to a constant positive factor). If there exists no trace on $\mathcal{M}$ then $\mathcal{M}$ is called a type III factor. If a trace τ exists on $\mathcal{M}$, the further classification proceeds according to the range $\tau(\mathrm{Proj}(\mathcal{M}))$ i.e. the set of values taken by τ on all projections, or equivalently, to the range of the corresponding dimension function: $\dim_\tau(\mathrm{Gr}_{\mathcal{M}}(\mathcal{H}))$.

The results of the classification are given in the following table (with a natural normalization of τ when necessary), type III included for the sake of completeness:

$\tau(\mathrm{Proj}(\mathcal{M}))$	Type	Other classifying names
$\{0, 1, \ldots, n\}$	I_n	Type I
$\{0, 1, \ldots, \infty\}$	I_∞	or discrete type
$[0, 1]$	II_1	Type II
$[0, \infty]$	II_∞	or continuous type
no trace	III	Infinite

Also factors of types I_n and II_1 are called finite, and factors of types I_∞ and II_∞ are called semifinite. A deep classification of type III factors exists but goes out of the scope of these lectures.

Any factor in a finite dimensional Hilbert space is isomorphic to $\mathcal{L}(V_0)$ for a finite dimensional Hilbert space V_0. Such a factor has type I_n where $n = \dim_{\mathbb{C}} V_0$. This factor can be represented in any infinite-dimensional Hilbert space too:

Exercise. For any infinite-dimensional Hilbert space $\mathcal{H}$ find a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ which is $*$-isomorphic to $\mathcal{L}(\mathbb{C}^n)$ (hence is a factor of type I_n).

Examples of factors of other types will be given later.

The most interesting new phenomenon for us here is the existence of factors of type II (or continuous type) where the range of the dimension function is either $[0, 1]$ or $[0, \infty]$. This means for example that we can find there subspaces of τ-dimension $\sqrt{2}$, $\pi/4$ etc. It is exactly the factors of type II which play an important role in the analysis on infinite covering spaces. By measuring some subspaces which are infinite-dimensional in the usual sense, the relative dimension function allows to construct some invariants of continuous spectrum of Laplacians, thereby providing topological invariants of the base manifolds.

These topics will be discussed in this book later.

2.8. Trace on $\mathcal{B}(\mathcal{H})$

Let $\mathcal{H}$ be a Hilbert space, $\{e_j | j \in J\}$ an orthonormal basis in $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})^+$ (i.e. $A \in \mathcal{B}(\mathcal{H})$ and $A \geq 0$) define

$$\mathrm{Tr}\, A = \sum_{j \in J} (Ae_j, e_j)$$

where the sum here (and in similar expressions below) is understood as the least upper bound of finite sums (i.e. sums over finite subsets in J).

Theorem. (i) Tr *is a faithful, normal and semifinite trace on* $\mathcal{B}(\mathcal{H})$. *It does not depend on the choice of the orthonormal basis in* $\mathcal{H}$.

(ii) *The set of its values on* $\mathrm{Proj}(\mathcal{B}(\mathcal{H}))$ *is* $\{0, 1, 2, \ldots, n\}$ *if* $\dim_{\mathbb{C}} \mathcal{H} = n < \infty$ *and* $\{0, 1, 2, \ldots, \infty\}$ *if* $\dim_{\mathbb{C}} \mathcal{H} = \infty$.

PROOF. 1) Tr is obviously linear. Now for any $C \in \mathcal{B}(\mathcal{H})$ we have

$$\begin{aligned}\operatorname{Tr}(C^*C) &= \sum_{j\in J}(C^*Ce_j, e_j) = \sum_{j\in J}\|Ce_j\|^2 = \sum_{j\in J}\sum_{k\in J}|(Ce_j, e_k)|^2 \\ &= \sum_{j,k\in J}|(Ce_j, e_k)|^2 = \sum_{j,k\in J}|(e_j, C^*e_k)|^2 = \sum_{k\in J}\sum_{j\in J}|(e_j, C^*e_k)|^2 \\ &= \sum_{k\in J}\|C^*e_k\|^2 = \operatorname{Tr}(CC^*),\end{aligned}$$

which proves that Tr is tracial.

2) To prove that Tr is faithful note first that $A \geq 0$ and $\operatorname{Tr} A = 0$ imply $(Ae_j, e_j) = 0$ for every $j \in J$. Now we can use the Cauchy-Schwarz inequality

$$|(Ax, y)|^2 \leq (Ax, x)(Ay, y), \quad x, y \in \mathcal{H},$$

which implies $(Ae_j, e_k) = 0$ for all $j, k \in J$. It follows that $Ae_j = 0$ for every $j \in J$, hence $A = 0$.

3) Let us check that Tr is normal. Suppose that $A_\gamma \nearrow A$, $A_\gamma \geq 0$. Then obviously $(A_\gamma e_j, e_j) \to (Ae_j, e_j)$ for every $j \in J$. Now for every $\epsilon > 0$ we can find a finite $S \subset J$ such that $|\operatorname{Tr} A - \sum_{j\in S}(Ae_j, e_j)| < \epsilon$ provided $\operatorname{Tr} A$ is finite, and $\sum_{j\in S}(Ae_j, e_j) > \epsilon^{-1}$ provided $\operatorname{Tr} A = \infty$. There exists γ_0 such that $\gamma \geq \gamma_0$ implies $|\sum_{j\in S}(A_\gamma e_j, e_j) - \sum_{j\in S}(Ae_j, e_j)| < \epsilon$. But this obviously implies that $|\operatorname{Tr} A - \operatorname{Tr} A_\gamma| < 2\epsilon$ if $\operatorname{Tr} A$ is finite, and $\operatorname{Tr} A_\gamma > \epsilon^{-1} - \epsilon$ if $\operatorname{Tr} A$ is infinite. In both cases we conclude that $\operatorname{Tr} A_\gamma \to \operatorname{Tr} A$.

4) Let us prove that Tr is semifinite. Let us take $A \geq 0$ such that $\operatorname{Tr} A = \infty$. Then for every $N > 0$ we can find a finite $S \subset J$ such that $\sum_{j\in S}(Ae_j, e_j) > N$. Now let P_S be the orthogonal projection on the closure of the linear span of the set $\{e_j | j \in S\}$. Then $0 \leq P_SAP_S \leq A$ and $\operatorname{Tr}(P_SAP_S) = \sum_{j\in S}(Ae_j, e_j) > N$ which proves the semifiniteness of the trace.

5) Note that since Tr is a trace, hence unitarily invariant, it does not depend on the choice of the orthonormal basis in $\mathcal{H}$ because change of basis reduces to a unitary similarity for A.

6) Let $P = P^2 = P^* \in \mathcal{B}(\mathcal{H})$. We can choose an orthonormal basis in $\mathcal{H}$ of the form $\{e_j | j \in J_0 \cup J_1\}$, where $\{e_j | j \in J_0\}$ is an orthonormal basis in the subspace $L = \operatorname{Im} P$, $J_0 \cap J_1 = \emptyset$. It follows that $\operatorname{Tr} P = \operatorname{Card}(J_0)$, so that the last statement follows. □

EXERCISE 1. Prove that $\mathcal{B}(\mathcal{H})$ is a factor and, moreover, that

$$(\mathcal{B}(\mathcal{H}))' = \{\lambda I | \lambda \in \mathbb{C}\}.$$

EXERCISE 2. Prove that any faithful, normal, semifinite trace on $\mathcal{B}(\mathcal{H})$ has the form $c\operatorname{Tr}$ with $c > 0$ (the constant c can be fixed e.g. by the requirement that the minimum positive value of the trace on $\operatorname{Proj}(\mathcal{B}(\mathcal{H}))$ is 1, or that the trace of a projection to a one-dimensional subspace is 1).

So according to the classification given earlier, for an infinite dimensional Hilbert space $\mathcal{H}$ the algebra $\mathcal{B}(\mathcal{H})$ is a factor of type I_∞. Actually it can be proved that any factor of type I_∞ is $*$-isomorphic to an algebra $\mathcal{B}(\mathcal{H})$, though it can act differently on another Hilbert space. An example of such situation is the algebra $\mathcal{B}(\mathcal{H}) \otimes I$ on $\mathcal{H} \otimes \mathbb{C}^N$. Instead of $\mathbb{C}^N$ we may also take an infinite-dimensional Hilbert space as will be explained later. On the other hand $\mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$ is a von Neumann algebra in $\mathcal{H}_1 \oplus \mathcal{H}_2$ for any Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, but it is not a factor (unless one of $\mathcal{H}_1$ is just a trivial 0-dimensional space); the traces on $\mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$ can be expressed in the form $c_1 \operatorname{Tr}_1 + c_2 \operatorname{Tr}_2$, where Tr_j is the trace on $\mathcal{B}(\mathcal{H}_j)$, $j = 1, 2$, $c_j > 0$.

2.9. Algebra L^∞

Let us consider the Hilbert space $\mathcal{H} = L^2([0,1])$ (with respect to the standard Lebesgue measure on the closed interval $[0,1]$). If $a \in L^\infty([0,1])$ then we can consider the multiplication operator $M_a \in \mathcal{B}(\mathcal{H})$ given by the formula $(M_a f)(x) = a(x)f(x)$. We shall identify a and M_a and so $L^\infty([0,1])$ becomes a subalgebra in $\mathcal{B}(L^2([0,1]))$.

THEOREM. *$L^\infty([0,1])$ is a maximal commutative subalgebra in $\mathcal{B}(L^2([0,1]))$ and $(L^\infty([0,1]))' = L^\infty([0,1])$. It is a von Neumann algebra.*

PROOF. Obviously $\mathcal{M} = L^\infty([0,1])$ is a $*$-subalgebra in $\mathcal{B}(L^2([0,1]))$. Hence the last statement will follow from the relation $\mathcal{M}' = \mathcal{M}$. This implies also that $\mathcal{M}$ is a maximal commutative subalgebra. Indeed if $\tilde{\mathcal{M}}$ is a commutative subalgebra in $\mathcal{B}(L^2([0,1])$ and $\mathcal{M} \subset \tilde{\mathcal{M}}$ then $\mathcal{M} \subset \tilde{\mathcal{M}} \subset \tilde{\mathcal{M}}' \subset \mathcal{M}' = \mathcal{M}$, hence $\tilde{\mathcal{M}} = \mathcal{M}$.

Now let us check that $\mathcal{M}' = \mathcal{M}$. Let $A \in \mathcal{M}'$ i.e. $A \in \mathcal{B}(L^2([0,1]))$ and A commutes with all operators M_a, $a \in L^\infty([0,1])$. Then for every $u \in L^\infty([0,1])$ we have $Au = AM_u 1 = M_u A1 = (A1) \cdot u$, so if $a = A1$, then $Au = M_a u$ for all $u \in L^\infty([0,1])$. Since A is bounded, it follows that $a \in L^\infty([0,1])$. Hence by continuity $Au = M_a u$ for all $u \in L^2([0,1])$. It means that $\mathcal{M}' \subset \mathcal{M}$ and the opposite inclusion is obvious. $\square$

EXERCISE 1. Generalize the previous Theorem to a general measure space (M, Σ, μ) (M is a set, Σ is a σ-algebra of subsets in M, μ is a positive σ-additive and σ-finite measure on Σ). Namely, prove that $L^\infty(M, \Sigma, \mu)$ is a maximal commutative subalgebra in $\mathcal{B}(L^2(M, \Sigma, \mu))$, $(L^\infty(M, \Sigma, \mu))' = L^\infty(M, \Sigma, \mu)$ and $L^\infty(M, \Sigma, \mu)$ is a commutative von Neumann algebra.

Hint. Choose $\varphi \in L^2(M, \Sigma, \mu)$ such that $\varphi > 0$ almost everywhere. Then take $A \in (L^\infty(M, \Sigma, \mu))'$ and define $a = \varphi^{-1} A \varphi$. Now check that $Au = M_a u$ provided $u \in L^2(M, \Sigma, \mu)$ and $\varphi^{-1} u \in L^\infty(M, \Sigma, \mu)$, and prove that the set of all such u is dense in $L^2(M, \Sigma, \mu)$.

EXERCISE 2. (i) Check that $C([0,1])$ is a norm-closed $*$-subalgebra in $\mathcal{B}(L^2([0,1]))$ (hence C^*-algebra) but not a von Neumann algebra.
(ii) Prove that the von Neumann algebra generated by operator M_x (multiplication by x) in $L^2([0,1])$ coincides with $L^\infty([0,1])$.

Now for a measure space (M, Σ, μ) take a measurable function $a : M \to [0, \infty]$ and define a map $\tau_a : (L^\infty(M, \Sigma, \mu))^+ \to [0, \infty]$ by

$$\tau_a(f) = \int_M af \, d\mu.$$

EXERCISE 3. (i) Prove that τ_a is always normal.
Hint. Use the monotone convergence theorem for the Lebesgue integral.
(ii) Prove that τ_a is faithful if and only if $a > 0$ almost everywhere.
(iii) Prove that τ_a is semifinite if and only if $a < \infty$ almost everywhere.

So every measurable function $a : M \to [0, \infty]$ which is almost everywhere positive and finite, defines a faithful normal and semifinite trace on $L^\infty(M, \Sigma, \mu)$. We see that there is a lot of such traces on this algebra unless μ is concentrated at a point (in which case $\dim_{\mathbb{C}} L^2(M, \Sigma, \mu) = 1$ and this is the only case when $L^\infty(M, \Sigma, \mu)$ is a factor).

We have seen that the Lebesgue integral is a particular case of a trace on a von Neumann algebra. This is the reason why some aspects of the von Neumann algebras theory are called Non-commutative Integration.

Actually it can be proved that any commutative von Neumann algebra is isometrically $*$-isomorphic to an algebra $L^\infty(M, \Sigma, \mu)$ for a measure space (M, Σ, μ), such that M is a locally compact topological space and μ is a Radon measure on M (see e.g. [**Tak**], Theorem 1.18 in Ch.III).

2.10. Algebras associated with a discrete group

Let Γ be a discrete group, $e \in \Gamma$ its neutral element. Consider the Hilbert space of L^2-functions on Γ:

$$L^2\Gamma = \{u \mid u : \Gamma \to \mathbb{C}, \sum_{g \in \Gamma} |u(g)|^2 < \infty\}.$$

On $L^2\Gamma$ consider the following left and right translation operators:

$$(L_g u)(h) = u(g^{-1}h), \quad (R_g u)(h) = u(hg), \qquad g, h \in \Gamma,$$

It is easy to see that L_g, R_g are unitary operators in $L^2\Gamma$,

$$L_g^* = (L_g)^{-1} = L_{g^{-1}}, \quad R_g^* = (R_g)^{-1} = R_{g^{-1}},$$
$$L_{g_1g_2} = L_{g_1}L_{g_2}, \quad R_{g_1g_2} = R_{g_1}R_{g_2},$$

so the maps $g \mapsto L_g$ and $g \mapsto R_g$ are unitary representations of Γ in $L^2\Gamma$. They are called *left and right regular representations* respectively.

DEFINITION. Define $\mathcal{M}_l$ or $\mathcal{M}_l(\Gamma)$ as the von Neumann algebra generated by $\{L_g | g \in \Gamma\}$ in $L^2\Gamma$, and $\mathcal{M}_r$ or $\mathcal{M}_r(\Gamma)$ as the von Neumann algebra generated by $\{R_g | g \in \Gamma\}$ in $L^2\Gamma$.

Obviously finite linear combinations of the operators L_g form a $*$-algebra which we shall denote $\overset{\circ}{\mathcal{M}}_l$. Similarly let $\overset{\circ}{\mathcal{M}}_r$ be the set of all finite linear combinations of the operators R_g. Obviously $\mathcal{M}_l$ and $\mathcal{M}_r$ are just strong (or weak) closures of $\overset{\circ}{\mathcal{M}}_l$ and $\overset{\circ}{\mathcal{M}}_r$ respectively due to the von Neumann Double Commutant Theorem (Theorem 1 in 2.5).

The Hilbert space $L^2\Gamma$ has an orthonormal basis $\{\delta_g | g \in \Gamma\}$ where $\delta_g(h) = 1$ if $g = h$ and 0 otherwise. It is easy to check that

$$L_g\delta_h = \delta_{gh}, \quad R_g\delta_h = \delta_{hg^{-1}}$$

In particular

$$L_g\delta_e = \delta_g, \quad R_g\delta_e = \delta_{g^{-1}}.$$

Suppose we have an operator $C \in \mathcal{B}(L^2\Gamma)$. With it we may associate a matrix $C_{a,b} = (C\delta_b, \delta_a)$, $a, b \in \Gamma$.

THEOREM. (i) $\mathcal{M}_l = \{C | C \in \mathcal{B}(L^2\Gamma), C_{ag,bg} = C_{a,b}$ *for all* $a, b, g \in \Gamma\}$, $\mathcal{M}_r = \{D | D \in \mathcal{B}(L^2\Gamma), D_{ga,gb} = D_{a,b}$ *for all* $a, b, g \in \Gamma\}$.

(ii) $\mathcal{M}_l' = \mathcal{M}_r$, $\mathcal{M}_r' = \mathcal{M}_l$.
(iii) *$\mathcal{M}_l$ (and $\mathcal{M}_r$) is a factor if and only if all conjugacy classes in Γ, except the trivial one $\{e\}$, are infinite.*

PROOF. 1) Let us introduce the temporary notations

$$\tilde{\mathcal{M}}_l = \{C \mid C \in \mathcal{B}(L^2\Gamma),\ C_{ag,bg} = C_{a,b} \text{ for all } a, b, g \in \Gamma\}$$
$$\tilde{\mathcal{M}}_r = \{D \mid D \in \mathcal{B}(L^2\Gamma), D_{ga,gb} = D_{a,b} \text{ for all } a, b, g \in \Gamma\}.$$

We have to prove that $\tilde{\mathcal{M}}_l = \mathcal{M}_l$, $\tilde{\mathcal{M}}_r = \mathcal{M}_r$.

We will first check that $\mathcal{M}_l \subset \tilde{\mathcal{M}}_l$, $\mathcal{M}_r \subset \tilde{\mathcal{M}}_r$. Note first that the matrix elements $C_{a,b}$ are continuous functions of C with respect to the weak operator topology. Therefore $\tilde{\mathcal{M}}_l$ and $\tilde{\mathcal{M}}_r$ are weakly closed linear subspaces in $\mathcal{B}(L^2\Gamma)$, and to prove the inclusion $\mathcal{M}_l \subset \tilde{\mathcal{M}}_l$ we only have to check that $(L_h)_{ag,bg} = (L_h)_{a,b}$ for all $a, b, g, h \in \Gamma$. But $(L_h)_{ag,bg} = (L_h\delta_{bg}, \delta_{ag}) = (\delta_{hbg}, \delta_{ag}) = (\delta_{hb}, \delta_a) = (L_h\delta_b, \delta_a) = (L_h)_{a,b}$ as required. The inclusion $\mathcal{M}_r \subset \tilde{\mathcal{M}}_r$ is checked similarly.

Note that since $L_gR_h = R_hL_g$ (all left translations commute with all right translations due to the associativity of multiplication in Γ), we also have the inclusions $\mathcal{M}_l \subset \mathcal{M}_r'$, $\mathcal{M}_r \subset \mathcal{M}_l'$. Now it is easy to check that actually

$$\mathcal{M}_r' = \tilde{\mathcal{M}}_l, \qquad \mathcal{M}_l' = \tilde{\mathcal{M}}_r.$$

Indeed, the inclusion $C \in \mathcal{M}_r'$ for $C \in \mathcal{B}(L^2\Gamma)$ is equivalent to the commutativity relations $CR_g = R_gC$ for all $g \in \Gamma$. The relation $CR_g = R_gC$ is equivalent to equalities $(CR_g\delta_b, \delta_a) = (R_gC\delta_b, \delta_a)$ for all $a, b \in \Gamma$, which are obviously rewritten as $C_{a,bg^{-1}} = C_{ag,b}$ for all $a, b \in \Gamma$. This reduces to $C_{a,b} = C_{ag,bg}$ if we replace b by bg, and gives the equality $\mathcal{M}_r' = \tilde{\mathcal{M}}_l$. The inclusion $\mathcal{M}_l' = \tilde{\mathcal{M}}_r$ is verified similarly.

Now let us check that $\tilde{\mathcal{M}}_l \subset \tilde{\mathcal{M}}_r'$ and $\tilde{\mathcal{M}}_r \subset \tilde{\mathcal{M}}_l'$, that is

$$\{C \in \tilde{M}_l, D \in \tilde{M}_r\} \Rightarrow CD = DC.$$

Indeed,

$$\begin{aligned}(CD)_{a,b} &= \sum_{m\in\Gamma} C_{a,m}D_{m,b} = \sum_{m\in\Gamma} C_{am^{-1},e}D_{b^{-1}m,e} = \sum_{n\in\Gamma} C_{n,e}D_{b^{-1}n^{-1}a,e} \\ &= \sum_{n\in\Gamma} D_{b^{-1}n^{-1}a,e}C_{n,e} = \sum_{n\in\Gamma} D_{a,nb}C_{n,e} = \sum_{k\in\Gamma} D_{a,k}C_{kb^{-1},e} = \sum_{k\in\Gamma} D_{a,k}C_{k,b} \\ &= (DC)_{a,b}\end{aligned}$$

as required. All these calculations make sense because for any $A \in \mathcal{B}(L^2\Gamma)$ we have

$$\sum_{m\in\Gamma} |A_{a,m}|^2 < \infty \quad \text{and} \quad \sum_{n\in\Gamma} |A_{n,b}|^2 < \infty$$

for every $a, b \in \Gamma$.

Now we have the following inclusions

$$\mathcal{M}_l \subset \mathcal{M}_r' = \tilde{\mathcal{M}}_l \subset (\tilde{\mathcal{M}}_r)' = (\mathcal{M}_l)'' = \mathcal{M}_l,$$

hence $\mathcal{M}_l = \tilde{\mathcal{M}}_l = \mathcal{M}_r'$ and similarly $\mathcal{M}_r = \tilde{\mathcal{M}}_r = \mathcal{M}_l'$ which proves statements (i) and (ii) in the Theorem.

2) Let us calculate the center $Z = Z(\mathcal{M}_l) = Z(\mathcal{M}_r) = \mathcal{M}_l \cap \mathcal{M}_r$ of both algebras $\mathcal{M}_l$ and $\mathcal{M}_r$. Obviously $C \in Z$ means that $C \in \mathcal{B}(L^2\Gamma)$ commutes with both left and right translations L_g, R_g, $g \in \Gamma$, or that the matrix elements of C satisfy both relations given in (i) (or in the definitions of $\tilde{\mathcal{M}}_l$ and $\tilde{\mathcal{M}}_r$). Define

$\tilde{C}(g) = C_{g,e}$, $g \in \Gamma$. Then $\tilde{C} \in L^2\Gamma$ and using the relations $C_{a,b} = C_{am,bm} = C_{na,nb}$ for all $a, b, m, n \in \Gamma$, we obtain

$$C_{a,b} = \tilde{C}(ab^{-1}) = \tilde{C}(b^{-1}a), \quad a, b \in \Gamma,$$

which gives $\tilde{C}(b^{-1}ab) = \tilde{C}(a)$ for all $a, b \in \Gamma$. This means that $\tilde{C}$ is constant on conjugacy classes. Now if $\tilde{C} = \delta_e$ then C is the identity operator. Non-trivial (i.e. not proportional to δ_e) functions $\tilde{C} \in L^2\Gamma$ which are constant on all conjugacy classes, exist if and only if Γ has non-trivial finite conjugacy classes. This proves the last statement of the Theorem. □

REMARK. So the matrix $[C_{a,b}]$ of an operator $C \in \mathcal{M}_l$ is uniquely defined by the function $\tilde{C}(a) = C_{a,e}$, $\tilde{C} \in L^2\Gamma$. For two operators $C, D \in \mathcal{M}_l$ the composition $C, D \mapsto CD$ in terms of the corresponding functions $\tilde{C}, \tilde{D}$ becomes the (left) convolution

$$\widetilde{CD}(a) = (C \underset{l}{*} D)(a) = \sum_{m \in \Gamma} \tilde{C}(am^{-1})\tilde{D}(m) = \sum_{n \in \Gamma} \tilde{C}(n)\tilde{D}(n^{-1}a).$$

Similarly if we define operators $C \in \mathcal{M}_r$ in terms of $\tilde{C}(a) = C_{a,e}$, then again $\tilde{C} \in L^2\Gamma$ but the corresponding composition operation becomes the (right) convolution

$$\widetilde{CD}(a) = (C \underset{r}{*} D)(a) = \sum_{m \in \Gamma} \tilde{C}(m^{-1}a)\tilde{D}(m) = \sum_{n \in \Gamma} \tilde{C}(n)\tilde{D}(an^{-1}).$$

Note however that the condition $\tilde{C} \in L^2\Gamma$ does not guarantee that the corresponding matrix $C_{a,b} = \tilde{C}(ab^{-1})$ (or $C_{a,b} = \tilde{C}(b^{-1}a)$) defines a bounded linear operator in $L^2\Gamma$. The boundedness condition has to be imposed specially and it is not known whether it can be explicitly formulated in terms of the function $\tilde{C}$. (The inclusion $\tilde{C} \in L^2\Gamma$ is necessary for the boundedness but it is not sufficient if Γ is infinite.)

EXERCISE. Assume that $\tilde{C} \in L^1\Gamma$, i.e. $\sum_{\gamma \in \Gamma} |\tilde{C}(\gamma)| < \infty$. Prove that the corresponding operators with the matrix elements $C_{a,b} = \tilde{C}(ab^{-1})$ and $C_{a,b} = \tilde{C}(b^{-1}a)$ are bounded in $L^2\Gamma$, hence belong to $\mathcal{M}_l$ and $\mathcal{M}_r$ respectively.

2.11. Trace on $\mathcal{M}_l$ and $\mathcal{M}_r$

THEOREM. *There is a canonical faithful, finite and normal trace on $\mathcal{M}_l$ (and on $\mathcal{M}_r$) given by*

$$\tau(A) = (A\delta_e, \delta_e), \quad A \in \mathcal{M}_l (\textit{or } \mathcal{M}_r).$$

This trace is weakly continuous and can also be written as

$$\tau(A) = (A\delta_g, \delta_g), \quad g \in \Gamma.$$

(The right hand side here does not depend on g).

PROOF. Obviously τ is linear, finite and weakly continuous (hence normal). Now if $A \in \mathcal{M}_l$ then

$$\begin{aligned}(A\delta_g, \delta_g) &= (AR_{g^{-1}}\delta_e, R_{g^{-1}}\delta_e) = (R_g AR_{g^{-1}}\delta_e, \delta_e) = (AR_g R_{g^{-1}}\delta_e, \delta_e) \\ &= (A\delta_e, \delta_e) = \tau(A)\end{aligned}$$

i.e. the matrix of A has equal diagonal elements (all of them equal $\tau(A)$).

Let us prove that τ is faithful. If $A \in \mathcal{M}_l^+$ and $\tau(A) = 0$ then $(A\delta_g, \delta_g) = 0$ for all $g \in \Gamma$. But then due to the Cauchy-Schwarz inequality $|(Af,g)|^2 \leq (Af,f)(Ag,g)$, $f,g \in L^2\Gamma$, we obtain $(A\delta_g, \delta_h) = 0$ for all $g, h \in \Gamma$, which implies $A = 0$.

It remains to prove that τ is tracial (e.g. on $\mathcal{M}_l$). We shall actually prove even more:

$$\tau(AB) = \tau(BA), \quad A, B \in \mathcal{M}_l.$$

Since τ is linear and weakly continuous it is sufficient to consider the case $A = L_g, B = L_h,\ g, h \in \Gamma$. But then

$$\tau(L_g L_h) = (L_g L_h \delta_e, \delta_e) = (L_{gh}\delta_e, \delta_e) = \left\{\begin{matrix} 1 & \text{if } gh = e \\ 0 & \text{otherwise} \end{matrix}\right\} = \tau(L_h L_g)$$

as required. $\square$

It is important to know the range of τ on $\mathrm{Proj}(\mathcal{M}_l)$ (or $\mathrm{Proj}(\mathcal{M}_r)$). We shall discuss this later.

Example 1. Finite groups. If Γ is finite, then $\dim_{\mathbb{C}} L^2\Gamma = |\Gamma|$ (=cardinality of Γ) and $\mathcal{M}_l$ and $\mathcal{M}_r$ are never factors. As we know from the consideration of the finite-dimensional case $\mathcal{M}_l$ (or $\mathcal{M}_r$) is the orthogonal direct sum of factors and each of them is isomorphic to $\mathcal{L}(V_j)$. Here V_j are Γ-invariant subspaces in $L^2\Gamma$ (invariant with respect to the left action of Γ if we consider $\mathcal{M}_l$). Actually each V_j is a subspace where an irreducible representation of Γ is given by its action. If $n_j = \dim V_j$ then the following decomposition is true

$$\mathcal{M}_l = \oplus_{j=1}^k \mathcal{L}(V_j)$$

(every V_j enters $\dim_{\mathbb{C}} V_j$ times in the decomposition of $L^2\Gamma$ into the direct sum of irreducible subspaces and $\mathcal{L}(V_j)$ acts naturally on each copy of this space). If $n_j = \dim_{\mathbb{C}} V_j$ then it follows that

$$|\Gamma| = \sum_{j=1}^k n_j^2.$$

Besides we know that $k = \dim_{\mathbb{C}} Z(\mathcal{M}_l)$, hence k equals the number of conjugacy classes in Γ.

It is well known from the representation theory of finite groups that each term $\mathcal{L}(V_j)$ (corresponding to a given irreducible representation of Γ) appears in the decomposition of the left regular representation $\mathcal{M}_l$ only once (see e.g. [**Se**], Sect 2.4). So k also equals to the number of equivalence classes of irreducible representations of Γ, whereas $n_1, \dots, n_k$ are the dimensions of these representations.

Now let us describe the range of the dimension function $\dim_\tau = \dim_{\mathcal{M}_l}$, i.e. the set

$$\tau(\mathrm{Proj}(\mathcal{M}_l)) = \dim_\tau(\mathrm{Gr}_{\mathcal{M}_l}(L^2\Gamma))$$

for our case. Note first that in the standard basis $\{\delta_g | g \in \Gamma\}$ all diagonal elements of $A \in \mathcal{M}_l$ (or $\mathcal{M}_r$) are equal to $\tau(A)$. Therefore $\tau(A) = \frac{1}{N}\mathrm{Tr}(A)$, where $N = |\Gamma| = \dim_{\mathbb{C}} L^2\Gamma$. Hence $\dim_{\mathcal{M}_l} L = \frac{1}{N}\dim_{\mathbb{C}} L$ for every subspace $L\eta\mathcal{M}_r$, that is, for every closed subspace $L \subset L^2\Gamma$ which is invariant under all operators L_g, $g \in \Gamma$. But L

should be isomorphic to a direct sum of several V_j each one with a multiplicity l_j, such that $0 \le l_j \le n_j$. Hence

$$\tau(\mathrm{Proj}(\mathcal{M}_l) = \left\{ \frac{1}{N} \sum_{j=1}^{n} l_j n_j | 0 \le l_j \le n_j, l_j \in \mathbb{Z} \right\}.$$

Actually the same numbers can be described as all possible dimensions of subrepresentations of the (left) regular representation, divided by N.

If Γ is abelian, then obviously $n_j = 1$ for all j, hence

$$\tau(\mathrm{Proj}(\mathcal{M}_l)) = \left\{ 0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}, 1 \right\}.$$

The same may be true for some non-commutative finite groups. If for instance $\Gamma = S_3$ (all permutations of 3 elements), then $N = |\Gamma| = 6$, $n_1 = n_2 = 1$ and $n_3 = 2$. Here again $\tau(\mathrm{Proj}(\mathcal{M}_l)) = \left\{0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1\right\}$.

Exercise 1. Find examples of finite groups Γ such that

$$\tau(\mathrm{Proj}(\mathcal{M}_l)) \ne \left\{ 0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}, 1 \right\}, N = |\Gamma|.$$

Exercise 2. (i) Prove that $|\Gamma|^{-1} \in \tau(\mathrm{Proj}(\mathcal{M}_l))$ for any finite group Γ.
(ii) Prove that the set $\tau(\mathrm{Proj}(\mathcal{M}_l))$ is invariant with respect to the involution $\lambda \mapsto 1 - \lambda$.

Example 2. $\underline{\Gamma = \mathbb{Z}}$. Denote $\mathcal{M} = \mathcal{M}_l = \mathcal{M}_r$. Since $\mathcal{M}' = \mathcal{M}$, the algebra $\mathcal{M}$ is a maximal commutative subalgebra in $\mathcal{B}(L^2\mathbb{Z})$. Consider the Fourier transformation $F : L^2\mathbb{Z} \to L^2(\mathbb{R}/2\pi\mathbb{Z})$, $(Fa)(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} a(n)e^{inx}$ for $a \in L^2\mathbb{Z}$. Then F is a unitary operator (we consider $L^2(\mathbb{R}/2\pi\mathbb{Z})$ with respect to the Lebesgue measure induced by the standard Lebesgue measure dx on $\mathbb{R}$). If L_1 is the "left" 1-translation on $\mathbb{Z}$ (and $L_1 = R_{-1}$) then obviously $L_1 = F^{-1}M_{e^{ix}}F$. More generally $L_n = F^{-1}M_{e^{inx}}F$. It easily follows that $F\mathcal{M}F^{-1} = L^\infty(\mathbb{R}/2\pi\mathbb{Z})$. Now it is easy to check that the trace τ transforms to the normalized integral:

$$\tau(A) = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{A}(x)dx \quad \text{if } A = F^{-1}M_{\widetilde{A}}F.$$

Obviously then $\tau(\mathrm{Proj}(\mathcal{M}))$ takes all values in [0,1].

Exercise 3. Describe $\mathcal{M} = \mathcal{M}_l = \mathcal{M}_r$ for the case $\Gamma = \mathbb{Z}^n$ and generalize all statements in Example 2 to this case.

Example 3. $\underline{\Gamma = \mathbb{R}}$. Here we consider $\mathbb{R}$ with the discrete topology. Hence $L^2\mathbb{R}$ is the set of functions $a : \mathbb{R} \to \mathbb{C}$ such that

$$\sum_{p\in\mathbb{R}} |a(p)|^2 < \infty$$

(hence $a(p) \ne 0$ only for at most countable set of $p \in \mathbb{R}$). For every $a \in L^2\mathbb{R}$ consider its Fourier transform, which is a formal sum

$$\tilde{a}(x) = \sum_{p\in\mathbb{R}} a(p)e^{ipx}.$$

In the case when only a finite set of $a(p)$ does not vanish, $\tilde{a}$ is a trigonometric polynomial. If $a, b \in L^2\mathbb{R}$ both vanish outside of a finite set, the scalar product in $L^2\mathbb{R}$ can be written as

$$(a,b) = \sum_{p\in\mathbb{R}} a(p)\overline{b(p)} = M_x(\tilde{a}(x)\overline{\tilde{b}(x)}) = \lim_{N\to\infty}\frac{1}{2N}\int_{-N}^{N}\tilde{a}(x)\overline{\tilde{b}(x)}dx.$$

Hence the space of all Fourier transforms is a Hilbert space $B^2(\mathbb{R}^n)$, which is a completion of the set of all trigonometric polynomials with respect to the norm induced by the scalar product in the right-hand side of the formula above (it is so-called Hilbert space of all Besikovich almost periodic functions). The Fourier transformation $a \mapsto \tilde{a}$ transforms a translation L_p to the multiplication by the exponent e^{ipx}. After taking linear span and weak closure of these multiplication operators we obtain the algebra $\mathcal{M}_l = \mathcal{M}_r = \mathcal{M}$ which is actually $L^\infty(\mathbb{R}_B)$ where $\mathbb{R}_B$ is a compact topological group of all characters of the group $\mathbb{R}$ (taken in discrete topology). The normalized Haar measure should be used on $(\mathbb{R}_B)$. Also $L^2\mathbb{R}$ is transformed to $L^2(\mathbb{R}_B)$ with respect to the same Haar measure. The set $\tau(\mathrm{Proj}(\mathcal{M}))$ is [0,1]. For more details, see e.g. [**Di**], [**Sh2**].

Similar statements are true for $\Gamma = \mathbb{R}^n$.

EXAMPLE 4. <u>Γ is a free group F_k with $k \geq 2$ generators.</u> Here obviously all non-trivial conjugacy classes are infinite, hence M_l (or M_r) is a factor. Since τ is a finite trace on M_l (or M_r), this is a factor of the type II_1 (i.e. $\tau(\mathrm{Proj}(\mathcal{M}_l)) = \tau(\mathrm{Proj}(\mathcal{M}_r)) = [0,1]$).

EXAMPLE 5. <u>$\Gamma = \pi_1(X)$ where X is a compact oriented surface of genus g.</u> Here topologically X is a sphere S^2 with g handles attached to it (i.e. if $g = 0$ then $X = S^2$, if $g = 1$ then $X = T^2$ is a 2-dimensional torus etc.), $\pi_1(X)$ is the fundamental group of X. It is well known that $\pi_1(S^2) = \{e\}$ is trivial (the sphere S^2 is simply connected), $\pi_1(T^2) = \mathbb{Z}^2$ and, more generally, if $g \geq 1$, then the group $\Gamma = \pi_1(X)$ has $2g$ generators $a_1, b_1, a_2, b_2, \dots, a_g, b_g$ with a single relation

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1} = e.$$

It can be proved (though non-trivially) that if $g \geq 2$, then all non-trivial conjugacy classes are infinite. Hence (again) $\mathcal{M}_l$ (or $\mathcal{M}_r$) is a II_1-factor and therefore $\tau(\mathrm{Proj}(\mathcal{M}_l)) = \tau(\mathrm{Proj}(\mathcal{M}_r)) = [0,1]$.

EXERCISE 4. Check that all non-trivial conjugacy classes of $\Gamma = \pi_1(X)$ are infinite in case when $g \geq 2$.

Hint: Γ acts on the hyperbolic plane $\mathbb{H}^2$, modeled as the upper half-plane in $\mathbb{C}$, as a discrete isometry group, which is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ $=\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. Prove that Γ is dense in $\mathrm{PSL}_2(\mathbb{R})$ in Zariski topology, hence existence of a non-trivial finite conjugacy class in Γ implies the same in $\mathrm{PSL}_2(\mathbb{R})$.

2.12. Tensor product of Hilbert spaces and von Neumann algebras

Let $\mathcal{H}$, $\mathcal{H}'$ be Hilbert spaces. Consider the algebraic tensor product $\mathcal{H} \underset{\mathrm{alg}}{\otimes} \mathcal{H}'$ and introduce there an inner product by the formula

$$(x \otimes x', y \otimes y') = (x,y)(x',y')$$

Now define $\mathcal{H}\otimes\mathcal{H}'$ as the completion of $\mathcal{H}\underset{\text{alg}}{\otimes}\mathcal{H}'$ with respect to this inner product. It is easy to check that if $\{e_j|j\in J\}$ and $\{e'_k : k\in J'\}$ are orthonormal bases in $\mathcal{H}$ and $\mathcal{H}'$ respectively, then $\{e_j\otimes e'_k|j\in J, k\in J'\}$ is an orthonormal basis in $\mathcal{H}\otimes\mathcal{H}'$, so

$$\mathcal{H}\otimes\mathcal{H}' = \{\sum_{j\in J,k\in J'} c_{jk}e_j\otimes e'_k|c_{jk}\in\mathbb{C}, \sum_{j,k}|c_{jk}|^2<\infty\}$$

Exercise. Let $\mathcal{H}=L^2(M,\Sigma,\mu)$, $\mathcal{H}'=L^2(M',\Sigma',\mu')$. Prove that then $\mathcal{H}\otimes\mathcal{H}'=L^2(M\times M',\Sigma\times\Sigma',\mu\times\mu')$ where $\Sigma\times\Sigma'$ is the σ-algebra generated by the sets of the form $S\times S'$, $S\in\Sigma$, $S'\in\Sigma'$, and $\mu\times\mu'$ is the unique σ-additive measure on $\Sigma\times\Sigma'$, satisfying $(\mu\times\mu')(S\times S)=\mu(S)\mu'(S')$.

Now consider Hilbert spaces $\mathcal{H},\mathcal{H}',\mathcal{H}_1,\mathcal{H}'_1$ and bounded linear operators $A:\mathcal{H}\to\mathcal{H}_1$ and $A':\mathcal{H}'\to\mathcal{H}'_1$. Then we can form the algebraic tensor product $A\otimes A':\mathcal{H}\underset{\text{alg}}{\otimes}\mathcal{H}'\to\mathcal{H}_1\underset{\text{alg}}{\otimes}\mathcal{H}'_1$.

Lemma 1. *$A\otimes A'$ can be extended by continuity to a bounded linear operator (denoted the same way)*

$$A\otimes A':\mathcal{H}\otimes\mathcal{H}'\to\mathcal{H}_1\otimes\mathcal{H}'_1.$$

Besides $||A\otimes A'||=||A||||A'||$.

Proof. We may factorize $A\otimes A'=(I\otimes A')\cdot(A\otimes I)$ where

$$A\otimes I:\mathcal{H}\underset{\text{alg}}{\otimes}\mathcal{H}'\to\mathcal{H}_1\underset{\text{alg}}{\otimes}\mathcal{H}' \qquad I\otimes A':\mathcal{H}_1\underset{\text{alg}}{\otimes}\mathcal{H}'\to\mathcal{H}_1\underset{\text{alg}}{\otimes}\mathcal{H}'_1$$

Now obviously it is sufficient to check that both $A\otimes I$ and $I\otimes A'$ are bounded, $||A\otimes I||=||A||$ and $||I\otimes A'||=||A'||$ (hence $||A\otimes A'||\le||A||||A'||$), and besides $||A\otimes A'||\ge||A||||A'||$.

Choose an orthonormal basis $\{e'_k|k\in J'\}$ in $\mathcal{H}'$. Then every vector in $\mathcal{H}\otimes\mathcal{H}'$ has the form $\sum_{k\in J'}x_k\otimes e'_k$ with $x_k\in\mathcal{H}$ such that $\sum_{k\in J'}||x_k||^2<\infty$; besides

$$||\sum_{k\in J'}x_k\otimes e'_k||^2=\sum_{k\in J'}||x_k||^2.$$

Now the required continuous extension of $A\otimes I$ is given by the formula

$$(A\otimes I)\left(\sum_{k\in J'}x_k\otimes e'_k\right)=\sum_{k\in J'}Ax_k\otimes e'_k,$$

and it is obvious from here that $||A\otimes I||=||A||$. (An equivalent argument: in the orthogonal decomposition

$$\mathcal{H}\otimes\mathcal{H}'=\underset{k\in J'}{\oplus}(\mathcal{H}\otimes e'_k)=\underset{k\in J'}{\oplus}\mathcal{H}^{(k)},$$

where each $\mathcal{H}^{(k)}$ is a copy of $\mathcal{H}$, the operator $A\otimes I$ acts diagonally i.e. all $\mathcal{H}^{(k)}$ are invariant, and $(A\otimes I)|\mathcal{H}^{(k)}\equiv A|\mathcal{H}$; see more details below.)

The inequality $||A\otimes A'||\ge||A||||A'||$ obviously follows from the identity $||x\otimes x'||=||x||||x'||$ for every $x\in\mathcal{H}$, $x'\in\mathcal{H}'$, and from the definition of the operator norm. □

DEFINITION. Let $\mathcal{M}_i \subset \mathcal{B}(\mathcal{H}_i)$ be von Neumann algebras in Hilbert spaces $\mathcal{H}_i$, $i = 1, 2$. Consider the algebraic tensor product $\mathcal{M}_1 \underset{\text{alg}}{\otimes} \mathcal{M}_2$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and denote by $\mathcal{M}_1 \otimes \mathcal{M}_2$ the von Neumann algebra in $\mathcal{H}_1 \otimes \mathcal{H}_2$ which is generated by $\mathcal{M}_1 \underset{\text{alg}}{\otimes} \mathcal{M}_2$ Then $\mathcal{M}_1 \otimes \mathcal{M}_2$ is called the *tensor product* of von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_2$.

Since $\mathcal{M}_1 \underset{\text{alg}}{\otimes} \mathcal{M}_2$ is obviously a $*$-subalgebra in $\mathcal{B}(\mathcal{H}_1 \otimes H_2)$, the algebra $\mathcal{M}_1 \otimes \mathcal{M}_2$ is just the weak (or strong) closure of $\mathcal{M}_1 \underset{\text{alg}}{\otimes} \mathcal{M}_2$.

Now we shall consider an important particular case when one of the algebras $\mathcal{M}_j$ is $\mathcal{B}(\mathcal{H}_j)$.

Consider again the Hilbert tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and let $\{e'_j | j \in J\}$ be an orthonormal basis in $\mathcal{H}_2$. Then we have an orthogonal decomposition

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \underset{j \in J}{\oplus} \mathcal{H}_1^{(j)},$$

where $\mathcal{H}_1^{(j)} = \mathcal{H}_1 \otimes e'_j$ is a copy of $\mathcal{H}_1$, and the decomposition means that every element of $\mathcal{H}_1 \otimes \mathcal{H}_2$ has the form $\sum_{j \in J} x_j \otimes e_j$ with $x_j \in \mathcal{H}_1$ and $\sum_{j \in J} ||x_j||^2 < \infty$ (in particular at most countable number of x_j may be non-zero), besides

$$||\sum_{j \in J} x_j \otimes e_j||^2 = \sum_{j \in J} ||x_j||^2.$$

Denote by P_j the orthogonal projection of $\mathcal{H}_1 \otimes \mathcal{H}_2$ to $\mathcal{H}_1^{(j)}$. Then for every $A \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_2)$ we may define bounded linear operators $A_{jk} = P_j A P_k$ as bounded linear operators from $\mathcal{H}_1^{(k)}$ to $\mathcal{H}_1^{(j)}$. Identifying all $\mathcal{H}_1^{(j)}$ with $\mathcal{H}_1$ we may also write $A_{jk} \in \mathcal{B}(\mathcal{H}_1)$ and so A becomes a block matrix $[A_{jk}]_{j,k \in J}$ with entries in $\mathcal{B}(\mathcal{H}_1)$.

Now let $\mathcal{M}$ be a von Neumann subalgebra in $\mathcal{B}(\mathcal{H}_1)$.

LEMMA 2. 1. *$\mathcal{M} \otimes \mathcal{B}(H_2)$ is the set of all operators $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that $A_{jk} \in \mathcal{M}$ for all $j, k \in J$ in the matrix representation described above.*

2. *$(\mathcal{M} \otimes \mathcal{B}(\mathcal{H}_2))' = \mathcal{M}' \otimes I$ i.e. the commutant of $\mathcal{M} \otimes \mathcal{B}(\mathcal{H}_2)$ is the set of operators with matrix elements $[\delta_{jk} A']_{j,k \in J}$ where $A' \in \mathcal{M}'$ is fixed and δ_{jk} is the Kronecker symbol ($\delta_{jk} = 1$ if $j = k$ and 0 otherwise.) In other words the commutant is the set of diagonal matrices, with a constant diagonal block $A' \in \mathcal{M}'$.*
3. *$\mathcal{M} \otimes \mathcal{B}(\mathcal{H}_2)$ is a factor if and only if $\mathcal{M}$ is a factor.*

PROOF. If $A = B \otimes C$ with $B \in \mathcal{M}$, $C \in \mathcal{B}(\mathcal{H}_2)$, then obviously $A_{jk} = C_{jk} B$ where $C_{jk} = (Ce_k, e_j)$ is a matrix element of C. Hence $A_{jk} \in \mathcal{M}$ and the same is true for every $A \in \mathcal{M} \underset{\text{alg}}{\otimes} \mathcal{B}(\mathcal{H}_2)$. Now if $w\text{-}\lim A_\gamma = A$ then it follows that $w\text{-}\lim (A_\gamma)_{jk} = A_{jk}$. Assuming $(A_\gamma)_{jk} \in \mathcal{M}$ for all γ, we obtain $A_{jk} \in \mathcal{M}$. It follows that $A_{jk} \in \mathcal{M}$ for every $A \in \mathcal{M} \otimes \mathcal{B}(\mathcal{H}_2)$ and every $j, k \in J$.

Vice versa, if $A \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $A_{jk} \in \mathcal{M}$ for all $j, k \in J$ then $A = w\text{-}\lim P_S A P_S$, where $P_S = \sum_{j \in S} P_j$, S is a finite subset in J and the limit is taken over the directed set of all finite subsets in J (ordered by inclusion). Since $P_S A P_S = \sum_{j,k \in S} P_j A_{jk} P_k \in \mathcal{M} \underset{\text{alg}}{\otimes} \mathcal{B}(\mathcal{H}_2)$, we obviously obtain that $A \in \mathcal{M} \otimes \mathcal{B}(\mathcal{H}_2)$. This proves the first statement of the lemma. The second statement is easily checked by a standard matrix calculations. The last statement is an obvious corollary of the first and second ones. □

2.13. Trace on $\mathcal{M}\otimes\mathcal{B}(\mathcal{H}_2)$

THEOREM 1. *Let τ be a trace on $\mathcal{M}$. For every $A\in(\mathcal{M}\otimes\mathcal{B}(\mathcal{H}_2))^+$, in the earlier notation, define*

$$\hat{\tau}(A)=(\tau\otimes\mathrm{Tr})(A)=\sum_{j\in J}\tau(A_{jj}).$$

Then $\hat{\tau}$ is a trace on $\mathcal{M}\otimes\mathcal{B}(\mathcal{H}_2)$.

PROOF. It is just repetition of the proof that the standard trace Tr on $\mathcal{B}(\mathcal{H})$ has all the required properties (i.e. that it is linear, tracial, faithful, normal and semi-finite). This is left to the reader as an exercise. □

REMARK. $\hat{\tau}$ will be finite if τ is finite and $\dim_{\mathbb{C}}\mathcal{H}_2<\infty$, otherwise $\hat{\tau}$ will be semi-finite. Such a situation occurs e.g. if $\mathcal{M}=\mathcal{M}_l$ or $\mathcal{M}_r$ for a discrete group Γ (then τ is finite) and $\mathcal{H}_2$ is infinite-dimensional. Now let us consider a particular case $\mathcal{M}=\mathcal{M}_l$ (or $\mathcal{M}_r$) for a given discrete group Γ. Then an alternative formula for the trace $\hat{\tau}=\tau\otimes\mathrm{Tr}$ on $\mathcal{M}_l\otimes\mathcal{B}(\mathcal{H}_2)$ (or $\mathcal{M}_r\otimes\mathcal{B}(\mathcal{H}_2)$) is possible. Namely the space $L^2\Gamma\otimes\mathcal{H}_2$ has an orthogonal decomposition

$$L^2\Gamma\otimes\mathcal{H}_2=\mathop{\oplus}_{g\in\Gamma}\mathcal{H}_2^{(g)},$$

where $\mathcal{H}_2^{(g)}=\delta_g\otimes\mathcal{H}_2$. In this decomposition any operator $A\in\mathcal{B}(L^2\Gamma\otimes\mathcal{H}_2)$ becomes a block matrix $[A_{g,h}]_{g,h\in\Gamma}$, $A_{g,h}\in\mathcal{B}(\mathcal{H}_2)$.

THEOREM 2. 1. *If $A\in\mathcal{B}(L^2\Gamma\otimes\mathcal{H}_2)$ then the following is true:*
(i) *$A\in\mathcal{M}_l\otimes\mathcal{B}(\mathcal{H}_2)$ if and only if $A_{gm,hm}=A_{g,h}$ for all $g,h,m\in\Gamma$;*
(ii) *$A\in\mathcal{M}_r\otimes\mathcal{B}(\mathcal{H}_2)$ if and only if $A_{ng,nh}=A_{g,h}$ for all $g,h,n\in\Gamma$;*
2. *If $A\in(\mathcal{M}_l\otimes\mathcal{B}(\mathcal{H}_2))^+$ or $(\mathcal{M}_r\otimes\mathcal{B}(\mathcal{H}_2))^+$) then*

$$\hat{\tau}(A)=(\tau\otimes\mathrm{Tr})(A)=\mathrm{Tr}\,A_{e,e}.$$

PROOF. 1. Inclusion $A\in\mathcal{M}_l\otimes\mathcal{B}(\mathcal{H}_2)$ is equivalent to the commutativity relations $(R_m\otimes I)A(R_m\otimes I)^{-1}=A$ that should hold for every $m\in\Gamma$. Now notice that $(A_{g,h}y,x)=(A(\delta_h\otimes y),\delta_g\otimes x)$, $x,y\in\mathcal{H}_2$, and the commutativity relations will hold if checked on all vectors of the form $\delta_g\otimes x$, $x\in\mathcal{H}_2$. This leads immediately to (i), and (ii) is proved similarly.

2. Obviously if $\{e_j|j\in J\}$ is an orthonormal basis in $\mathcal{H}_2$ then

$$\mathrm{Tr}\,A_{e,e}=\sum_{j\in J}(A(\delta_e\otimes e_j),\delta_e\otimes e_j)=\sum_{j\in J}\tau(A_{jj})=\hat{\tau}(A)=(\tau\otimes\mathrm{Tr})(A)$$

which proves the second statement in Theorem. □

2.14. Von Neumann algebra of free discrete group actions

Let Γ be a discrete group that acts freely on a (possibly non-compact) smooth manifold M, possibly with a boundary. This means that a map

$$\Gamma\times M\to M,\qquad \{g,x\}\mapsto gx,$$

is given, satisfying the following properties:

(i) $ex=x$, $g_1(g_2x)=(g_1g_2)x$ for all $x\in M, g_1,g_2\in\Gamma$;

(ii) for every $g \in \Gamma$ the map $x \mapsto gx$ is a diffeomorphism of M;
(iii) every $x \in M$ has a neighborhood U such that $gU \cap hU = \emptyset$ for every $g, h \in \Gamma$ such that $g \neq h$.

In this case we shall say that M is a Γ-*manifold*. Obviously then the orbit space $X = M/\Gamma$ is a C^∞-manifold (possibly with a boundary).

EXAMPLE. Let X be a compact manifold, $\Gamma = \pi_1(X)$ its fundamental group. Let M be the universal covering of X. Then Γ acts on M (by so-called deck transformations) freely, so that $M/\Gamma = X$.

Let F be a fundamental domain of the action of Γ on M. This means that F is an open subset $F \subset M$, such that

(i) $gF \cap hF = \emptyset$ for every $g, h \in \Gamma$ such that $g \neq h$;
(ii) $\bigcup_{g\in\Gamma} g\bar{F} = M$ (here $\bar{F}$ means the closure of F);
(iii) $\mathrm{mes}(\bar{F}\backslash F) = 0$ where mes means the Lebesgue measure in local coordinates.

We shall always suppose that the fundamental domain exists (this can easily be proved if for instance $X = M/\Gamma$ is compact). Now let μ be a positive measure with a smooth positive density on M, such that μ is Γ-invariant. (It can be obtained, e.g. by lifting any measure with a smooth positive density from X to M). Then we may consider the space $L^2(M) = L^2(M, d\mu)$ (we shall omit $d\mu$ for the brevity of notations). The group Γ has a unitary representation in $L^2(M)$ given by $g \mapsto L_g$, where $(L_g u)(x) = u(g^{-1}x)$. If we chose a fundamental domain $F \subset M$ for the action of Γ, then we have a decomposition $M \simeq \Gamma \times F$ (up to a set of measure 0), $gx \leftrightarrow \{g, x\}$, $x \in F$, $g, \in \Gamma$. The action of Γ on M becomes then the action given by $h\{g, x\} = \{hg, x\}$ on $\Gamma \times F$. It gives a unitary isomorphism

$$U_F : L^2(M) \xrightarrow{\sim} L^2\Gamma \otimes L^2(F),$$

and the action of L_g on $L^2(M)$ becomes $L_g \otimes I$ on $L^2\Gamma \otimes L^2(F)$.

DEFINITION. $\mathcal{M}_\Gamma$ is the von Neumann algebra consisting of all operators $A \in \mathcal{B}(L^2(M))$ which commute with all left translations $\{L_g | g \in \Gamma\}$ in $L^2(M)$.

It follows that $\mathcal{M}'_\Gamma$ is exactly the von Neumann algebra generated by $\{L_g | g \in \Gamma\}$ in $L^2(M)$.

Representing $L^2(M) = L^2\Gamma \otimes L^2(F)$ we get $\mathcal{M}_\Gamma = \mathcal{M}_r \otimes \mathcal{B}(L^2(F))$ and $\mathcal{M}'_\Gamma = \mathcal{M}_l \otimes I$.

The natural trace $\tau \otimes \mathrm{Tr}$ on $\mathcal{M}_\Gamma$ will be denoted Tr_Γ.

THEOREM. Tr_Γ *does not depend on the choice of the fundamental domain* F.

PROOF. Consider the canonical projection $\pi : M \to X = M/\Gamma$. Then $\pi : F \to X$ is an isomorphism of the measure spaces (F, μ) and (X, μ), up to a subset of measure 0 in X (the measure μ on X is induced by the same density and is well-defined because the corresponding density on M is Γ-invariant). This gives a unitary isomorphism of Hilbert spaces, $V_F : L^2(F) \xrightarrow{\sim} L^2(X)$. Hence, if F_1, F_2 are two fundamental domains, we obtain a unitary isomorphism

$$V_{F_1,F_2} : L^2(F_2) \xrightarrow{\sim} L^2(F_1), \;\; V^*_{F_1,F_2} = V^{-1}_{F_1,F_2} = V_{F_2,F_1}.$$

Now let us consider the following commutative diagram

$$
\begin{tikzcd}
& L^2\Gamma \otimes L^2(F_1) \arrow[r, "I\otimes V_{F_1}"] & L^2\Gamma \otimes L^2(X) \\
L^2(M) \arrow[ur, "U_{F_1}"] \arrow[dr, "U_{F_2}"'] & & \\
& L^2\Gamma \otimes L^2(F_2) \arrow[uu, "U_{F_1,F_2}"'] \arrow[r, "I\otimes V_{F_2}"] & L^2\Gamma \otimes L^2(X) \arrow[uu, "W_{F_1,F_2}"']
\end{tikzcd}
$$

where U_{F_1,F_2} and W_{F_1,F_2} are unitary isomorphisms which are defined from the condition of commutativity of the diagram (i.e. $U_{F_1,F_2} = U_{F_1}U_{F_2}^{-1}$ etc.). Let us calculate the operators U_{F_1,F_2} and W_{F_1,F_2}. Obviously $L^2\Gamma \otimes L^2F_j \cong L^2(\Gamma \times F_j)$ and the unitary isomorphisms in the diagram are induced by the maps in the following geometric commutative diagram:

$$
\begin{tikzcd}
& \Gamma \times F_1 \arrow[dl, "u_{F_1}"'] \arrow[dd, "u_{F_1,F_2}"] & \Gamma \times X \arrow[l, "(1\times\pi_1)^{-1}"'] \arrow[dd, "w_{F_1,F_2}"] \\
M & & \\
& \Gamma \times F_2 \arrow[ul, "u_{F_2}"] & \Gamma \times X \arrow[l, "(1\times\pi_2)^{-1}"']
\end{tikzcd}
$$

where all maps are measure isomorphisms defined up to measure zero sets, $\pi_j : F_j \to X$ are natural projections, $u_{F_j}(\{g, x\}) = gx$, $x \in F_j$, $g \in \Gamma$, the maps u_{F_1,F_2} and w_{F_1,F_2} are defined from the commutativity condition. Namely $u_{F_1,F_2}(\{g,x\}) = \{h,y\}$ if $x \in F_1$, $y \in F_2$ and $gx = hy$. This means in particular that $h^{-1}gx = y$, hence x and y are in the same Γ-orbit i.e. $\pi_1(x) = \pi_2(y)$. Now let $s = h^{-1}g \in \Gamma$. Then s is uniquely defined by the condition that $sx \in F_2$, so $s = s(x)$ and, obviously, s is a measurable function on F_1 with values in Γ (i.e. every set $\{x | s(x) = s_0\}$ is a measurable subset in F_1). So we get $g = hs(x)$ and, therefore, $u_{F_1,F_2}(\{g,x\}) = \{gs(x)^{-1}, y\}$, $w_{F_1,F_2}(\{g,x\}) = \{gt(x)^{-1}, x\}$, where $t : X \to \Gamma$ is a measurable function defined by the condition $t(x)\tilde{x} \in F_2$ provided $\tilde{x} \in F_1$ and $\pi(\tilde{x}) = x$.

Moving to the operator diagram we see that W_{F_1,F_2} commutes with the operators $L_g \otimes I$ in $L^2\Gamma \otimes L^2(X)$, hence belongs to $\mathcal{M}_r \otimes \mathcal{B}(L^2(X))$. Now suppose that $A \in \mathcal{M}_\Gamma$ i.e. A commutes with every $L_g, g \in \Gamma$, in $L^2(M)$. Operators U_{F_j}, $I \otimes V_{F_j}$ commute with the left action on the corresponding spaces too. Hence if we shift A to $L^2\Gamma \otimes L^2(F_j)$ and further on to $L^2\Gamma \otimes L^2(X)$ by considering operators

$$A_j = U_{F_j} A U_{F_j}^{-1} \in \mathcal{B}(L^2\Gamma \otimes L^2(F_j)),\ \tilde{A}_j = (I \otimes V_{F_j}) A_j (I \otimes V_{F_j})^{-1} \in \mathcal{B}(L^2\Gamma \otimes L^2(X)),$$

then $A_j, \tilde{A}_j$ will commute with the left action of Γ also. Now the choice of the fundamental domain F_j defines the trace τ_j on $\mathcal{M}_\Gamma$ by

$$\tau_j(A) = (\tau \otimes \mathrm{Tr})(A_j) = (\tau \otimes \mathrm{Tr})(\tilde{A}_j), \ j = 1, 2.$$

These traces are exactly two variants of Tr_Γ given by the two choices of a fundamental domain. But from the operator commutative diagram we obtain the following connection between $\tilde{A}_1$ and $\tilde{A}_2$:

$$\begin{aligned}
\tilde{A}_2 &= (I \otimes V_{F_2}) U_{F_2} A U_{F_2}^{-1} (I \otimes V_{F_2})^{-1} \\
&= (I \otimes V_{F_2}) U_{F_2} U_{F_1}^{-1} (I \otimes V_{F_1})^{-1} \tilde{A}_1 (I \otimes V_{F_1}) U_{F_1} U_{F_2}^{-1} (I \otimes V_{F_2})^{-1} \\
&= (I \otimes V_{F_2}) U_{F_1,F_2}^{-1} (I \otimes V_{F_1})^{-1} \tilde{A}_1 (I \otimes V_{F_1}) U_{F_1,F_2} (I \otimes V_{F_2})^{-1} \\
&= W_{F_1,F_2}^{-1} \tilde{A}_1 W_{F_1,F_2}.
\end{aligned}$$

Now W_{F_1,F_2} is a unitary operator which commutes with the left action of Γ on $L^2\Gamma \otimes L^2(X)$ (i.e. with every operator $L_g \otimes I$, $g \in \Gamma$). Hence $W_{F_1,F_2} \in \mathcal{M}_\Gamma$ in $L^2\Gamma \otimes L^2(X)$ and if $\tilde{A}_1 \in \mathcal{M}_\Gamma^+$ then $\tilde{A}_2 \in \mathcal{M}_\Gamma^+$ and $(\tau \otimes \mathrm{Tr})((\tilde{A}_1) = (\tau \otimes \mathrm{Tr})(\tilde{A}_2)$ as required. □

2.15. Generalization to vector Γ -bundles

Let $p : E \to M$ be a complex vector bundle over a Γ-manifold M. (All bundles are supposed to be C^∞ too). Suppose that an action of Γ on E is given such that if E_x denotes the fiber of E over a point x then every element $g \in \Gamma$ defines for every $x \in M$ a linear isomorphism $\hat{g}_x : E_x \to E_{gx}$, so the action on E covers the action on M. We may then identify E with $\pi^* E_0$ where E_0 is a vector bundle over $X = M/\Gamma$ and $\pi : M \to X$ is the canonical projection. (Recall that $\pi^* E_0$ has the fiber at $x \in M$ identified with the fiber of E_0 at $\pi(x)$; this identification can be done by the action of Γ on E). Then we can introduce an hermitian positive inner product on every E_x, $x \in M$, such that $\hat{g}_x : E_x \to E_{gx}$ is an isometry) e.g. by lifting to E a Hermitian metric given on E_0. For simplicity we shall always suppose that such a metric is already given. Let a positive Γ-invariant C^∞-density on M is also given. Then the Hilbert space $L^2(M, E)$ of square integrable sections of E is well defined. The group Γ then acts on $L^2(M, E)$ by isometries, if we define the action by the formula

$$(L_g f)(x) = \hat{g}_x f(g^{-1}x).$$

Then again we can define a von Neumann algebra $\mathcal{M}_\Gamma = \mathcal{M}_\Gamma(E)$ as the algebra of all operators $A \in \mathcal{B}(L^2(M, E))$ that commute with L_g for every $g \in \Gamma$. The results of the previous section are easily generalized to this case. Namely, representing $L^2(M, E) = L^2\Gamma \otimes L^2(F, E)$ we obtain a trace $\mathrm{Tr}_\Gamma = \tau \otimes \mathrm{Tr}$ on $\mathcal{M}_\Gamma$ which does not depend on the choice of the fundamental domain.

An example of this situation arises as follows. Let M be an arbitrary Γ-manifold, i.e. a smooth manifold with a free action of a discrete group Γ and with a compact quotient M/Γ. Then Γ acts in the tangent bundle TM, so TM becomes a (real) Γ-bundle. We can take now a complexified tensor bundle E over M. The action of Γ on TM canonically induces an action of Γ on E, which makes E a Γ-bundle. Then a Γ-invariant Riemannian metric induces an hermitian inner product on E, so we arrive at the situation described above. The most important example: complexified exterior degrees of the cotangent bundle, which lead to Hilbert spaces of L^2 differential forms on M.

2.16. Hilbert Γ-modules and Γ-dimension.

DEFINITION 1. (i) A *(general) Hilbert* Γ*-module* is a Hilbert space with a left unitary action of Γ i.e. with a unitary representation of Γ.

(ii) A *free Hilbert* Γ*-module* is a Hilbert Γ-module which is unitarily isomorphic to a Γ-module of the form $L^2\Gamma \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ is a Hilbert space, and the representation of Γ on $L^2\Gamma \otimes \mathcal{H}_0$ is given by $g \mapsto L_g \otimes I$.

(iii) A *(projective) Hilbert* Γ*-module* is a (general) Hilbert Γ-module which is unitarily isomorphic to a closed submodule of a free Hilbert Γ-module (i.e. to a closed Γ-invariant subspace in $L^2\Gamma \otimes \mathcal{H}_0$ for a Hilbert space $\mathcal{H}_0$).

In this definition unitary isomorphisms of Hilbert Γ-modules are always understood as unitary maps which commute with the action of Γ (or, in other words, intertwining unitary maps between two representations of Γ).

Obviously a projective Hilbert Γ-module is always a direct summand in a free Hilbert Γ-module since the orthogonal complement will be a Γ-module again.

In other words a projective Hilbert module is a subrepresentation of a direct sum of several (or even infinitely many) copies of the left regular representation. If Γ is finite then every Γ-module is projective, but for infinite Γ it is not always the case.

In the future we will only need projective Hilbert Γ-modules, so we will skip the word "projective" for the sake of brevity.

So from now on the words "Hilbert Γ*-module" will always mean projective Hilbert* Γ*-module unless otherwise indicated.*

Let V be a Hilbert Γ-module, $V \subset L^2\Gamma \otimes \mathcal{H}_0$. Let P_V be the orthogonal projection onto V in $L^2\Gamma \otimes \mathcal{H}_0$. Then $P_V \in \mathcal{M}_\Gamma$, since P_V commutes with $L_g \otimes I$ for every $g \in \Gamma$.

DEFINITION 2. The Γ-dimension of V is defined as

$$\dim_\Gamma V = \operatorname{Tr}_\Gamma P_V$$

THEOREM. $\dim_\Gamma V$ *does not depend on the choice of the Hilbert* Γ*-module inclusion of* V *into a free Hilbert* Γ*-module.*

PROOF. Suppose that V is unitarily isomorphic to Hilbert Γ-submodules $V_1 \subset L^2\Gamma \otimes \mathcal{H}_1$, $V_2 \subset L^2\Gamma \otimes \mathcal{H}_2$, and P_{V_j} is the orthogonal projection of $L^2\Gamma \otimes \mathcal{H}_j$ onto V_j. We have to prove that $\operatorname{Tr}_\Gamma P_{V_1} = \operatorname{Tr}_\Gamma P_{V_2}$. Obviously there exists a unitary Γ-isomorphism $B : V_1 \to V_2$. By replacing $\mathcal{H}_1$ and $\mathcal{H}_2$ by $\mathcal{H}_1 \oplus \mathcal{H}_2$ we may reduce the proof to the case when $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_0$. Then consider $U = P_{V_2} B P_{V_1}$ as an operator in $L^2\Gamma \otimes \mathcal{H}_0$. It commutes with the left action of Γ in $L^2\Gamma \otimes \mathcal{H}_0$ since this is the case for all operators P_{V_1}, B, P_{V_2}. Moreover, U is a partial isometry, $U^*U = P_{V_1}$, $UU^* = P_{V_2}$. Hence

$$\operatorname{Tr}_\Gamma P_{V_1} = \operatorname{Tr}_\Gamma(U^*U) = \operatorname{Tr}_\Gamma(UU^*) = \operatorname{Tr}_\Gamma P_{V_2}$$

due to the standard property of Tr_Γ in $\mathcal{M}_\Gamma$. □

It is a convenient to know that unitary isomorphisms in Definition 1 can be replaced by linear topological isomorphisms:

PROPOSITION. *Let* V_1, V_2 *be (general) Hilbert* Γ*-modules,* $A : V_1 \to V_2$ *a linear topological isomorphism of* Γ*-modules. Then there exists a unitary isomorphism of* Γ*-modules* $U : V_1 \to V_2$*.*

PROOF. If we take the polar decomposition $A = US$, then $U : V_1 \to V_2$ will be the desired unitary isomorphism of Γ-modules. □

COROLLARY 1. *Assume that V is a (general) Hilbert Γ-module and there exists a topological Γ-isomorphism of V onto a closed Γ-submodule of a free Hilbert Γ-module. Then V is a (projective) Hilbert Γ-module.*

COROLLARY 2. *Let V be a (projective) Hilbert Γ-module with a Hilbert structure given by the inner product $(\cdot,\cdot)$. Denote by $\|\cdot\|$ the corresponding norm. Let $(\cdot,\cdot)'$ be another Γ-invariant inner product on V such that the corresponding norm $\|\cdot\|'$ is equivalent to $\|\cdot\|$. Then V remains a (projective) Hilbert Γ-module if we keep the same action of Γ but replace the inner product $(\cdot,\cdot)$ by $(\cdot,\cdot)'$.*

EXAMPLES OF HILBERT Γ-MODULES.

(a) If M is a Γ-manifold with a Γ-invariant measure, defined by a positive smooth density, then $L^2(M)$ is a free Hilbert Γ-module (with a natural action of Γ) because of the isomorphism $L^2(M) = L^2\Gamma \otimes L^2(F)$ where F is a fundamental domain of the action of Γ on M (see 2.14).

Therefore any closed Γ-invariant subspace of $L^2(M)$ is a Hilbert Γ-module.

(b) More generally, let us take a vector Γ-bundle E over M with a Γ-invariant hermitian metric on E. Then the Hilbert space $L^2(M, E)$ (see 2.15) is again a free Hilbert Γ-module, whereas any closed Γ-invariant subspace in $L^2(M, E)$ is a Hilbert Γ-module.

Important examples of closed Γ-invariant subspaces in $L^2(M, E)$ are spaces of weak solutions of Γ-invariant differential equations of the form $Au = 0$ where A is a Γ-invariant differential operator. We will discuss such examples in more detail in Chapter 3.

2.17. Properties of the Γ-dimension

The Γ-dimension $\dim_\Gamma$ has the following properties:

1. $\dim_\Gamma L^2\Gamma = 1$ (here $L^2\Gamma$ is considered as a left Γ-module), $\dim_\Gamma\{0\} = 0$, $\dim_\Gamma(L^2\Gamma \otimes \mathcal{H}_0) = \dim_{\mathbb{C}} \mathcal{H}_0$.
2. $\dim_\Gamma(M_1 \oplus M_2) = \dim_\Gamma M_1 + \dim_\Gamma M_2$ for two Γ-modules M_1, M_2.
3. $M_1 \subset M_2$ implies $\dim_\Gamma M_1 \le \dim_\Gamma M_2$, equality holds here if and only if $M_1 = M_2$.

It follows that the set of all possible values of Γ-dimension is an additive subsemigroup in $[0, \infty]$. This subsemigroup contains $\mathbb{Z}_+ \cup \{+\infty\}$, where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ is the set of all non-negative integers.

EXAMPLE 1. If $\Gamma = \{e\}$ (i.e. Γ is trivial), then $\dim_\Gamma$ coincides with the usual dimension $\dim_{\mathbb{C}}$. So in this case the values of $\dim_\Gamma$ are all non-negative integers and $+\infty$.

EXAMPLE 2. If Γ is finite, then the possible values of $\dim_\Gamma$ are $\{\frac{k}{N} | k \in \mathbb{Z}_+\}$ and $+\infty$, where $N = \mathrm{Card}(\Gamma)$.

To see this note first that for the submodules of $L^2\Gamma$ the possible values of $\dim_\Gamma$ are in $\{\frac{k}{N} | k = 0, 1, \dots, N\}$ (see Example 1 in 2.11). Moreover, $1/N$ is indeed a possible value of $\dim_\Gamma$ (it is the Γ-dimension of the submodule in $L^2\Gamma$ which consists of all constant functions). Due to the semigroup property of the range of $\dim_\Gamma$ on all Hilbert modules, this range contains all numbers k/N with $k \in \mathbb{Z}$

and also $+\infty$ (the latter because we can take a free Γ-module $L^2\Gamma \otimes \mathcal{H}_0$ with an infinite-dimensional $\mathcal{H}_0$).

On the other hand, any irreducible Γ-module of Γ is a submodule of $L^2\Gamma$ as is well known from the representation theory of finite groups (see e.g. [**Se**], Sect. 2.4). It follows that the Γ-dimension can not take any values in the interval $(0, 1/N)$. Hence, $1/N$ is the minimal positive value of $\dim_\Gamma$. Therefore the set of all possible values is $\{\frac{k}{N} | k \in \mathbb{Z}_+\} \cup \{+\infty\}$.

Example 3. If all conjugacy classes of Γ are infinite, the set of all possible values of $\dim_\Gamma$ is $[0, +\infty]$.

Indeed, in this case the algebra $\mathcal{M}_l$ in $L^2\Gamma$ is a factor due to Theorem in 2.10. Now let us use the classification of factors which is explained in 2.7. Clearly, $\mathcal{M}_l$ can not be type I_n because $\mathcal{M}_l$ is infinite-dimensional. Since it has a finite trace, the only remaining possibility is that $\mathcal{M}_l$ has type II_1, so on the submodules of $L^2\Gamma$ it takes all values from $[0, 1]$. therefore due to the semigroup property the set of all values of $\dim_\Gamma$ is $[0, +\infty]$.

It can be also proved in this situation that two Hilbert Γ-modules V_1, V_2 are unitarily isomorphic if and only if $\dim_\Gamma V_1 = \dim_\Gamma V_2$. We will discuss this in 2.28 in a more general context of arbitrary factors.

2.18. Ideals in von Neumann algebras

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. A left (right) ideal in $\mathcal{M}$ is a linear subspace $\mathcal{J} \subset \mathcal{M}$ such that $A \in \mathcal{M}, B \in \mathcal{J}$ implies $AB \in \mathcal{J}$ (resp. $BA \in \mathcal{J}$). An ideal (or two-sided ideal) is a linear subspace $\mathcal{J} \subset \mathcal{M}$ which is a left and right ideal at the same time. Now let $\mathcal{J}^* = \{B^* \mid B \in \mathcal{J}\}$. A subset $\mathcal{J} \subset \mathcal{M}$ is called self-adjoint if $B \in \mathcal{J}$ implies $B^* \in \mathcal{J}$ or equivalently, if $\mathcal{J}^* = \mathcal{J}$.

Lemma 1. (i) *If $\mathcal{J}$ is a (two-sided) ideal in a von Neumann algebra $\mathcal{M}$ then $\mathcal{J}$ is self-adjoint.*

(ii) *If $\mathcal{J}$ is a left (or right) self-adjoint ideal in $\mathcal{M}$ then $\mathcal{J}$ is a two-sided ideal.*

(iii) *If $\mathcal{J}$ is a (two-sided) ideal in $\mathcal{M}$ then for $B \in \mathcal{M}$ the inclusion $B \in \mathcal{J}$ is equivalent to $|B| \in \mathcal{J}$.*

Proof. If $\mathcal{J}$ is a left (or right) ideal in $\mathcal{M}$ then $\mathcal{J}^*$ is a right (resp. left) ideal. Hence (ii) follows. Now suppose that $\mathcal{J}$ is a (two-sided) ideal in $\mathcal{M}$. For any $B \in \mathcal{J}$ consider its polar decomposition $B = US$ with $S = |B| \geq 0$, U partial isometry such that $\operatorname{Ker} U = \operatorname{Ker} S$. Then $U \in \mathcal{M}$, $S \in \mathcal{M}$. Moreover $S = U^*US = U^*B \in \mathcal{J}$ and, obviously, $S \in \mathcal{J}$ implies $B \in \mathcal{J}$ which proves (iii). Now $B^* = SU^* = U^*BU^* \in \mathcal{J}$ which proves (i). □

Later on the word ideal will always mean two-sided (or, equivalently, self-adjoint) ideal.

Lemma 2. *Let $\mathcal{J}$ be an ideal in $\mathcal{M}$, $\mathcal{J}^+ = \{A \mid A \in \mathcal{J}, A \geq 0\}$. Then $\mathcal{J}$ is a linear span of $\mathcal{J}^+$. More exactly every element of $\mathcal{J}$ is a linear combination of four elements of $\mathcal{J}^+$.*

Proof. If $A \in \mathcal{J}$, then $A = A_1 + iA_2$ with $A_1 = (A + A^*)/2$, $A_2 = (A - A^*)/(2i)$, $A_j = A_j^*$, $j = 1, 2$. Now it is sufficient to prove that every self-adjoint operator $A \in \mathcal{J}$ is a linear combination of two operators from $\mathcal{J}^+$. Let $E_+ = E([0, \infty))$, $E_- = E((-\infty, 0))$ are spectral projections of A. Then $A =$

$E_+A + E_-A$; $E_+, E_- \in \mathcal{M}$; $E_+A \geq 0$, $(-E_-)A \geq 0$. Hence E_+A and $(-E_-)A$ are in $\mathcal{J}^+$ which proves Lemma. □

Now we want to describe subsets $\mathcal{J}_0 \subset \mathcal{M}^+$ which can be represented in the form $\mathcal{J}^+$ for an ideal $\mathcal{J}$ in $\mathcal{M}$, or, equivalently, such that $\mathrm{span}(\mathcal{J}_0)$ is an ideal in $\mathcal{M}$.

THEOREM. 1) *Let $\mathcal{J}_0 \subset \mathcal{M}^+$. An ideal $\mathcal{J} \subset \mathcal{M}$ such that $\mathcal{J}^+ = \mathcal{J}_0$ exists if and only if $\mathcal{J}_0$ has the following three properties:*

(i) *If $S \in \mathcal{J}_0$ and $U \in \mathcal{M}$ is unitary, then $U^{-1}SU \in \mathcal{J}_0$ (i.e. $\mathcal{J}_0$ is $\mathcal{M}$-unitarily invariant);*

(ii) *If $S \in \mathcal{J}_0$, $T \in \mathcal{M}^+$ and $T \leq S$ then $T \in \mathcal{J}_0$ (star-shape property);*

(iii) *If $S_1, S_2 \in \mathcal{J}_0$ then $S_1 + S_2 \in \mathcal{J}_0$ (semigroup property).*

2) *If the conditions (i)–(iii) are satisfied then $\mathcal{J} = \{A \mid A \in \mathcal{M}, |A| \in \mathcal{J}_0\}$.*

3) *If the conditions (i)–(iii) are satisfied then the set*

$$\mathcal{J}_{(2)} = \{A \mid A \in \mathcal{M},\ A^*A \in \mathcal{J}_0\}$$

is an ideal in $\mathcal{M}$ and $\mathcal{J} = (\mathcal{J}_{(2)})^2$ i.e. $\mathcal{J}$ is the set of all linear combinations of operators of the form BC with $B, C \in \mathcal{J}_{(2)}$.

The proof will be divided in a series of lemmas.

LEMMA 3. *Let $\mathcal{J}_0$ satisfies (i)–(iii) in Theorem. Then*

$$\mathcal{J}_{(2)} = \{A \mid A \in \mathcal{M},\ A^*A \in \mathcal{J}_0\}$$

is an ideal in $\mathcal{M}$ and $(\mathcal{J}_{(2)}^2)^+ = \mathcal{J}_0$.

PROOF. 1) Let $A \in \mathcal{J}_{(2)}$, $U \in \mathcal{M}$ is a unitary operator. Then

$$(UA)^*(UA) = A^*U^*UA = A^*A \in \mathcal{J}_0$$

and

$$(AU)^*(AU) = U^*A^*AU = U^{-1}A^*AU \in \mathcal{J}_0$$

due to (i).

Hence UA and AU are in $\mathcal{J}_{(2)}$. Therefore BA and AB are in $\mathcal{J}_{(2)}$ for every $B \in \mathcal{M}$ due to Theorem 3 in 2.5.

Now let us check that $A_1, A_2 \in \mathcal{J}_{(2)}$ implies $A_1 + A_2 \in \mathcal{J}_{(2)}$. Indeed,

$$(A_1 + A_2)^*(A_1 + A_2) \leq 2A_1^*A_1 + 2A_2^*A_2 \in \mathcal{J}_0$$

due to (iii). Hence $A_1 + A_2 \in \mathcal{J}_{(2)}$ due to (ii). Hence, we have proved that $\mathcal{J}_{(2)}$ is an ideal in $\mathcal{M}$.

2) Let us prove that $\mathcal{J}_0 = ((\mathcal{J}_{(2)})^2)^+$. If $S \in \mathcal{J}_0$ then $\sqrt{S} \in \mathcal{J}_{(2)}$, so $S \in ((J_{(2)})^2)^+$. Conversely, let $S \in ((\mathcal{J}_{(2)})^2)^+$ i.e. $S \geq 0$ and S is a finite sum of operators of the form C^*D with $C, D \in \mathcal{J}_{(2)}$. Now using the polarization identity

$$\begin{aligned} 4C^*D &= (C+D)^*(C+D) - (C-D)^*(C-D) \\ &\quad - i(C+iD)^*(C+iD) + i(C-iD)^*(C-iD), \end{aligned}$$

we see that $S \leq \sum_{i=1}^N C_i^*C_i$ for some operators $C_i \in \mathcal{J}_{(2)}$. Since $C_i^*C_i \in \mathcal{J}_0$ it follows that $S \in \mathcal{J}_0$ due to (ii) in Theorem. Thus $((\mathcal{J}_{(2)})^2)^+ = \mathcal{J}_0$. □

LEMMA 4. *Let $S, T \in \mathcal{M}^+$ and $S \leq T$. There exists a unique operator $A \in \mathcal{B}(\mathcal{H})$ such that*

(i) $\sqrt{S} = A\sqrt{T}$;
(ii) $\operatorname{Ker} A \supset \operatorname{Ker} T$.

If A is this operator then $A \in \mathcal{M}$.

PROOF. Note that $\|\sqrt{S}x\|^2 = (Sx, x) \le (Tx, x) = \|\sqrt{T}x\|^2$. Hence the *equality* $A\sqrt{T}x = \sqrt{S}x$ defines A as a bounded linear operator on $\overline{\operatorname{Im}\sqrt{T}} = (\operatorname{Ker}\sqrt{T})^\perp = (\operatorname{Ker} T)^\perp$. Now (ii) requires to extend A by putting $A|_{\operatorname{Ker} T} = 0$. The uniqueness of A follows and inclusion $A \in \mathcal{M}$ follows from the unitary invariance of the construction. □

PROOF OF THEOREM. Lemma 3 implies that if (i)–(iii) are satisfied, then there exists an ideal $\mathcal{J} = (\mathcal{J}_{(2)})^2$ such that $\mathcal{J}^+ = \mathcal{J}_0$. (This proves also part 3 in Theorem and part 2 is contained in Lemma 1.) It remains to prove that if $\mathcal{J}_0 = \mathcal{J}^+$ for an ideal $\mathcal{J}$ then (i)–(iii) are satisfied. Note that (i) and (iii) are obvious. To prove (ii) suppose that $S \in \mathcal{M}^+$, $S \le T \in \mathcal{J}_0$. Then there exist $A \in \mathcal{M}$ such that $\sqrt{S} = A\sqrt{T}$, hence

$$S = \sqrt{S}\sqrt{S} = A\sqrt{T}\sqrt{T}A^* = ATA^* \in \mathcal{J}_0$$

which proves the Theorem. □

2.19. Extension of a trace

Let $\mathcal{M}$ be a von Neumann algebra and $\tau : \mathcal{M}^+ \to [0, \infty]$ be a trace on $\mathcal{M}$. It is natural to try and extend τ linearly to a maximal possible domain. We shall discuss now what is the natural extension of this kind.

Consider the following subsets in $\mathcal{M}^+$:

$$\mathcal{S}_1^+ = \{S \mid S \in \mathcal{M}^+,\ \tau(S) < \infty\},$$
$$\mathcal{S}_2^+ = \{S \mid S \in \mathcal{M}^+,\ \tau(S^2) < \infty\}.$$

Obviously they depend on $\mathcal{M}$ and τ, so we shall write sometimes $\mathcal{S}_j^+(\mathcal{M})$ or $\mathcal{S}_j^+(\tau)$ or $\mathcal{S}_j^+(\mathcal{M}, \tau)$ if $\mathcal{M}$ or/and τ have to be specified explicitly.

Now define $\mathcal{S}_j = \operatorname{span}(\mathcal{S}_j^+)$, $j = 1, 2$. Then by the results of the previous section, both $\mathcal{S}_1$ and $\mathcal{S}_2$ are ideals in $\mathcal{M}$ that can be described as follows:

$$\mathcal{S}_1 = \{A \mid A \in \mathcal{M}, \tau(|A|) < \infty\}$$
$$\mathcal{S}_2 = \{A \mid A \in \mathcal{M}, \tau(A^*A) < \infty\}.$$

Besides $(\mathcal{S}_2)^2 = \mathcal{S}_1$.

Here again we will write $\mathcal{S}_j(\mathcal{M})$ or $\mathcal{S}_j(\tau)$ or $\mathcal{S}_j(\mathcal{M}, \tau)$ instead of $\mathcal{S}_j$ if $\mathcal{M}$ or/and τ have to be specified explicitly.

The trace τ uniquely extends by linearity to a linear map $\tau : \mathcal{S}_1 \to \mathbb{C}$.

EXERCISE 1. Check that the linear extension of τ to $\mathcal{S}_1$ from $\mathcal{S}_1^+$ is possible and well defined.

EXAMPLES. 1) If $A \in \mathcal{S}_1(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$ and $\{e_j \mid j \in \mathcal{J}\}$ is an orthonormal basis in $\mathcal{H}$, then $\sum_{j \in \mathcal{J}} |(Ae_j, e_j)| < \infty$ and $\operatorname{Tr} A = \sum_{j \in \mathcal{J}} (Ae_j, e_j)$. (Indeed, this is true on $\mathcal{S}_1^+(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$, hence on $\mathcal{S}_1(\mathcal{B}(\mathcal{H}), \operatorname{Tr})$).

2) If $A \in \mathcal{S}_1(\mathcal{M} \otimes \mathcal{B}(\mathcal{H}_2), \tau \otimes \operatorname{Tr})$, then in notations described earlier (see 2.12 and 2.13) $\sum_j |\tau(A_{jj})| < \infty$ and $(\tau \otimes \operatorname{Tr})(A) = \sum_j \tau(A_{jj})$

3) If $A \in \mathcal{S}_1(\mathcal{M}_r \otimes \mathcal{B}(\mathcal{H}_2), \tau \otimes \mathrm{Tr})$ (or the same with $\mathcal{M}_l$ instead of $\mathcal{M}_r$) then $A_{e,e} \in \mathcal{S}_1(\mathcal{B}(\mathcal{H}_2)), \mathrm{Tr})$ and $(\tau \otimes \mathrm{Tr})(A) = \mathrm{Tr}\, A_{e,e}$.

4) If $\mathcal{M} = L^\infty(M, \Sigma, \mu)$ considered as a von Neumann algebra in $L^2(M, \Sigma, \mu)$ and $\tau(a) = \tau(M_a) = \int_M a\, d\mu$ for $a \in \mathcal{M}^+$, then $\mathcal{S}_1 = L^1(M, \Sigma, \mu) \cap L^\infty(M, \Sigma, \mu)$ and the extension of τ is naturally given by the same formula as for $a \in M^+$.

The following theorem gives the most important algebraic property of the extended trace.

THEOREM. *Let $\mathcal{M}$ be a von Neumann algebra with a trace τ. Then for the extended trace*

$$\tau(AB) = \tau(BA) \text{ provided } A \in \mathcal{M}, B \in \mathcal{S}_1 \text{ or } A \in \mathcal{S}_2, B \in \mathcal{S}_2.$$

PROOF. 1) Consider first the case $A \in \mathcal{M},\ B \in \mathcal{S}_1$. Then representing A as a linear combination of 4 unitary operators in $\mathcal{M}$, we reduce the proof to the case when $A = U = (U^*)^{-1}$ is unitary. But then $AB = UB = UBAU^{-1}$, so denoting $C = BA$ we see that it is sufficient to prove that $\tau(UCU^{-1}) = \tau(C)$ if $U \in \mathcal{M}$ is unitary and $C \in \mathcal{S}_1$. Due to linearity it is sufficient to consider the case $C \in \mathcal{S}_1^+$ where the result is due to the unitary invariance of τ.

2) Let $A \in \mathcal{S}_2,\ B \in \mathcal{S}_2$. If for instance $B = A^*$ the result follows from the traciality of τ. The general case reduces to this particular one by use of the following two polarization identities:

$$\begin{aligned} 4AB^* &= (A+B)(A+B)^* - (A-B)(A-B)^* \\ &\quad + i(A+iB)(A+iB)^* - i(A-iB)(A-iB)^*, \\ 4B^*A &= (A+B)^*(A+B) - (A-B)^*(A+B) \\ &\quad + i(A+iB)^*(A+iB) - i(A-iB)^*(A-iB). \end{aligned}$$

□

On $\mathcal{S}_2$ there exists a natural inner product given by

$$(A, B)_2 = \tau(B^*A) = \tau(AB^*) = \overline{\tau(A^*B)} = \overline{\tau(BA^*)}, \quad A, B \in \mathcal{S}_2.$$

It induces the norm $\|A\|_2 = [\tau(A^*A)]^{1/2}$. (For the case $\mathcal{M} = L^\infty(M, \Sigma, \mu)$ and $\tau(a) = \int_M a\, d\mu$ this will be the standard L^2-norm. Note that $\mathcal{S}_2$ is not necessarily complete in this case!). So $\mathcal{S}_2$ has a structure of a pre-Hilbert space (but is not generally a Hilbert space).

EXERCISE 2. (i) Prove that $|\tau(A)| \le \tau(|A|)$ for any $A \in \mathcal{S}_1$.

(ii) Introduce $\|B\|_1 = \tau(|B|),\ B \in \mathcal{S}_1$. Prove that $\|B^*\|_1 = \|B\|_1$ and

$$\|AB\|_1 \le \|A\|\|B\|_1, \quad \|BA\|_1 \le \|A\|\|B\|_1,$$

provided $A \in \mathcal{M}, B \in \mathcal{S}_1$.

(iii) Prove that $\|B\|_1 = \sup\{|\tau(AB)| \mid A \in \mathcal{M}, \|A\| \le 1\}$

(iv) Prove that $\|\cdot\|_1$ is a semi-norm on $\mathcal{S}_1$ i.e.

$$\|A + B\|_1 \le \|A\|_1 + \|B\|_1, \quad A, B \in \mathcal{S}_1.$$

DEFINITION. 1) Operators in $\mathcal{S}_2(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$ are called *Hilbert-Schmidt* operators in $\mathcal{H}$. If $\{e_j \mid j \in \mathcal{J}\}$ is an orthonormal basis in $\mathcal{H}$ then the Hilbert-Schmidt operators are exactly operators that satisfy

$$\sum_{j,k \in \mathcal{J}} |(Ae_j, e_k)|^2 < \infty$$

they form a Hilbert space with respect to the inner product $(\cdot,\cdot)_2$. This Hilbert space is naturally untarily isomorphic to $\mathcal{H}\otimes\mathcal{H}$.

2) Operators in $\mathcal{S}_1(\mathcal{B}(\mathcal{H}),\mathrm{Tr})$ are called *trace-class* operators. Here again $\mathcal{S}_1(\mathcal{B}(\mathcal{H}),\mathrm{Tr})$ is a Banach space with respect to the norm $\|\cdot\|_1$ (see e.g. [**BeSh**], Proposition 4.8 in Appendix 1 or [**ReSi**], Theorem VI.20). But generally $\mathcal{S}_1(\mathcal{M},\tau)$ may be not complete with respect to the norm $\|\cdot\|_1$.

2.20. Schwartz kernels of Hilbert-Schmidt operators

Let us recall that Hilbert-Schmidt operators in a Hilbert space $\mathcal{H}$ are operators that belong to $\mathcal{S}_2(\mathcal{B}(\mathcal{H}),\mathrm{Tr})$, or operators $A\in\mathcal{B}(\mathcal{H})$ such that $\mathrm{Tr}(A^*A)<\infty$. To simplify notations we shall write $\mathcal{S}_2(\mathcal{H})$ instead of $\mathcal{S}_2(\mathcal{B}(\mathcal{H}),\mathrm{Tr})$.

Let us choose an orthonormal basis $\{e_j \mid j\in\mathcal{J}\}$ in $\mathcal{H}$. Then

$$\mathrm{Tr}(A^*A)=\sum_{j\in\mathcal{J}}(A^*Ae_j,e_j)=\sum_{j\in\mathcal{J}}\|Ae_j\|^2=\sum_{j,k\in\mathcal{J}}|(Ae_j,e_k)|^2$$

and, hence, the right hand side does not depend on the choice of the orthonormal basis. Consider the norm (often called the Hilbert-Schmidt norm)

$$\|A\|_2=[\mathrm{Tr}(A^*A)]^{1/2}$$

Obviously $A\in\mathcal{S}_2(\mathcal{H})$ if and only if $A\in\mathcal{B}(\mathcal{H})$ and $\|A\|_2<\infty$. The norm $\|\cdot\|_2$ is induced by the inner product

$$(A,B)_2=\mathrm{Tr}(AB^*).$$

It is a positive definite hermitian inner product. Obviously $\|A^*\|_2=\|A\|_2$.

Lemma 1. $\|A\|\le\|A\|_2$ *if* $A\in\mathcal{S}_2(\mathcal{H})$.
(Here $\|\cdot\|$ means the usual operator norm.)

Proof. If $x=\sum_{j\in\mathcal{J}}x^je_j\in\mathcal{H}$ then due to the Cauchy-Schwartz inequality

$$\begin{aligned}\|Ax\|^2=\|\sum_{j\in\mathcal{J}}x^jAe_j\|^2&\le\left(\sum_{j\in\mathcal{J}}|x^j|\|Ae_j\|\right)^2\\&\le\left(\sum_{j\in\mathcal{J}}|x^j|^2\right)\left(\sum_{j\in\mathcal{J}}\|Ae_j\|^2\right)=\|A\|_2^2\|x\|^2\end{aligned}$$

which implies the desired inequality. □

It follows that if A is a densely defined operator with a domain that includes all vectors $e_j,\ j\in\mathcal{J}$, and $\sum_{j\in\mathcal{J}}\|Ae_j\|^2<\infty$, then A can be extended by continuity to a Hilbert-Schmidt operator in $\mathcal{H}$. It follows also that $\mathcal{S}_2(\mathcal{H})$ is a Hilbert space which is unitarily isomorphic to the space of all function $a:\mathcal{J}\times\mathcal{J}\to\mathbb{C}$ such that $\sum_{j,k\in\mathcal{J}}|a(j,k)|^2<\infty$, the isomorphism is given by $A\mapsto a,\ a(j,k)=(Ae_j,e_k)$. This property can be essentially generalized so as to make an explicit description of Hilbert-Schmidt operators in any Hilbert space of the form $\mathcal{H}=L^2(M,\Sigma,\mu)$.

Theorem. *If* $\mathcal{H}=L^2(M,\Sigma,\mu)$ *and* $A\in\mathcal{S}_2(\mathcal{H})$ *then there exists a function* $K_A\in L^2(M\times M,\Sigma\times\Sigma,\mu\times\mu)$ *such that* A *is expressed via* K_A *as follows:*

$$(Au)(x)=\int_M K_A(x,y)u(y)d\mu(y),\qquad u\in L^2(M,\Sigma,\mu).$$

Conversely, if $K_A \in L^2(M \times M, \Sigma \times \Sigma, \mu \times \mu)$ *is given, then the formula above defines a Hilbert-Schmidt operator* A *in* $\mathcal{H}$. *The function* K_A *is uniquely defined by* A *(up to its values on a set of* $\mu \times \mu$*-measure zero) and*

$$\|A\|_2^2 = \iint\limits_{M \times M} |K_A(x,y)|^2 \, d\mu(x) d\mu(y).$$

REMARK 1. The function K_A is called *kernel*, or *Schwartz kernel*, or *L. Schwartz kernel*, of the operator A. The last two names are useful if we need to distinguish between $\operatorname{Ker} A$ (the null space of A which is also called kernel of A) and the Schwartz kernel which is the function K_A or, more generally, a distribution which plays a similar role.

Let X be a C^∞ manifold with a smooth positive measure μ (i.e. a measure given by a smooth positive density). Let us recall that a *distribution* u on X is a linear continuous functional on $C_c^\infty(X)$, which is considered as a space of test functions. The value of a distribution u on a test function $\phi \in C_c^\infty(X)$ is usually denoted $\langle u, \phi \rangle$.

The continuity can be understood by use of the natural convergence of sequences $\{\phi_k | k = 1, 2, \dots\}$ in $C_c^\infty(X)$. Namely, one says that $\phi_k \to \phi$ in $C_c^\infty(X)$ as $k \to \infty$ if there exists a compact $K \subset X$ such that $\operatorname{supp} \phi_k \subset K$ for all k, and $P\phi_k \to P\phi$ as $k \to \infty$ uniformly on X for every fixed differential operator P with smooth coefficients on X. So a linear functional $u : C_c^\infty(X) \to \mathbb{C}$ is a distribution if $\phi_k \to \phi$ in $C_c^\infty(X)$ implies $\langle u, \phi_k \rangle \to \langle u, \phi \rangle$. The set of all distributions on X will be denoted $\mathcal{D}'(X)$.

We will identify a locally integrable function u on X with a distribution defined by

$$\langle u, \phi \rangle = \int_X u(x)\phi(x) d\mu(x).$$

(This is where the measure μ on X is needed.)

We will use the weak topology on $\mathcal{D}'(X)$. This topology is generated by the seminorms $p_\phi(u) = |\langle u, \phi \rangle|$, $\phi \in C_c^\infty(X)$.

Now let Y be another C^∞ manifold with a smooth positive measure ν. We will use the measure $\mu \times \nu$ on $X \times Y$. We will say that a distribution $K_A \in \mathcal{D}'(X \times Y)$ is the *Schwartz kernel* of an operator $A : C_c^\infty(Y) \to \mathcal{D}'(X)$ if

$$\langle A\psi, \phi \rangle = \langle K_A, \phi \otimes \psi \rangle, \quad \phi \in C_c^\infty(X), \psi \in C_c^\infty(Y),$$

where $(\phi \otimes \psi)(x,y) = \phi(x)\psi(y)$. Then K_A is uniquely defined by A because the linear span of the set of the functions of the form $\phi \otimes \psi$ is dense in $C_c^\infty(X \times Y)$. Vice versa, it is obvious that A is uniquely defined by K_A.

The Laurent Schwartz Kernel Theorem states that in fact there is a one-one correspondence between all linear continuous operators $A : C_c^\infty(Y) \to \mathcal{D}'(X)$ and the distributions on $X \times Y$, and this correspondence $A \leftrightarrow K_A$ is given by the formula above. In particular, every continuous linear operator $A : C_c^\infty(Y) \to \mathcal{D}'(X)$ has a Schwartz kernel which is a distribution. This holds for example for any bounded linear operator $A : L^2(Y) \to L^2(X)$.

In fact measures on X and Y are not necessary. Instead we can define distributions as linear continuous functionals on the set of all smooth densities with compact support.

Also the L. Schwarz kernel theorem is easily generalized to operators in sections of vector bundles.

For the proof of the L.Schwartz kernel theorem and further comments see e.g. [**GV**], Ch. 1, [**Ho**], Vol. I, Sect. 5.2, [**Tr**], Theorem 51.7, and [**Sch1, Sch2**].

Proof of Theorem. Choose an orthonormal basis $\{e_j \mid j \in \mathcal{J}\}$ in $L^2(M, \Sigma, \mu)$, so $e_j = e_j(x),\ x \in M$. Then for any $u \in L^2(M, \Sigma, \mu),\ A \in \mathcal{S}_2$ we obtain

$$Au = A\left(\sum_{j\in\mathcal{J}}(u, e_j)e_j\right) = \sum_{j\in\mathcal{J}}(u, e_j)Ae_j = \sum_{j,k\in\mathcal{J}}(u, e_j)(Ae_j, e_k)e_k$$

Now define

$$K_A(x, y) = \sum_{j,k\in\mathcal{J}}(Ae_j, e_k)e_k(x)\overline{e_j(y)}.$$

Since functions $\{e_k \otimes \bar{e}_j \mid k, j \in \mathcal{J}\}$ (here $(e_k \otimes \bar{e}_j)(x, y) = e_k(x)\overline{e_j(y)}$) form an orthonormal basis in $L^2(M \times M, \Sigma \times \Sigma, \mu \times \mu)$ and $\sum_{j,k\in\mathcal{J}}|Ae_j, e_k)|^2 < \infty$, we obtain $K_A \in L^2(M\times M, \Sigma\times\Sigma, \mu\times\mu)$ and A is really defined by K_A as its Schwartz kernel. Moreover

$$\|K_A\|^2_{L^2(M\times M,\Sigma\times\Sigma,\mu\times\mu)} = \sum_{j,k\in\mathcal{J}}|(Ae_j, e_k)|^2 = \|A\|_2^2.$$

Conversely if $K \in L^2(M \times M, \Sigma \times \Sigma, \mu \times \mu)$ then the formula

$$Au(x) = \int_M K(x, y)u(y)\, d\mu(y)$$

defines a Hilbert-Schmidt operator with $K_A = K$. Indeed, if we expand the function K in $L^2(M \times M, \Sigma \times \Sigma, \mu \times \mu)$

$$K(x, y) = \sum_{j,k\in\mathcal{J}} c_{jk}e_k(x)\overline{e_j(y)},$$

then $\sum_{j,k\in\mathcal{J}}|c_{jk}|^2 < \infty$ and $Ae_j = \sum_{k\in\mathcal{J}} c_{jk}e_k$. This implies that $\|A\|_2^2 = \sum_{j,k\in\mathcal{J}}|c_{jk}|^2 < \infty$. It remains to note that the maps $A \mapsto K_A$ and $K \mapsto A$ defined by the formulas given above are mutually inverse as maps between $\mathcal{S}_2(L^2(M, \Sigma, \mu))$ and $L^2(M \times M, \Sigma \times \Sigma, \mu \times \mu)$. $\square$

Later on we will also need

Lemma 2. *Assume that* $T \in \mathcal{S}_2(\mathcal{H})$ *and that* $A \in \mathcal{B}(\mathcal{H})$. *Then*

$$\|AT\|_2 \le \|A\|\|T\|_2 \quad \textit{and} \quad \|TA\|_2 \le \|A\|\|T\|_2$$

Proof. Let us choose an orthonormal basis $\{e_j|j \in \mathcal{J}\}$ in $\mathcal{H}$. Then

$$\|AT\|_2 = \sum_{j\in\mathcal{J}}\|ATe_j\|^2 \le \|A\|^2\sum_{j\in\mathcal{J}}\|Te_j\|^2 = \|A\|^2\|T\|_2^2$$

which proves the first inequality. Now we get

$$\|TA\|_2 = \|A^*T^*\|_2 \le \|A^*\|\|T^*\|_2 = \|A\|\|T\|,$$

as required. $\square$

Exercise 1. Prove that any Hilbert-Schmidt operator in a Hilbert space $\mathcal{H}$ can be approximated (in norm topology) by operators of finite rank (i.e. with a finite-dimensional range), hence it is a compact operator (i.e. maps the unit ball in $\mathcal{H}$ to a precompact subset in $\mathcal{H}$).

EXERCISE 2. Prove that if A is a self-adjoint Hilbert-Schmidt operator in $\mathcal{H}$, then there exists an orthonormal basis $\{e_j \mid j \in \mathcal{J}\}$ in $\mathcal{H}$ such that $Ae_j = \lambda_j e_j$, $\lambda_j \in \mathbb{R}$, for every $j \in \mathcal{J}$. Besides $\|A\|_2^2 = \sum_{j\in\mathcal{J}} \lambda_j^2$.

Hint: Use the Spectral Theorem.

REMARK 2. By polarization, we obtain from the Theorem that if $A, B \in \mathcal{S}_2(\mathcal{H})$ K_A, K_B are the corresponding Schwartz kernels then

$$(A, B)_2 = \iint\limits_{M\times M} K_A(x,y)\overline{K_B(x,y)},\, d\mu(x)d\mu(y)$$

2.21. Trace in terms of the Schwartz kernel

Let $\mathcal{H} = L^2(M, \Sigma, \mu)$, $A \in \mathcal{S}_1(\mathcal{H})$. Since $\mathcal{S}_1(\mathcal{H}) \subset \mathcal{S}_2(\mathcal{H})$, the operator A has a Schwartz kernel $K_A \in L^2(M \times M, \Sigma \times \Sigma, \mu \times \mu)$. The question arises: how to express $\operatorname{Tr} A$ in terms of K_A? We shall first answer it heuristically and then give rigorous answers in some important particular cases.

We can represent A as $A = BC$ with $B, C \in \mathcal{S}_2(\mathcal{H})$ (e.g. if $A = US$ is the polar decomposition of A then we can take $B = U\sqrt{S}$, $C = \sqrt{S}$). Then K_A can be given by the formula

$$K_A(x,y) = \int_M K_B(x,z)K_C(z,y)\, d\mu(z)$$

(this easily follows from the Fubini theorem). Certainly K_A is almost everywhere defined by the operator A but now some information about $K_A(x,x)$ will be necessary, and the diagonal $\Delta = \{(x,x) \mid x \in M\}$ has measure 0 in $(M\times M, \Sigma\times\Sigma, \mu\times\mu)$. Hence the formula for K_A should carry non-trivial information beyond just saying that K_A is the Schwartz kernel of A. Now assuming that this formula is true, we obtain

$$\begin{aligned}\operatorname{Tr} A = \operatorname{Tr}(BC) = (B, C^*)_2 &= \iint\limits_{M\times M} K_B(x,z)\overline{K_{C^*}(x,z)}\, d\mu(x)d\mu(z)\\ &= \iint\limits_{M\times M} K_B(x,z)K_C(z,x)\, d\mu(x)d\mu(z),\end{aligned}$$

hence

$$\operatorname{Tr} A = \int_M K_A(x,x)\, d\mu(x).$$

Now we can try to make sense of this formula without any special choice of K_A. To do this suppose that we have a family of mollifying functions

$$\psi_\epsilon \in L^2(M \times M, \Sigma \times \Sigma, \mu \times \mu),\ 0 < \epsilon < 1,$$

satisfying the following condition: for any $f \in L^2(M \times M, \Sigma \times \Sigma, \mu \times \mu)$

$$\int_M f(z,y)\psi_\epsilon(x,y)\, d\mu(y) \to f(z,x) \text{ in } L^2(M \times M, \Sigma \times \Sigma, \mu \times \mu) \text{ as } \epsilon \to 0.$$

such a family always exists in analytic situations that we shall need. But then we have

THEOREM 1. *If $A \in \mathcal{S}_1(\mathcal{H})$ where $\mathcal{H} = L^2(M, \Sigma, \mu)$ and ψ_ϵ satisfies the condition given above, then*

$$\operatorname{Tr} A = \lim_{\epsilon \to 0} \iint\limits_{M \times M} K_A(x, y)\psi_\epsilon(x, y)\, d\mu(x) d\mu(y).$$

PROOF. Representing $A = BC$ as before (with $B, C \in \mathcal{S}_2(\mathcal{H})$) we get

$$\begin{aligned} &\iint_{M \times M} K_A(x, y)\psi_\epsilon(x, y)\, d\mu(x) d\mu(y) \\ &= \iiint_{M \times M \times M} K_B(x, z) K_C(z, y)\psi_\epsilon(x, y)\, d\mu(z) d\mu(x) d\mu(y) \\ &= \iint_{M \times M} K_B(x, z) \left(\int_M K_C(z, y)\psi_\epsilon(x, y)\, d\mu(y)\right) d\mu(z) d\mu(x) \end{aligned}$$

Taking the limit as $\epsilon \to 0$, we get $\operatorname{Tr} A = \operatorname{Tr}(BC)$ as required. □

Roughly speaking, the condition on ψ_ϵ means that $\int_M \psi_\epsilon(x, y)\, d\mu(y) \to 1$ and the support of $\psi_\epsilon(x, \cdot)$ shrinks to the point x. We shall give now several examples of such situations.

EXAMPLE 1. Let M be a bounded open subset in $\mathbb{R}^n$, μ the Lebesgue measure on M. Choose $\varphi \in C_c^\infty(\mathbb{R}^n)$ (i.e. φ is a C^∞-function with a compact support in $\mathbb{R}^n$) such that $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. Then define $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(\epsilon^{-1}x)$, so again $\varphi_\epsilon \geq 0$ and $\int_{\mathbb{R}^n} \varphi_\epsilon(x)\, dx = 1$. Now we can put $\psi_\epsilon(x, y) = \varphi_\epsilon(x - y)$. It is easy to prove that ψ_ϵ satisfies the required condition. Indeed, define for every $f \in L^2(M \times M)$

$$f_\epsilon(x, y) = \int_M f(x, z)\varphi_\epsilon(y - z) dz, \quad x, y \in \mathbb{R}^n;$$

where we consider $f(x, y)$ extended by 0 if x or y is outside M. We obviously also have

$$f_\epsilon(x, y) = \int_M f(x, y - z)\varphi_\epsilon(z) dz, \quad x, y \in \mathbb{R}^n.$$

It follows that

$$\begin{aligned} \|f_\epsilon(x, \cdot)\|_{L^2(\mathbb{R}^n)} &\leq \int_{\mathbb{R}^n} \|f(x, \cdot - z)\|_{L^2(\mathbb{R}^n)}\varphi_\epsilon(z)\, dz = \int_{\mathbb{R}^n} \|f(x, \cdot)\|_{L^2(M)}\varphi_\epsilon(z)\, dz \\ &= \|f(x, \cdot)\|_{L^2(M)} \int_{\mathbb{R}^n} \varphi_\epsilon(z)\, dz = \|f(x, \cdot)\|_{L^2(M)}, \end{aligned}$$

therefore

$$\|f_\epsilon\|_{L^2(M \times M)} \leq \|f\|_{L^2(M \times M)}.$$

Now obviously $f_\epsilon \underset{\epsilon \to 0}{\to} f$ uniformly (hence in $L^2(M \times M)$) if $f \in C_c^\infty(M \times M)$. But $C_c^\infty(M \times M)$ is dense in $L^2(M \times M)$. It easily follows that $f_\epsilon \to f$ as $\epsilon \to 0$ in $L^2(M \times M)$ for every $f \in L^2(M \times M)$. Namely, if $\delta > 0$, $g \in C_c^\infty(M \times M)$ is chosen so that $\|f - g\|_{L^2(M \times M)} < \delta/3$ and $\epsilon_0 > 0$ is chosen so that $\|g_\epsilon - g\|_{L^2(M \times M)} < \delta/3$ provided $0 < \epsilon < \epsilon_0$, then $\|f_\epsilon - f\| \leq \|f_\epsilon - g_\epsilon\| + \|g_\epsilon - g\| + \|g - f\| < \delta$ for any $\epsilon \in (0, \epsilon_0)$ (all norms in $L^2(M \times M)$) since $\|f_\epsilon - g_\epsilon\| = \|(f - g)_\epsilon\| \leq \|f - g\| < \delta/3$.

EXAMPLE 2. Let $M = \mathbb{R}^n$, μ the Lebesgue measure on M. Choose φ_ϵ as before and let $\chi_\epsilon \in C_c^\infty(\mathbb{R}^n)$, $0 \le \chi_\epsilon(x) \le 1$, $\chi_\epsilon(x) = 1$ if $|x| \le 1/\epsilon$. Then we can take $\psi_\epsilon(x,y) = \chi_\epsilon(x)\varphi_\epsilon(x-y)$ and the same arguments apply.

EXAMPLE 3. Let M be a compact manifold, μ is the measure given by a smooth positive density on M. Let us introduce a Riemannian metric and the corresponding distance $d = d(x,y)$, $x, y \in M$. We can consider then $\tilde{\psi}_\epsilon(x,y) = \max\{\epsilon - d(x,y), 0\}$ and define

$$\psi_\epsilon(x,y) = \left(\int_M \tilde{\psi}_\epsilon(x,y)\, d\mu(y)\right)^{-1} \tilde{\psi}_\epsilon(x,y).$$

EXAMPLE 4. Let M be a (non-compact) manifold, μ is a measure given by a smooth positive density on M. We may find a family of cut-off functions $\chi_t \in C_c^\infty(M)$ such that $0 \le \chi_t \le 1$, $\chi_t(x) = 1$ in a neighborhood of x if $t \ge t_0(x)$. Then we may take $\psi_\epsilon(x,y)$ constructed as in the previous example and then multiply it by $\chi_{\epsilon^{-1}}(x)$, thus coming to a new function satisfying all required conditions.

Now we may give some conditions which are sufficient for the trace formula to be true.

THEOREM 2. *Let M be a compact manifold, μ is a measure given by a smooth positive density on M, $A \in \mathcal{S}_1(L^2(M,\mu))$ and A has a Schwartz kernel K_A which is continuous on $M \times M$. Then*

$$\operatorname{Tr} A = \int_M K_A(x,x)\, d\mu(x).$$

PROOF. Using ψ_ϵ constructed in Example 3 we obtain due to the uniform continuity of K_A on $M \times M$ that uniformly with respect to $x \in M$

$$\lim_{\epsilon \to 0} \int_M K_A(x,y)\psi_\epsilon(x,y)\, d\mu(y) = K_A(x,x),$$

hence the required formula immediately follows. □

REMARK 1. The continuity of K_A can be replaced by a weaker requirement, e.g. continuity outside submanifolds which are transverse to the diagonal in $M \times M$, provided K_A is bounded on $M \times M$.

REMARK 2. Suppose that A acts not in $L^2(M,\mu)$ but in a similar space of vector-functions (with values in a finite-dimensional vector space) or, more generally, in the space of square-integrable sections of a complex vector bundle over M with an hermitian inner product on fibers. Then $K_A(x,y)$ becomes a linear operator (acting from the fiber over y to the fiber over x) and the trace formula generalizes as follows:

$$\operatorname{Tr} A = \int_M \operatorname{tr} K_A(x,x)\, d\mu(x)$$

where tr means the matrix trace (or the trace of the corresponding fiber endomorphism).

EXERCISE. Let $A \in \mathcal{S}_1(L^2(\mathbb{R}^n))$. Then

(i) $K_A(x, x+t) \in L^1(\mathbb{R}^n_x)$ for almost every $t \in \mathbb{R}^n$.

(ii) $K_A(x, x+t)$ regarded as a function of t with values in $L^1(\mathbb{R}^n_x)$, coincides almost everywhere with a continuous function of t with values in $L^1(\mathbb{R}^n_x)$;

(iii) denote by $K_A(x,x)$ the value of the above continuous function at $t=0$, then $\operatorname{Tr} A = \int_{\mathbb{R}^n} K_A(x,x)\,dx$.

Hint: Prove the estimate $\sup_{t\in\mathbb{R}^n}\int_{\mathbb{R}^n} |K_A(x,x+t)|\,dx \le \|B\|_2\|C\|_2$ if $A=BC$ and then pass to the limit from B,C such that $K_B,K_C \in C_c^\infty(\mathbb{R}^{2n})$.

REMARK 3. Let $A \in \mathcal{B}(L^2(M,\mu))$, where M is a compact C^∞-manifold, μ is given by a smooth positive density. Suppose that A is given by a continuous Schwartz kernel $K_A \in C(M\times M)$ and $A \ge 0$. Then $K_A(x,x)\ge 0$ for all $x\in M$ and

$$\operatorname{Tr} A = \int_M K_A(x,x)\,d\mu(x).$$

(In particular A is in the trace class.) This follows from the Mercer Theorem (see e.g. [**RiNa**], Ch.VI, Sect.98) which guarantees that K_A has an absolutely and uniformly convergent decomposition

$$K_A(x,y) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x)\overline{\phi_j(y)},$$

where λ_j runs over all non-zero eigenvalues of A, ϕ_j are corresponding eigenfunctions which are normalized in $L^2(M,d\mu)$.

2.22. Analytic expression for the Γ-trace

Let M be a Γ-manifold such that $X = M/\Gamma$ is compact, μ is a Γ-invariant measure on M given by a positive smooth density. Then we have a von Neumann algebra $\mathcal{M}_\Gamma$ in $L^2(M,\mu)$ and a trace Tr_Γ on $\mathcal{M}_\Gamma$. We want to describe the trace in terms of the Schwartz kernel.

Let us represent $L^2(M)\cong L^2\Gamma\otimes L^2(F)$ where we use the notation $L^2(M) = L^2(M,\mu)$ for brevity, and F is a fundamental domain of the action of Γ on M (and $L^2(F) = L^2(F,\mu)$). As we discussed above, there is a natural decomposition into an orthogonal direct sum

$$L^2(M) \cong \bigoplus_{g\in\Gamma} L^2(F)$$

and any operator $A\in\mathcal{B}(L^2(M))$ is represented there by a block matrix $[A_{g,h}]_{g,h\in\Gamma}$ with the entries $A_{g,h}\in\mathcal{B}(L^2(F))$. Now if $A\in\mathcal{M}_\Gamma^+$ then $\operatorname{Tr}_\Gamma A = \operatorname{Tr} A_{e,e}$. It follows that $A\in\mathcal{S}_1(\mathcal{M}_\Gamma,\operatorname{Tr}_\Gamma)$ implies $A_{e,e}\in\mathcal{S}_1(L^2(F))$ and then again $\operatorname{Tr}_\Gamma A = \operatorname{Tr} A_{e,e}$.

Let us describe the Schwartz kernel of $A_{e,e}$ in terms of the Schwartz kernel of A. We have

$$(A_{e,e}f,g) = (A(\delta_e\otimes f),\delta_e\otimes g),\quad f,g\in L^2(F)$$

But the element $\delta_e\otimes f \in L^2\Gamma\otimes L^2(F)$ corresponds in $L^2(M)$ to the function $f^0 = \operatorname{ext}^0(f)$ which is defined as follows: $f^0(x) = f(x)$ if $x\in F$, $f^0(x)=0$ if $x\notin F$, that is, f^0 is the extension of f by 0 on M. Since $A_{e,e}\in\mathcal{S}_1(L^2(F))\subset\mathcal{S}_2(L^2(F))$, we see that $A_{e,e}$ is a Hilbert-Schmidt operator in $L^2(F)$. Hence there exists $K\in L^2(F\times F,\mu\times\mu)$ such that K is the Schwartz kernel of $A_{e,e}$. But then obviously if K_A is the Schwartz kernel of A, we should have $K_A|_{F\times F} = K$. Hence we should have

$$\operatorname{Tr}_\Gamma A = \int_F K_A(x,x)\,d\mu(x)$$

in one of the senses described earlier for the usual trace Tr.

Note also that the previous reasoning implies that $K_A \in L^2_{\rm loc}(M)$ (the space of functions on $M \times M$ which are square-integrable on any compact subset in $M \times M$) for any $A \in \mathcal{S}_1(\mathcal{M}_\Gamma, \mathrm{Tr}_\Gamma)$. Indeed, if $x_0, y_0 \in M$ are arbitrary points and U, V are their sufficiently small neighborhoods then $U \cup V$ can be included in a fundamental domain, hence $K_A \in L^2$ near (x_0, y_0).

LEMMA. *Let $A \in \mathcal{B}(L^2(M))$, K_A be the Schwartz kernel of A. Then the inclusion $A \in \mathcal{M}_\Gamma$ is equivalent to relations*

$$K_A(gx, gy) = K_A(x, y) \quad \textit{for all } g \in \Gamma \textit{ identically with respect to } (x, y).$$

PROOF. For simplicity let us consider the case when $K_A \in L^2_{\rm loc}(M \times M)$. If $u \in L^2(M)$ has a compact support then

$$\begin{aligned}(L_g^{-1} A L_g u)(x) &= L_g^{-1} \int_M K_A(x, y) u(g^{-1}y)\, d\mu(y) = L_g^{-1} \int_M K_A(x, gy) u(y)\, d\mu(y) \\ &= \int_M K_A(gx, gy) u(y)\, d\mu(y).\end{aligned}$$

Hence $(x, y) \mapsto K_A(gx, gy)$ is the Schwartz kernel of $L_{g^{-1}} A L_g$ and it remains to notice that the coincidence of operators is equivalent to the coincidence of their Schwartz kernels. □

This Lemma shows in particular that the formula for the trace Tr gives the result that is independent on the choice of the fundamental domain F. Moreover, this formula can be rewritten in the form

$$\mathrm{Tr}_\Gamma A = \int_X K_A(x, x)\, d\mu(x), \quad X = M/\Gamma.$$

Adjustments similar to the case of the usual trace are easily made. In particular, this formula can be understood literally in the case when $A \in \mathcal{S}_1(\mathcal{M}_\Gamma, \mathrm{Tr}_\Gamma)$ and $K_A \in C(M \times M)$. This case often appears in analytic applications.

REMARK 1. In case of vector-functions or sections of a Γ-bundle E the trace formula should be rewritten as

$$\mathrm{Tr}_\Gamma A = \int_X \mathrm{tr}\, K_A(x, x)\, d\mu(x), \quad X = M/\Gamma,$$

where tr means the matrix trace. The condition $A \in \mathcal{M}_\Gamma(E)$ in terms of the Schwartz kernel looks in this case as follows:

$$K_A(gx, gy) = \hat{g}_x K_A(x, y) \hat{g}_y^{-1}, \quad x, y \in M,$$

where $\hat{g}_x : E_x \to E_{gx}$ is the linear map on fibers that is defined by the action of g in E. It follows that

$$\mathrm{tr}\, K_A(gx, gx) = \mathrm{tr}\, K_A(x, x), \quad x \in M,$$

hence the integral in the trace formula is well defined.

REMARK 2. If we define an action of Γ on $M \times M$ by

$$g(x, y) = (gx, gy),$$

then for any $A \in \mathcal{M}_\Gamma$ the Schwartz kernel K_A will be a function (or a distribution) on the orbit space $\frac{M \times M}{\Gamma}$. The same is true for $A \in \mathcal{M}_\Gamma(E)$ (in sections of a vector Γ-bundle E) if we identify the fibers E_x and E_{gx} by means of the isomorphism $\hat{g}_x : E_x \to E_{gx}$.

2.23. Γ-Hilbert-Schmidt and Γ-trace class operators

DEFINITION. 1) Operators $A \in \mathcal{S}_2(\mathcal{M}_\Gamma, \mathrm{Tr}_\Gamma)$ are called Γ-*Hilbert-Schmidt* operators.

2) Operators $A \in \mathcal{S}_1(\mathcal{M}_\Gamma, \mathrm{Tr}_\Gamma)$ are called Γ-*trace class* operators.

Now we shall give more explicit conditions for the operators $A \in \mathcal{M}_\Gamma$ to be Γ-Hilbert-Schmidt or Γ-trace class in terms of their kernels. As before we shall begin with the Γ-Hilbert-Schmidt operators.

THEOREM 1. *Let $X = M/\Gamma$ be compact and a fundamental domain $F \subset M$ is chosen so that its closure $\bar{F}$ is compact. Let a Γ-invariant measure μ with a positive smooth density is fixed on M. Then for an operator $A \in \mathcal{M}_\Gamma$ the following conditions are equivalent:*

(i) *A is a Γ-Hilbert-Schmidt operator;*
(ii) *A has a Schwartz kernel $K_A \in L^2\left(\frac{M\times M}{\Gamma}\right)$, where L^2 is taken with respect to the measure induced by $\mu \times \mu$ on $M \times M$;*
(iii) *ϕA is a Hilbert-Schmidt operator in $L^2(M)$ for every bounded measurable function $\phi : M \to \mathbb{C}$ with a compact support.*
(iv) *there exists $\phi \in C_c^\infty(M)$ such that $\phi > 0$ on $\bar{F}$ and ϕA is a Hilbert-Schmidt operator in $L^2(M)$;*
(v) *$A\phi$ is a Hilbert-Schmidt operator in $L^2(M)$ for every bounded measurable function $\phi : M \to \mathbb{C}$ with a compact support;*
(vi) *there exists $\phi \in C_c^\infty(M)$ such that $\phi > 0$ on $\bar{F}$ and $A\phi$ is a Hilbert-Schmidt operator in $L^2(M)$.*

PROOF. Obviously a fundamental domain of the action of Γ in $M \times M$ can be taken in the form $F \times M$ or $M \times F$. This clearly implies that the conditions (ii)–(vi) are equivalent.

Now let A be a Γ-Hilbert-Schmidt operator. Let $\chi_F : M \to \mathbb{R}$ be the characteristic function of the set F. Then for any $B \in \mathcal{M}_\Gamma$ the operator $B_{e,e} \in \mathcal{B}(L^2(F))$ can be interpreted as $\chi_F B \chi_F$. Hence $\chi_F A^* A \chi_F$ should be a trace class operator which is equivalent to saying that $A\chi_F$ is a Hilbert-Schmidt operator in $L^2(M)$. This is equivalent to (v) because we can change the fundamental domain and use the fact that the Hilbert-Schmidt operators form an ideal in $\mathcal{B}(L^2(M))$. This proves the Theorem. □

COROLLARY. *Let A be in the Γ-trace class, $\phi, \psi : M \to \mathbb{C}$ are bounded measurable functions with a compact support. Then $\phi A \psi$ is in the trace class in $L^2(M)$.*

PROOF. Choosing a factorization $A = B_1 B_2$ where B_1, B_2 are Γ-Hilbert-Schmidt operators we obtain $\phi A \psi = (\phi B_1)(B_2 \psi)$ and $\phi B_1, B_2\psi$ will be Hilbert-Schmidt operators due to the Theorem, hence $\phi A \psi$ will be of trace class. □

The converse is true for positive operators:

THEOREM 2. *Let $A \in \mathcal{M}_\Gamma^+$. Then the following conditions are equivalent:*

(i) *A is of Γ-trace class;*
(ii) *$\phi A \psi$ is of trace class for all $\phi, \psi \in C_c^\infty(M)$;*
(iii) *$A^{1/2}$ is Γ-Hilbert-Schmidt.*

PROOF. (i) is equivalent to (iii) due to the general theory developed earlier. Implication (i)⇒(ii) was proved in the corollary above. Now if (ii) is true then taking

$\phi = \bar{\psi}$ we obtain $\phi A \psi = (A^{1/2}\psi)^*(A^{1/2}\psi)$, hence $A^{1/2}\psi$ is Hilbert-Schmidt for every $\psi \in C_c^\infty(M)$ and $A^{1/2}$ is Γ-Hilbert-Schmidt, which proves the Theorem. □

Now we shall give another expression for the Γ-trace.

LEMMA. (i) *There exists $\phi \in C_c^\infty(M)$ such that*

$$\sum_{g\in\Gamma} L_g\phi = 1.$$

(ii) *There exist $\phi, \psi \in C_c^\infty(M)$ such that*

$$\sum_{g\in\Gamma} L_g(\phi\psi) = 1.$$

PROOF. (i) First take $\phi_1 \in C_c^\infty(M)$ so that $\phi_1 > 0$ on a closure of a fundamental domain and define $\tilde{\phi}_1 = \sum_{g\in\Gamma} L_g\phi_1$. Then $\tilde{\phi}_1 \in C^\infty(M)$, $\tilde{\phi}_1 > 0$ on M and $L_g\tilde{\phi}_1 = \tilde{\phi}_1$ for all $g \in \Gamma$. Now $\phi = (\tilde{\phi}_1)^{-1}\phi_1$ satisfies the condition in (i).

(ii) If we take ϕ as in the part (i), then we can take $\psi \in C_c^\infty(M)$ such that $\psi = 1$ on $\operatorname{supp}\phi$ and the condition in (ii) will be obviously fulfilled. □

THEOREM 3. *Let A be a Γ-trace class operator, and let ϕ, ψ be bounded measurable functions with compact support on M such that*

$$\sum_{g\in\Gamma} L_g(\phi\psi) = 1.$$

Then

$$\operatorname{Tr}_\Gamma A = \operatorname{Tr}(\phi A \psi).$$

PROOF. The formula is obviously true if $\phi = \psi = \chi_F$ is the characteristic function of a fundamental domain with a compact closure. So the Theorem will follow if we prove that $\operatorname{Tr}(\phi A\psi)$ does not depend on the choice of functions ϕ, ψ satisfying the condition in Theorem. Let ϕ', ψ' be another couple of functions satisfying the condition. There exists a finite subset $S \subset \Gamma$ such that

$$\{\operatorname{supp} L_g\phi' \cap \operatorname{supp}\phi \neq \emptyset\} \Rightarrow \{g \text{ and } g^{-1} \text{ are in } S\}.$$

Hence

$$\sum_{g\in S} L_g(\phi'\psi')\phi = \sum_{g\in\Gamma} L_g(\phi'\psi')\phi = \phi.$$

Therefore

$$\begin{aligned}
\operatorname{Tr}(\phi A\psi) &= \operatorname{Tr}\sum_{g\in S} L_g(\phi'\psi')\phi A\psi = \sum_{g\in S}\operatorname{Tr}[L_g(\phi'\psi')\phi A\psi] = \\
&= \sum_{g\in S}\operatorname{Tr}[L_g(\phi')\phi A\psi L_g(\psi')] = \sum_{g\in S}\operatorname{Tr}\left[(L_g\phi' L_g^{-1})\phi A\psi(L_g\psi' L_g^{-1}\right] = \\
&= \sum_{g\in S}\operatorname{Tr}\left[(L_g\phi' L_g^{-1})\phi(L_g A L_g^{-1})\psi(L_g\psi' L_g^{-1})\right] \\
&= \sum_{g\in S}\operatorname{Tr}\left[L_g\phi'(L_g^{-1}\phi L_g)A(L_g^{-1}\psi L_g)\psi' L_g^{-1}\right] = \\
&= \sum_{g\in S}\operatorname{Tr}\left[\phi' L_g^{-1}(\phi)AL_g^{-1}(\psi)\psi'\right] = \sum_{g\in S}\operatorname{Tr}\left[L_g^{-1}(\phi\psi)\phi' A\psi'\right] = \\
&= \operatorname{Tr}\sum_{g\in S} L_g^{-1}(\phi\psi)\phi' A\psi' = \operatorname{Tr}\left[\left(\sum_{g\in S} L_g^{-1}(\phi\psi)\right)\phi' A\psi'\right] = \operatorname{Tr}(\phi' A\psi'),
\end{aligned}$$

where we used the relation $L_g A L_g^{-1} = A$, traciality of Tr and the coincidence of $L_g\phi L_g^{-1}$ with the multiplication operator by $L_g(\phi)$. $\square$

Later we shall also give sufficient conditions for an operator A in $L^2(M)$ to be in trace class or in Γ-trace class. Now we will just mention some of them without proof:

(a) If A has a Schwartz kernel $K_A \in C_c^\infty(M\times M)$ (i.e. C^∞ with compact support in $M\times M$) then A is in trace class.
(b) If A has a Schwartz kernel $K_A \in C_c^\infty\left(\frac{M\times M}{\Gamma}\right)$ (i.e. C^∞ with compact support in $\frac{M\times M}{\Gamma}$), then A is in Γ-trace class.

Note that the compactness of support for K_A in $\frac{M\times M}{\Gamma}$ actually means that $K_A(x,y)=0$ if (x,y) is far from the diagonal in $M\times M$, that is, when $\operatorname{dist}(x,y) > C = C(A)$, where dist is the distance induced by a Γ-invariant Riemannian metric on M.

2.24. Dominated convergence theorems for traces

THEOREM 1. *Let $\mathcal{H}$ be a Hilbert space, S a trace class operator in $\mathcal{H}$, $\{A_\kappa \mid \kappa\in K\}$ a net of bounded linear operators in $\mathcal{H}$ such that*

(i) *there exists $C>0$, such that $\|A_\kappa\| \le C$ for every $\kappa\in K$;*
(ii) *$w\text{-}\lim\limits_K A_\kappa = A$.*

Then $\lim\limits_K \operatorname{Tr}(SA_\kappa) = \operatorname{Tr}(SA)$.

PROOF. Let us choose an orthonormal basis $\{e_j \mid j\in\mathcal{J}\}$ in $\mathcal{H}$. Then

$$\operatorname{Tr}(SA_\kappa) = \sum_{j\in\mathcal{J}}(SA_\kappa e_j, e_j) = \sum_{j\in\mathcal{J}}(A_\kappa e_j, S^*e_j).$$

Every term in the last sum converges to $(Ae_j, S^*e_j) = (SAe_j, e_j)$, so to apply the usual Lebesgue dominated convergence theorem it is sufficient to check that for every $\epsilon>0$ there exists a finite subset $\mathcal{J}_\epsilon\subset\mathcal{J}$ such that $\sum_{j\in\mathcal{J}\setminus\mathcal{J}_\epsilon}|(A_\kappa e_j, S^*e_j)|<\epsilon$ for all $\kappa\in K$ (or, in other words, that the series $\sum_{j\in\mathcal{J}}|(A_\kappa e_j, S^*e_j)|$ converges uniformly with respect to κ). To do this represent $S = T_1T_2$, where T_1 and T_2 are Hilbert-Schmidt operators. Then

$$|(SA_\kappa e_j, e_j)| = |(T_2A_\kappa e_j, T_1^*e_j)| \leq \|T_2A_\kappa e_j\|\|T_1^*e_j\|$$

and

$$\sum_{j\in\mathcal{J}_1} |(SA_\kappa e_j, e_j)| \leq \left(\sum_{j\in\mathcal{J}_1} \|T_2A_\kappa e_j\|^2\right)^{1/2} \left(\sum_{j\in\mathcal{J}_1} \|T_1^*e_j\|^2\right)^{1/2}$$
$$\leq \|T_2A_\kappa\|_2 \left(\sum_{j\in\mathcal{J}_1} \|T_1^*e_j\|^2\right)^{1/2} \leq C\|T_2\|_2 \left(\sum_{j\in\mathcal{J}_1} \|T_1^*e_j\|^2\right)^{1/2},$$

where $\mathcal{J}_1 \subset \mathcal{J}$ is an arbitrary subset, $\|B\|_2 = (\mathrm{Tr}(B^*B))^{1/2}$ and we used the inequality $\|TA\|_2 \leq \|T\|_2\|A\|$ (see Lemma 2 in 2.20). Now the uniform convergence follows from the convergence of the series $\sum_{j\in\mathcal{J}} \|T_1^*e_j\|^2$. □

Corollary 1. *If S is a trace class operator in $\mathcal{H}$ and $\{A_k \mid k = 1, 2, \dots\}$ is a sequence of operators in $\mathcal{B}(\mathcal{H})$ such that $w\text{-}\lim_{k\to\infty} A_k = A$, then*

$$\lim_{k\to\infty} \mathrm{Tr}(SA_k) = \mathrm{Tr}(SA).$$

Proof. The inequality $\|A_k\| \leq C$ follows from the well known uniform boundedness or Banach–Steinhaus principle in functional analysis (see e.g. [**ReSi**], Theorems III.9 and VI.1). □

Theorem 2. *Let M be a Γ-manifold with a Γ-invariant measure μ having a positive C^∞ density, such that $X = M/\Gamma$ is compact. Let S be a Γ-trace class operator in $L^2(M, \mu)$ and $\{A_\kappa \mid \kappa \in K\}$ a net of operators in $\mathcal{M}_\Gamma$ such that*

(i) *there exists $C > 0$ such that $\|A_\kappa\| \leq C$ for every $\kappa \in K$;*
(ii) *$w\text{-}\lim_K A_\kappa = A$.*

Then $\lim_K \mathrm{Tr}_\Gamma(SA_\kappa) = \mathrm{Tr}_\Gamma(SA)$.

Proof. Let F be a fundamental domain with a compact closure $\bar{F}$, χ_F the characteristic function of F. Then

$$\mathrm{Tr}_\Gamma(SA_\kappa) = \mathrm{Tr}(\chi_F SA_\kappa \chi_F)$$

Now choose T_1, T_2 in the Γ-Hilbert-Schmidt class such that $S = T_1T_2$. Then

$$\chi_F SA_\kappa \chi_F = (\chi_F T_1)(T_2A_\kappa\chi_F),$$

and both $\chi_F T_1$, $T_2A_\kappa\chi_F$ are Hilbert-Schmidt operators since T_2A_κ is a Γ-Hilbert-Schmidt operator. Now we can repeat the arguments from the proof of the previous theorem because the Hilbert-Schmidt norms $\|T_2A_\kappa\chi_F\|$ are bounded by a constant which does not depend on κ. □

Corollary 2. *If S is in the Γ-trace class and $\{A_k \mid k = 1, 2, \dots\}$ is a sequence of operators in $\mathcal{M}_\Gamma$ such that $w\text{-}\lim_{k\to\infty} A_k = A$. Then*

$$\lim_{k\to\infty} \mathrm{Tr}_\Gamma(SA_k) = \mathrm{Tr}_\Gamma(SA).$$

2.25. Generalized polar decomposition

There are several versions of polar decomposition that we might need. We shall list them now and indicate the necessary proofs.

1°. *Operators from one Hilbert space to another one.*
Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $A : \mathcal{H}_1 \to \mathcal{H}_2$ a bounded linear operator. Its polar decomposition is its factorization of the form $A = US$, where S is a bounded positive self-adjoint operator in $\mathcal{H}_1$, U is a partial isometry from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $\operatorname{Ker} U = \operatorname{Ker} S$ ($U : \mathcal{H}_1 \to \mathcal{H}_2$ is called *partial isometry* if U is an isometry from $(\operatorname{Ker} U)^\perp$ to $\operatorname{Im} U$). The proof given for the case $\mathcal{H}_1 = \mathcal{H}_2$ automatically extends to this case, thus showing that this polar decomposition exists and is unique. Also $S = \sqrt{A^*A}$ is denoted $|A|$.
Applying this polar decomposition (sometimes called *left polar decomposition*) to A^* we see that another decomposition (called *right polar decomposition*) also exists and is unique: $A = TV$ where $V : \mathcal{H}_1 \to \mathcal{H}_2$ is a partial isometry with $\operatorname{Ker} V^* = \operatorname{Ker} T$ and T is a self-adjoint positive bounded operator in $\mathcal{H}_2$.

2°. *Operators in* Γ*-modules.*
Let V_1, V_2 be Hilbert Γ-modules, $A : V_1 \to V_2$ a morphism of Γ-modules (i.e. a bounded linear operator that commutes with the action of Γ on V_1 and V_2). If we take left or right polar decompositions $A = US = TV$, then U, S, V, T will be morphisms of Γ-modules as well due to the unitary invariance and uniqueness of the polar decomposition (action of Γ in V_1 and V_2 is unitary by the definition of Hilbert module).

2.26. Almost isomorphisms

DEFINITION. 1) Let V_1, V_2 be Hilbert spaces. A bounded linear operator $A : V_1 \to V_2$ is called *almost isomorphism* (of V_1 and V_2) if $\operatorname{Ker} A = 0$ and $\operatorname{Im} A$ is dense in V_2.

2) Let V_1, V_2 be Hilbert Γ-modules and $A : V_1 \to V_2$ a bounded linear operator. Then A is called *almost isomorphism of* Γ*-modules* or *almost* Γ*-isomorphism* if A is a morphism of Γ-modules and an almost isomorphism at the same time.

3) Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator, $\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces. Assume that $V_1 \subset \mathcal{H}_1, V_2 \subset \mathcal{H}_2$ are closed subspaces. Then we shall say that A is *an almost isomorphism of* V_1 *and* V_2 if $AV_1 \subset V_2$, $\operatorname{Ker} A \cap V_1 = \{0\}$ and AV_1 is dense in V_2 (i.e. if $A|_{V_1}$ is an almost isomorphism of V_1 and V_2). The same applies to almost Γ-isomorphisms.

The following Lemma gives the most important property of almost isomorphisms.

LEMMA. 1) *Let* $A : V_1 \to V_2$ *be an almost isomorphism (or almost* Γ*-isomorphism),* $A = US$ *its polar decomposition. Then* $U : V_1 \to V_2$ *gives a unitary isomorphism (resp.* Γ*-isomorphism) of* V_1 *and* V_2*.*

2) *Let* $A : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear operator that gives almost isomorphism of* $V_1 \subset \mathcal{H}_1$ *and* $V_2 \subset \mathcal{H}_2$*,* P_{V_j} *the orthogonal projection on* V_j *in* $\mathcal{H}_j$*,* $j = 1, 2$*. Let* $P_{V_2} A P_{V_1} = US$ *be the polar decomposition of* $P_{V_2} A P_{V_1}$*. Then* U *is a partial isometry with* $\operatorname{Ker} U = (V_1)^\perp$, $\operatorname{Ker} U^* = (\operatorname{Im} U)^\perp = (V_2)^\perp$ *(so* U

restricts to a unitary isomorphism of V_1 *and* V_2*). If* $\mathcal{H}_1, \mathcal{H}_2, V_1, V_2$ *are* Γ*-modules and* A *is a* Γ*-morphism, then* U *restricts to a unitary* Γ*-isomorphism of* V_1 *and* V_2*.*

PROOF. is obvious due to the fact that if $A = US$ is a polar decomposition and $\operatorname{Ker} A = 0$ then $\operatorname{Ker} U = 0$, and besides, $\operatorname{Im} U \supset \operatorname{Im} A$ □

COROLLARY. *If* V_1, V_2 *are Hilbert* Γ*-modules and there exists an almost isomorphism of* V_1 *and* V_2 *then* $\dim_\Gamma V_1 = \dim_\Gamma V_2$*.*

EXERCISE. Prove the following statement that may be considered a generalization of the Corollary: if $\mathcal{H}$ is a Hilbert space, $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra in $\mathcal{H}$, V_1 and V_2 are closed subspaces in $\mathcal{H}$ which are affiliated to $\mathcal{M}$ (i.e. the corresponding orthogonal projections are in $\mathcal{M}$) such that there exists an operator $A \in \mathcal{M}$ giving an almost isomorphism of V_1 and V_2, τ is a trace on $\mathcal{M}$ and $\dim_\tau$ is the corresponding dimension function, then $\dim_\tau V_1 = \dim_\tau V_2$.

2.27. Hilbert complexes and almost exact sequences

DEFINITION 1. Let

$$V : 0 \to V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} \cdots \to V_k \xrightarrow{d_k} V_{k+1} \to \dots \xrightarrow{d_{N-1}} V_N \to 0$$

be a sequence of Hilbert spaces V_k and bounded linear operators $d_k : V_k \to V_{k+1}$. It is called a *Hilbert complex* if $d_{k+1} \cdot d_k = 0$ for all $k = 0, 1, \dots, N-2$.

The *cohomology* or *cohomology groups* or *cohomology spaces* of the Hilbert complex are linear spaces

$$H^k(V) = \operatorname{Ker} d_k / \operatorname{Im} d_{k-1}, \quad k = 0, \dots N.$$

(By definition we put $d_{-1} = d_N = 0$.)

The *reduced cohomology* or *reduced cohomology groups* or *reduced cohomology spaces* of the Hilbert complex are Hilbert spaces

$$\bar{H}^k(V) = \operatorname{Ker} d_k / \overline{\operatorname{Im} d_{k-1}}, \quad k = 0, \dots, N.$$

The complex V is called an *exact sequence* if $H^k(V) = 0$ (i.e. $\operatorname{Ker} d_k = \operatorname{Im} d_{k-1}$) for all k, and an *almost exact sequence* if $\bar{H}^k(V) = 0$ (i.e. $\operatorname{Im} d_{k-1}$ is dense in $\operatorname{Ker} d_k$) for all k.

Note that $\bar{H}^k(V)$ can be identified with a closed subspace

$$\mathcal{H}_k(V) = \operatorname{Ker} d_k \ominus \overline{\operatorname{Im} d_{k-1}} = \operatorname{Ker} d_k \cap (\operatorname{Im} d_{k-1})^\perp$$

in V_k. Hence if all V_k are Hilbert Γ-modules and d_k are Γ-isomorphisms, then $\bar{H}^k(V)$ will be Hilbert Γ-modules too.

DEFINITION 2. A *Hilbert* Γ*-complex* is a Hilbert complex

$$V : 0 \to V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} \dots \xrightarrow{d_{N-1}} V_N \to 0$$

such that all V_k are Hilbert Γ modules and all d_k are Γ-morphisms (or morphisms of Hilbert Γ-modules). Its Γ-*Betti numbers* are defined as

$$\bar{b}_k = \bar{b}_k(V) = \dim_\Gamma \bar{H}^k(V).$$

Its Γ-*Euler characteristic* is defined as

$$\chi_\Gamma(V) = \sum_{j=0}^{N} (-1)^j \bar{b}_j$$

provided all $\bar{b}_j,\ j = 0, 1, \dots, N$, are finite.

LEMMA. *Suppose that* $\dim_\Gamma V_k < \infty,\ k = 0, 1, \dots, N$, *in the previous definition. Then*

$$\sum_{k=0}^{N}(-1)^k \dim_\Gamma V_k = \chi_\Gamma(V) = \sum_{k=0}^{N}(-1)^k \dim_\Gamma \bar{H}^k(V).$$

PROOF. We prove this by induction with respect to N. If $N = 0$ then $\bar{H}^0(V) = V_0$ and the statement is obvious. Assuming that it is true for $N-1$, consider the Hilbert complex

$$\tilde{V} : 0 \to V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} \dots \xrightarrow{d_{N-3}} V_{N-2} \xrightarrow{\tilde{d}_{N-2}} \tilde{V}_{N-1} \to 0$$

where $\tilde{V}_{N-1} = \operatorname{Ker} d_{N-1}$, $\tilde{d}_{N-2}$ is d_{N-2} considered as a map from V_{N-2} to $\operatorname{Ker} d_{N-1}$ (it is well defined since $\operatorname{Im} d_{N-2} \subset \operatorname{Ker} d_{N-1}$). We obviously have

$$\bar{H}^k(\tilde{V}) = \bar{H}^k(V),\ k = 0, \dots, N-1;$$

therefore $\chi_\Gamma(\tilde{V}) = \sum_{k=0}^{N-1}(-1)^k \dim_\Gamma \bar{H}^k(V)$. By the induction assumption

$$\chi_\Gamma(\tilde{V}) = \sum_{k=0}^{N-2}(-1)^k \dim_\Gamma V_k + (-1)^{N-1} \dim_\Gamma \operatorname{Ker} d_{N-1}.$$

Now d_{N-1} defines an almost isomorphism of $(\operatorname{Ker} d_{N-1})^\perp$ and $\overline{\operatorname{Im} d_{N-1}}$, therefore

$$\begin{aligned}\dim_\Gamma V_{N-1} &= \dim_\Gamma \operatorname{Ker} d_{N-1} + \dim_\Gamma (\operatorname{Ker} d_{N-1})^\perp \\ &= \dim_\Gamma \operatorname{Ker} d_{N-1} + \dim_\Gamma \overline{\operatorname{Im} d_{N-1}}.\end{aligned}$$

But on the other hand

$$\dim_\Gamma \overline{\operatorname{Im} d_{N-1}} = \dim_\Gamma V_N - \dim_\Gamma V_N/\overline{\operatorname{Im} d_{N-1}} = \dim_\Gamma V_N - \dim_\Gamma \bar{H}^N(V).$$

Hence

$$\begin{aligned}\dim_\Gamma \operatorname{Ker} d_{N-1} &= \dim_\Gamma V_{N-1} - \dim_\Gamma \overline{\operatorname{Im} d_{N-1}} \\ &= \dim_\Gamma V_{N-1} - \dim_\Gamma V_N + \dim_\Gamma \bar{H}^N(V).\end{aligned}$$

Substituting this to the formula for $\chi_\Gamma(\tilde{V})$ we get

$$\sum_{k=0}^{N-1}(-1)^k \bar{b}_k = \sum_{k=0}^{N-2}(-1)^k \dim_\Gamma V_k + (-1)^{N-1}\left[\dim_\Gamma V_{N-1} - \dim_\Gamma V_N + \bar{b}_N\right],$$

which yields the desired result. □

COROLLARY. *Let*

$$0 \to V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} \dots \xrightarrow{d_{N-1}} V_N \to 0$$

be an almost exact sequence of Hilbert Γ*-modules and* Γ*-morphisms, and*

$$\dim_\Gamma V_k < \infty,\ k = 0, 1, \dots, N.$$

Then

$$\sum_{k=0}^{N}(-1)^k \dim_\Gamma V_k = 0.$$

An important particular case of the Euler characteristic is the index (the case $N = 1$, so we have only one non-trivial operator d_0):

DEFINITION 3. Let V_0, V_1 be Hilbert Γ-modules, $A : V_0 \to V_1$ a Γ-morphism such that

$$\dim_\Gamma \operatorname{Ker} A < \infty \quad \text{and} \quad \dim_\Gamma V_1/\overline{AV_0} = \dim_\Gamma (AV_0)^\perp = \dim_\Gamma \operatorname{Ker} A^* < \infty.$$

Then the Γ-*index* of A is defined by the formula

$$\operatorname{ind}_\Gamma A = \dim_\Gamma \operatorname{Ker} A - \dim_\Gamma \operatorname{Ker} A^*.$$

It follows from the Lemma above that if we assume that $\dim_\Gamma V_j < \infty$, $j = 0, 1$, then $\operatorname{ind}_\Gamma A = \dim_\Gamma V_0 - \dim_\Gamma V_1$, so in this case $\operatorname{ind}_\Gamma A$ does not depend of A. Generally $\operatorname{ind}_\Gamma A$ has some stability properties with respect to norm-continuous deformations of A provided we restrict ourselves to a class of so-called Γ-Fredholm morphisms A. This topics will be will be discussed in the next chapter.

2.28. Measurement of projections and types of factors

We shall sketch the proof of the classification of factors which was mentioned earlier. This classification relies on a measurement procedure for projections.

In this part $\mathcal{M}$ will always denote a factor in a Hilbert space $\mathcal{H}$.

Recall that if L is a closed linear subspace in $\mathcal{H}$ then we write $L\eta\mathcal{M}$ (and say that L is affiliated to $\mathcal{M}$) if $P_L \in \mathcal{M}$ where P_L is the orthogonal projection in $\mathcal{H}$ with $\operatorname{Im} P_L = L$.

DEFINITION 1. Let M, N be closed subspaces in $\mathcal{H}$. We shall write $M \sim N$ (and say that M and N are *equivalent with respect to* $\mathcal{M}$) if there exists a partial isometry $U \in \mathcal{M}$ such that

$$U^*U = P_M, \qquad UU^* = P_N.$$

It follows that $M\eta\mathcal{M}$ and $N\eta\mathcal{M}$.

EXERCISE 1. Prove that this is really an equivalence relation: (i) $M \sim M$ provided $M\eta\mathcal{M}$; (ii) $M \sim N$ implies $N \sim M$; (iii) if $M \sim N$ and $N \sim L$, then $M \sim L$.

EXERCISE 2. Prove that if $\{M_\kappa \mid \kappa \in K\}, \{N_\kappa \mid \kappa \in K\}$ are families of subspaces $M_\kappa\eta\mathcal{M}$, $N_\kappa\eta\mathcal{M}$ such that $M_\alpha \perp M_\beta$ if $\alpha \neq \beta$, $N_\alpha \perp N_\beta$ if $\alpha \neq \beta$, and $M_\kappa \sim N_\kappa$ for all $\kappa \in K$, then $\bigoplus_{\kappa\in K} M_\kappa \sim \bigoplus_{\kappa\in K} N_\kappa$.

EXERCISE 3. Prove that if $M\eta\mathcal{M}$, $N\eta\mathcal{M}$ and there exists $A \in M$ such that $M \cap \operatorname{Ker} A = \{0\}, A(M) \subset N$ and $A(M)$ is dense in N, then $M \sim N$.

Hint: Use the polar decomposition of $P_N A P_M$.

The following Lemma is a simplified version of the infinite-dimensional Cancellation Lemma (see 1.10 and Exercise 1 in 2.5).

LEMMA 1. *If* $P, Q \in \operatorname{Proj}(\mathcal{M})$, $P \neq 0$, $Q \neq 0$, *then there exists* $A \in \mathcal{M}$ *such that* $QAP \neq 0$.

PROOF. Denote $M = \operatorname{Im} P$. Then $\overline{\mathcal{M}M}$ is a $\mathcal{M}$-invariant and also $\mathcal{M}'$-invariant subspace in $\mathcal{H}$. Therefore the corresponding orthogonal projection should be in the center of $\mathcal{M}$, hence equal I because $M \neq \{0\}$. This means that $\mathcal{M}M$ is dense in $\mathcal{H}$. In particular the inclusion $\mathcal{M}M \subset \operatorname{Ker} Q$ is impossible, hence there exists $A \in \mathcal{M}$ such that $QAP \neq 0$. □

DEFINITION 2. Let $M, N\eta\mathcal{M}$. We shall write $M \lesssim N$ if there exists $M_1 \subset N$ such that $M \sim M_1$. We shall write $M < N$ if $M \lesssim N$ and M is not equivalent to N.

EXERCISE 4. Prove that if $M \lesssim N$ and $N \lesssim L$ then $M \lesssim L$.

The following property is of crucial importance.

LEMMA 2. *Let M, N be closed subspaces in $\mathcal{H}$, $M\eta\mathcal{M}$, $N\eta\mathcal{M}$, $M \neq \{0\}$, $N \neq \{0\}$. Then there exist $M_1 \subset M$, $N_1 \subset N$, such that $M_1\eta\mathcal{M}$, $N_1\eta\mathcal{M}$, $M_1 \sim N_1$ and $M_1 \neq \{0\}$, $N_1 \neq \{0\}$.*

PROOF. Let P, Q be orthogonal projections on M, N respectively. Then we can find $A \in \mathcal{M}$ such that $QAP \neq 0$. Now let $QAP = US$ be the polar decomposition of QAP. Then $\operatorname{Ker} S = \operatorname{Ker} U \supset \operatorname{Ker} P$ and $\operatorname{Im} U \subset \operatorname{Im} Q = N$. Hence we may take $M_1 = \operatorname{Im} U^*$, $N_1 = \operatorname{Im} U$. $\square$

LEMMA 3. *Let $M\eta\mathcal{M}$, $N\eta\mathcal{M}$. Then either $M \lesssim N$ or $N \lesssim M$.*

PROOF. We may assume that $M \neq \{0\}$, $N \neq \{0\}$. Then we can find $M_1 \subset M$, $N_1 \subset N$ such that $M_1 \neq \{0\}$, $N_1 \neq \{0\}$, $M_1\eta\mathcal{M}$, $N_1\eta\mathcal{M}$ and $M_1 \sim N_1$. Now let us consider subspaces $M \ominus M_1 = M \cap M_1^{\perp}$ and $N \ominus N_1 = N \cap N_1^{\perp}$. If both of them are non-zero subspaces then we can find non-zero subspaces $M_2 \subset M \ominus M_1$, $N_2 \subset N \ominus N_1$ affiliated to $\mathcal{M}$ such that $M_2 \sim N_2$. Obviously $M_1 \perp M_2$, $N_1 \perp N_2$, hence $M_1 \oplus M_2 \sim N_1 \oplus N_2$. Repeating this reasoning and applying the Zorn lemma, we can construct subspaces M_κ, N_κ, $\kappa \in K$, such that $M_\kappa \sim N_\kappa$ for every κ and either $\oplus_{\kappa\in K} M_\kappa = M$ or $\oplus_{\kappa\in K} N_\kappa = N$. Since $\oplus_{\kappa\in K} M_\kappa \sim \oplus_{\kappa\in K} N_\kappa$ we see that $M \lesssim N$ in the first case and $N \lesssim M$ in the second case. $\square$

LEMMA 4. *If $M \lesssim N$ and $N \lesssim M$ then $M \sim N$.*

PROOF. The proof is similar to the proof of the well known Cantor–Bernstein theorem about cardinalities. Suppose that $M \sim N' \subset N$, $N \sim M' \subset M$.

Let $U \in \mathcal{M}$ be a partial isometry with $\operatorname{Im} U^* = N$, $\operatorname{Im} U = M'$. Then U maps $N' \subset N$ isometrically onto $M'' \subset M'$ and $M'' \sim M$. Hence we can find a partial isometry $V \in \mathcal{M}$ such that $\operatorname{Im} V^* = M$, $\operatorname{Im} V = M''$. Define then $M^{(2k)} = V^k M$, $M^{(2k+1)} = V^k M'$. Since $M'' \subset M' \subset M$, we obviously obtain

$$M = M^{(0)} \supset M' = M^{(1)} \supset M'' = M^{(2)} \supset M^{(3)} \supset M^{(4)} \supset \cdots .$$

and $M^{(k)}\eta\mathcal{M}$ for all $k = 0, 1, 2, \ldots$. Moreover V maps $M^{(p)}$ isometrically onto $M^{(p+2)}$, $M^{(p+1)}$ onto $M^{(p+3)}$, hence $M^{(p)} \ominus M^{(p+1)}$ onto $M^{(p+2)} \ominus M^{(p+3)}$. Now consider equalities

$$\begin{aligned} M^{(0)} &= \left[M^{(0)} \ominus M^{(1)}\right] \oplus \left[M^{(1)} \ominus M^{(2)}\right] \oplus \left[M^{(2)} \ominus M^{(3)}\right] \oplus \cdots \\ &\quad \cdots \oplus \left(M^{(0)} \cap M^{(1)} \cap M^{(2)} \cap \cdots\right) \\ M^{(1)} &= \left[M^{(1)} \ominus M^{(2)}\right] \oplus \left[M^{(2)} \ominus M^{(3)}\right] \oplus \left[M^{(3)} \ominus M^{(4)}\right] \oplus \cdots \\ &\quad \cdots \oplus \left(M^{(1)} \cap M^{(2)} \cap M^{(3)} \cap \cdots\right) \end{aligned}$$

Here the subspaces in odd positions of the first decomposition are equivalent to the subspaces in even positions of the second decomposition whereas the subspaces in even positions of the first decomposition are just equal to the subspaces in odd positions of the second decomposition; the last subspaces in these decompositions also coincide. Hence $M^{(0)} \sim M^{(1)}$ which implies that $M \sim N$. □

Lemmas 3 and 4 immediately imply

THEOREM. *If M, N are two closed subspaces affiliated to $\mathcal{M}$ then exactly one of the following relations holds:*

$$M \sim N, \qquad M < N, \qquad N < M.$$

DEFINITION 3. A subspace $M\eta\mathcal{M}$ is said to be *infinite* if it is equivalent to a proper subspace of itself, and *finite* otherwise. A projection $P \in \mathcal{M}$ is said to be *infinite* (*finite*) if $\operatorname{Im} P$ is infinite (respectively finite).

EXERCISE 5. Prove that if $M\eta\mathcal{M}$ is finite, $N\eta\mathcal{M}$ and $N \subset M$ then N is finite too.

EXERCISE 6. If $M\eta\mathcal{M}$, $N\eta\mathcal{M}$, $M \neq \{0\}$, $N \neq \{0\}$, then M can be represented in the form

$$M = (\oplus_{\kappa \in K} N_\kappa) \oplus L,$$

where $N_\kappa \eta\mathcal{M}$, $N_\kappa \sim N$ for every κ, $L\eta\mathcal{M}$ and $L < N$. Besides if K is infinite then we can obtain such a decomposition with $L = \{0\}$.

EXERCISE 7. Every infinite subspace M can be represented in the form $M = \oplus_{\kappa \in K} N_\kappa$ where the subspaces N_κ are mutually equivalent and the index set K is infinite.

EXERCISE 8. Every infinite subspace M contains a subspace N such that $M \sim N \sim M \ominus N$.

EXERCISE 9. The orthogonal sum of two (or a finite number) of finite subspaces is finite.

EXERCISE 10. Consider a representation $M = (\oplus_{\kappa \in K} N_\kappa) \oplus L$ as in Exercise 6. Now assume that N is finite. Then M is finite if and only if the set K is finite. In this case $\operatorname{Card} K$ does not depend on the choice of the representation.

Now let us introduce the following important notation:

$$\left[\frac{M}{N}\right] = \operatorname{Card} K \quad \text{in the representation of Exercise 6.}$$

EXERCISE 11. $\left[\frac{M}{N}\right] \cdot \left[\frac{N}{L}\right] \leq \left[\frac{M}{L}\right] \leq \left(\left[\frac{M}{N}\right] + 1\right)\left(\left[\frac{N}{L}\right] + 1\right)$ for any three non-zero subspaces M, N, L affiliated to $\mathcal{M}$.

Now note that if we have a trace τ on $\mathcal{M}$ and the corresponding dimension function is $\dim_\tau$, then $M \sim N$ would necessarily imply $\dim_\tau M = \dim_\tau N$. Moreover, since $\dim_\tau(M \oplus N) = \dim_\tau M + \dim_\tau N$, we should obviously have

$$\left[\frac{M}{N}\right] \leq \frac{\dim_\tau M}{\dim_\tau N} \leq \left[\frac{M}{N}\right] + 1$$

for any non-zero subspaces M, N affiliated to $\mathcal{M}$, provided M is finite. In this case also $\dim_\tau M = k \dim_\tau N$ if $M = N_1 \oplus \cdots \oplus N_k$ with $N_j \sim N$ for every $j = 1, \ldots, k$. Actually these two properties imply the uniqueness of the dimension function (up to a positive constant factor), hence the uniqueness of the trace as we shall see below. They also provide the way to prove the existence (when the trace exists). But first we will distinguish the case when the (faithful, normal, semifinite) trace does not exist.

DEFINITION 4. $\mathcal{M}$ is called a *type* III *factor* if all non-zero subspaces affiliated to $\mathcal{M}$ are infinite.

Then we should obviously have $\dim_\tau M = \infty$ for any non-zero subspace $M \eta \mathcal{M}$ and for any trace τ. Hence there can be no (faithful, normal, semifinite) trace on type III factors.

From now on let us assume that there exists a non-zero finite subspace $M \eta \mathcal{M}$. Then also two possibilities arise.

DEFINITION 5. Suppose that there exists a minimal subspace $M_0 \eta \mathcal{M}$ i.e. a non-zero subspace $M_0 \subset \mathcal{H}$ affiliated to $\mathcal{M}$, such that the conditions $N \eta \mathcal{M}$, $N < M_0$ imply that $N = \{0\}$. Then $\mathcal{M}$ is called *type* I *factor*.

In this case obviously M_0 is finite and any finite $M \eta \mathcal{M}$ can be represented in the form $M = N_1 \oplus \cdots \oplus N_k$ where $N_j \sim M_0$, $j = 1, \dots, k$. Then we define $\frac{M}{M_0} = k$ and putting $\dim_\tau M = \frac{M}{M_0}$ we obtain the dimension function with the set of values either $\{0, 1, 2, \dots, n\}$ (then $\mathcal{M}$ is called I_n-*factor*) or $\{0, 1, 2, \dots, \infty\}$ (then $\mathcal{M}$ is called I_∞-*factor*). Obviously in this case any other dimension function (induced by a trace) is given by $\dim_\tau M = c\frac{M}{M_0}$, where $c > 0$ is a positive constant.

Now let us consider the last possibility.

DEFINITION 6. Suppose that finite subspaces $M \eta \mathcal{M}$ exist but there is no minimal one. Then $\mathcal{M}$ is called *type* II *factor*.

In this case we can construct a sequence of finite subspaces $M_1, M_2, M_3, \dots$ such that $\left[\frac{M_k}{M_{k+1}}\right] \geq 2$. To do this we can start from an arbitrary finite subspace M_1. Then find $N < M_1$, $N \neq \{0\}$. Then $N \sim L \subset M_1$. Now either $N \precsim M_1 \ominus L$ or $M_1 \ominus L \precsim L$. In the first case, we put $M_2 = N$, and in the second, we put $M_2 = M_1 \ominus L$, so we obtain $\left[\frac{M_1}{M_2}\right] \geq 2$. Applying the same procedure to M_2 we can obtain M_3 such that $\left[\frac{M_2}{M_3}\right] \geq 2$ etc.

EXERCISE 12. Let M, N be two arbitrary finite non-zero subspaces. Prove that there exists a limit

$$\frac{M}{N} = \lim_{k\to\infty} \frac{\left[\frac{M}{M_k}\right]}{\left[\frac{N}{M_k}\right]}$$

and $0 < \frac{M}{N} < \infty$.

Now we have the following two possibilities:

1. $\mathcal{H}$ is finite. Then $\mathcal{M}$ is called a II_1-*factor*.
2. $\mathcal{H}$ is infinite. Then $\mathcal{M}$ is called a II_∞-*factor*.

EXERCISE 13. (i) Let $\mathcal{M}$ be a II_1-factor. Prove that $\frac{M}{\mathcal{H}}$ takes all values in $[0, 1]$ (on all subspaces M affiliated to $\mathcal{M}$).
(ii) Let $\mathcal{M}$ be a II_∞-factor. Prove that if M_0 is any finite non-zero subspace in $\mathcal{H}$ then $\frac{M}{M_0}$ takes all values in $[0, \infty]$.

Now if $\dim_\tau$ is a dimension function induced by a trace τ on $\mathcal{M}$ then we should have

$$\dim_\tau M = \frac{M}{M_0} \dim_\tau M_0$$

for any finite non-zero subspace M_0 and for any $M \eta \mathcal{M}$. Hence this dimension function is unique up to a normalization that can be fixed e.g. if we choose a finite non-zero M_0 and put $\dim_\tau M_0 = 1$ (in II_1-case it is natural to take $M_0 = \mathcal{H}$, then the range of the dimension function will be $[0,1]$).

These arguments prove the uniqueness of the dimension function associated with a factor $\mathcal{M}$ (up to a constant positive factor), hence the uniqueness of the trace on $\mathcal{M}$, because the trace can be reconstructed from the dimension due to the Spectral Theorem.

The existence can be proved (in case of types I or II) by using the formulas given in the proof.

Exercise 14. Show that the types (I_n), (I_∞), (II_1), (II_∞), (III), the dimension function and the trace (up to a positive constant factor) are invariant with respect to arbitrary $*$-isomorphisms of algebras.

Hint: Consider orthogonal projections instead of subspaces and formulate everything in terms of projections.

Remark 1. Any factor of type I_n is $*$-isomorphic to $\mathcal{B}(\mathbb{C}^n)$. Any factor of type I_∞ is $*$-isomorphic to $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is an infinite-dimensional Hilbert space.

Any II_∞-factor is $*$-isomorphic to a factor of the form

$$\mathcal{M} = \mathcal{M}_0 \otimes \mathcal{B}(\mathcal{H}_0)$$

where $\mathcal{M}_0$ is a II_1-factor, $\mathcal{H}_0$ is an infinite-dimensional Hilbert space. If $\{e_j \mid j \in \mathcal{J}\}$ is an orthonormal basis in $\mathcal{H}_0$ then $\mathcal{M}$ becomes an algebra of bounded linear operators in $\mathcal{H} = \oplus_{j\in\mathcal{J}} \mathcal{H}_1^{(j)}$, where $\mathcal{H}_1^{(j)} \cong \mathcal{H}_1$ is the Hilbert space where $\mathcal{M}_0$ is acting. Namely, if $A \in \mathcal{B}(\mathcal{H})$ is given by the matrix $[A_{j,k}]_{j,k\in\mathcal{J}}$, $A_{j,k} \in \mathcal{B}(\mathcal{H}_1)$, then the inclusion $A \in \mathcal{M}$ is equivalent to inclusions $A_{j,k} \in \mathcal{M}_0$ for all $j,k \in \mathcal{J}$. If τ_0 is the trace on $\mathcal{M}_0$ then the trace on $\mathcal{M}$ can be given by

$$\tau(A) = \sum_{j\in\mathcal{J}} \tau_0(A_{j,j}).$$

Remark 2. More details about the proofs and further developments of the abstract theory of factors and von Neumann algebras, as discussed in this section, can be found in the original papers by F. Murray and J. von Neumann [**MN**], as well as in monographs by J. Dixmier [**Di**] and M.A. Naimark [**Na**].