Chapter III
Positive Currents and Lelong Numbers

In 1957, P. Lelong introduced natural positivity concepts for currents of pure bidimension (p,p) on complex manifolds. With every analytic subset is associated a current of integration over its set of regular points and all such currents are positive and closed. The important closedness property is proved here via the Skoda-El Mir extension theorem. Positive currents have become an important tool for the study of global geometric problems as well as for questions related to local algebra and intersection theory. We develope here a differential geometric approach to intersection theory through a detailed study of wedge products of closed positive currents (Monge-Ampère operators). The Lelong-Poincaré equation and the Jensen-Lelong formula are basic in this context, providing a useful tool for studying the location and multiplicities of zeroes of entire functions on $\mathbb{C}^n$ or on a manifold, in relation with the growth at infinity. Lelong numbers of closed positive currents are then introduced; these numbers can be seen as a generalization to currents of the notion of multiplicity of a germ of analytic set at a singular point. We prove various properties of Lelong numbers (e.g. comparison theorems, semi-continuity theorem of Siu, transformation under holomorphic maps). As an application to Number Theory, we prove a general Schwarz lemma in $\mathbb{C}^n$ and derive from it Bombieri's theorem on algebraic values of meromorphic maps and the famous theorems of Gelfond-Schneider and Baker on the transcendence of exponentials and logarithms of algebraic numbers.

1. Basic Concepts of Positivity

1.A. Positive and Strongly Positive Forms

Let V be a complex vector space of dimension n and $(z_1,\ldots,z_n)$ coordinates on V. We denote by $(\partial/\partial z_1,\ldots,\partial/\partial z_n)$ the corresponding basis of V, by $(dz_1,\ldots,dz_n)$ its dual basis in $V^\star$ and consider the exterior algebra

$$\Lambda V^\star_{\mathbb{C}} = \bigoplus \Lambda^{p,q}V^\star, \qquad \Lambda^{p,q}V^\star = \Lambda^p V^\star \otimes \Lambda^q\overline{V^\star}.$$

We are of course especially interested in the case where $V = T_xX$ is the tangent space to a complex manifold X, but we want to emphasize here that our considerations only involve linear algebra. Let us first observe that V has a canonical orientation, given by the (n,n)-form

$$\tau(z) = \mathrm{i}dz_1\wedge d\overline{z}_1\wedge\ldots\wedge \mathrm{i}dz_n\wedge d\overline{z}_n = 2^n\,dx_1\wedge dy_1\wedge\ldots\wedge dx_n\wedge dy_n$$

where $z_j = x_j + iy_j$. In fact, if $(w_1, \ldots, w_n)$ are other coordinates, we find

$$dw_1 \wedge \ldots \wedge dw_n = \det(\partial w_j/\partial z_k)\, dz_1 \wedge \ldots \wedge dz_n,$$
$$\tau(w) = \left|\det(\partial w_j/\partial z_k)\right|^2 \tau(z).$$

In particular, a complex manifold always has a canonical orientation. More generally, natural positivity concepts for (p,p)-forms can be defined.

(1.1) Definition. *A (p,p)-form $u \in \Lambda^{p,p}V^\star$ is said to be positive if for all $\alpha_j \in V^\star$, $1 \le j \le q = n-p$, then*

$$u \wedge \mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_q \wedge \overline{\alpha}_q$$

is a positive (n,n)-form. A (q,q)-form $v \in \Lambda^{q,q}V^\star$ is said to be strongly positive if v is a convex combination

$$v = \sum \gamma_s \ \mathrm{i}\alpha_{s,1} \wedge \overline{\alpha}_{s,1} \wedge \ldots \wedge \mathrm{i}\alpha_{s,q} \wedge \overline{\alpha}_{s,q}$$

where $\alpha_{s,j} \in V^\star$ and $\gamma_s \ge 0$.

(1.2) Example. Since $\mathrm{i}^p(-1)^{p(p-1)/2} = \mathrm{i}^{p^2}$, we have the commutation rules

$$\mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p = i^{p^2} \alpha \wedge \overline{\alpha}, \qquad \forall \alpha = \alpha_1 \wedge \ldots \wedge \alpha_p \in \Lambda^{p,0}V^\star,$$
$$i^{p^2} \beta \wedge \overline{\beta} \wedge i^{m^2} \gamma \wedge \overline{\gamma} = i^{(p+m)^2} \beta \wedge \gamma \wedge \overline{\beta \wedge \gamma}, \qquad \forall \beta \in \Lambda^{p,0}V^\star,\ \forall \gamma \in \Lambda^{m,0}V^\star.$$

Take $m = q$ to be the complementary degree of p. Then $\beta \wedge \gamma = \lambda dz_1 \wedge \ldots \wedge dz_n$ for some $\lambda \in \mathbb{C}$ and $i^{n^2} \beta \wedge \gamma \wedge \overline{\beta \wedge \gamma} = |\lambda|^2 \tau(z)$. If we set $\gamma = \alpha_1 \wedge \ldots \wedge \alpha_q$, we find that $i^{p^2} \beta \wedge \overline{\beta}$ is a positive (p,p)-form for every $\beta \in \Lambda^{p,0}V^\star$; in particular, strongly positive forms are positive. □

The sets of positive and strongly positive forms are closed convex cones, i.e. closed and stable under convex combinations. By definition, the positive cone is dual to the strongly positive cone via the pairing

$$(1.3) \qquad \begin{aligned} \Lambda^{p,p}V^\star \times \Lambda^{q,q}V^\star &\longrightarrow \mathbb{C} \\ (u,v) &\longmapsto u \wedge v/\tau, \end{aligned}$$

that is, $u \in \Lambda^{p,p}V^\star$ is positive if and only if $u \wedge v \ge 0$ for all strongly positive forms $v \in \Lambda^{q,q}V^\star$. Since the bidual of an arbitrary convex cone Γ is equal to its closure $\overline{\Gamma}$, we also obtain that v is strongly positive if and only if $v \wedge u = u \wedge v$ is ≥ 0 for all positive forms u. Later on, we will need the following elementary lemma.

(1.4) Lemma. *Let $(z_1, \ldots, z_n)$ be arbitrary coordinates on V. Then $\Lambda^{p,p}V^\star$ admits a basis consisting of strongly positive forms*

$$\beta_s = \mathrm{i}\beta_{s,1} \wedge \overline{\beta}_{s,1} \wedge \ldots \wedge \mathrm{i}\beta_{s,p} \wedge \overline{\beta}_{s,p}, \qquad 1 \le s \le \binom{n}{p}^2$$

where each $\beta_{s,l}$ is of the type $dz_j \pm dz_k$ or $dz_j \pm \mathrm{i}dz_k$, $1 \le j,k \le n$.

Proof. Since one can always extract a basis from a set of generators, it is sufficient to see that the family of forms of the above type generates $\Lambda^{p,p}V^\star$. This follows from the identities

$$\begin{aligned} 4dz_j \wedge d\overline{z}_k = \;& (dz_j + dz_k) \wedge \overline{(dz_j + dz_k)} - (dz_j - dz_k) \wedge \overline{(dz_j - dz_k)} \\ & +\mathrm{i}(dz_j + \mathrm{i}dz_k) \wedge \overline{(dz_j + \mathrm{i}dz_k)} - \mathrm{i}(dz_j - \mathrm{i}dz_k) \wedge \overline{(dz_j - \mathrm{i}dz_k)}, \end{aligned}$$

$$dz_{j_1} \wedge \ldots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \ldots \wedge d\overline{z}_{k_p} = \pm \bigwedge_{1 \le s \le p} dz_{j_s} \wedge d\overline{z}_{k_s}. \qquad \square$$

(1.5) Corollary. *All positive forms u are real, i.e. satisfy $\overline{u} = u$. In terms of coordinates, if $u = i^{p^2} \sum_{|I|=|J|=p} u_{I,J}\, dz_I \wedge d\overline{z}_J$, then the coefficients satisfy the hermitian symmetry relation $\overline{u_{I,J}} = u_{J,I}$.*

Proof. Clearly, every strongly positive (q,q)-form is real. By Lemma 1.4, these forms generate over $\mathbb{R}$ the real elements of $\Lambda^{q,q}V^\star$, so we conclude by duality that positive (p,p)-forms are also real. $\square$

(1.6) Criterion. *A form $u \in \Lambda^{p,p}V^\star$ is positive if and only if its restriction $u_{\restriction S}$ to every p-dimensional subspace $S \subset V$ is a positive volume form on S.*

Proof. If S is an arbitrary p-dimensional subspace of V we can find coordinates $(z_1, \ldots, z_n)$ on V such that $S = \{z_{p+1} = \ldots = z_n = 0\}$. Then

$$u_{\restriction S} = \lambda_S\, \mathrm{i}dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}dz_p \wedge d\overline{z}_p$$

where λ_S is given by

$$u \wedge \mathrm{i}dz_{p+1} \wedge d\overline{z}_{p+1} \wedge \ldots \wedge \mathrm{i}dz_n \wedge d\overline{z}_n = \lambda_S\, \tau(z).$$

If u is positive then $\lambda_S \ge 0$ so $u_{\restriction S}$ is positive for every S. The converse is true because the $(n-p, n-p)$-forms $\bigwedge_{j>p} \mathrm{i}dz_j \wedge d\overline{z}_j$ generate all strongly positive forms when S runs over all p-dimensional subspaces. $\square$

(1.7) Corollary. *A form $u = \mathrm{i} \sum_{j,k} u_{jk}\, dz_j \wedge d\overline{z}_k$ of bidegree $(1,1)$ is positive if and only if $\xi \mapsto \sum u_{jk} \xi_j \overline{\xi}_k$ is a semi-positive hermitian form on $\mathbb{C}^n$.*

Proof. If S is the complex line generated by ξ and $t \mapsto t\xi$ the parametrization of S, then $u_{\restriction S} = \big(\sum u_{jk} \xi_j \overline{\xi}_k\big)\, \mathrm{i}dt \wedge d\overline{t}$. $\square$

Observe that there is a canonical one-to-one correspondence between hermitian forms and real $(1,1)$-forms on V. The correspondence is given by

$$(1.8)\quad h=\sum_{1\le j,k\le n} h_{jk}(z)\,dz_j\otimes d\overline{z}_k \longmapsto u=\mathrm{i}\sum_{1\le j,k\le n} h_{jk}(z)\,dz_j\wedge d\overline{z}_k$$

and does not depend on the choice of coordinates: indeed, as $\overline{h}_{jk}=h_{kj}$, one finds immediately

$$u(\xi,\eta)=\mathrm{i}\sum h_{jk}(z)(\xi_j\overline{\eta}_k-\eta_j\overline{\xi}_k)=-2\,\mathrm{Im}\,h(\xi,\eta),\qquad \forall\xi,\eta\in TX.$$

Moreover, h is ≥ 0 as a hermitian form if and only if $u\ge 0$ as a $(1,1)$-form. A diagonalization of h shows that every positive $(1,1)$-form $u\in\Lambda^{1,1}V^\star$ can be written

$$u=\sum_{1\le j\le r}\mathrm{i}\gamma_j\wedge\overline{\gamma}_j\ ,\qquad \gamma\in V^\star,\quad r=\text{rank of}\quad u,$$

in particular, every positive $(1,1)$-form is strongly positive. By duality, this is also true for $(n-1,n-1)$-forms.

(1.9) Corollary. *The notions of positive and strongly positive (p,p)-forms coincide for $p=0,1,n-1,n$.* □

(1.10) Remark. It is not difficult to see, however, that positivity and strong positivity differ in all bidegrees (p,p) such that $2\le p\le n-2$. Indeed, a positive form $i^{p^2}\beta\wedge\overline{\beta}$ with $\beta\in\Lambda^{p,0}V^\star$ is strongly positive if and only if β is decomposable as a product $\beta_1\wedge\ldots\wedge\beta_p$. To see this, suppose that

$$i^{p^2}\beta\wedge\overline{\beta}=\sum_{1\le j\le N} i^{p^2}\gamma_j\wedge\overline{\gamma}_j$$

where all $\gamma_j\in\Lambda^{p,0}V^\star$ are decomposable. Take N minimal. The equality can be also written as an equality of hermitian forms $|\beta|^2=\sum|\gamma_j|^2$ if β,γ_j are seen as linear forms on Λ^pV. The hermitian form $|\beta|^2$ has rank one, so we must have $N=1$ and $\beta=\lambda\gamma_j$, as desired. Note that there are many non decomposable p-forms in all degrees p such that $2\le p\le n-2$, e.g. $(dz_1\wedge dz_2+dz_3\wedge dz_4)\wedge\ldots\wedge dz_{p+2}$: if a p-form is decomposable, the vector space of its contractions by elements of $\bigwedge^{p-1}V$ is a p-dimensional subspace of $V^\star$; in the above example the dimension is $p+2$.

(1.11) Proposition. *If $u_1,\ldots,u_s$ are positive forms, all of them strongly positive (resp. all except perhaps one), then $u_1\wedge\ldots\wedge u_s$ is strongly positive (resp. positive).*

Proof. Immediate consequence of Def. 1.1. Observe however that the wedge product of two positive forms is not positive in general (otherwise we would infer that positivity coincides with strong positivity). □

(1.12) Proposition. *If $\Phi : W \longrightarrow V$ is a complex linear map and $u \in \Lambda^{p,p}V^\star$ is (strongly) positive, then $\Phi^\star u \in \Lambda^{p,p}W^\star$ is (strongly) positive.*

Proof. This is clear for strong positivity, since

$$\Phi^\star(\mathrm{i}\alpha_1 \wedge \overline{\alpha}_1 \wedge \ldots \wedge \mathrm{i}\alpha_p \wedge \overline{\alpha}_p) = \mathrm{i}\beta_1 \wedge \overline{\beta}_1 \wedge \ldots \wedge \mathrm{i}\beta_p \wedge \overline{\beta}_p$$

with $\beta_j = \Phi^\star\alpha_j \in W^\star$, for all $\alpha_j \in V^\star$. For u positive, we may apply Criterion 1.6: if S is a p-dimensional subspace of W, then $u_{\restriction\Phi(S)}$ and $(\Phi^\star u)_{\restriction S} = (\Phi_{\restriction}S)^\star u_{\restriction\Phi(S)}$ are positive when $\Phi_{\restriction S} : S \longrightarrow \Phi(S)$ is an isomorphism; otherwise we get $(\Phi^\star u)_{\restriction S} = 0$. □

1.B. Positive Currents

The duality between the positive and strongly positive cones of forms can be used to define corresponding positivity notions for currents.

(1.13) Definition. *A current $T \in \mathcal{D}'_{p,p}(X)$ is said to be positive (resp. strongly positive) if $\langle T, u\rangle \ge 0$ for all test forms $u \in \mathcal{D}_{p,p}(X)$ that are strongly positive (resp. positive) at each point. The set of positive (resp. strongly positive) currents of bidimension (p,p) will be denoted*

$$\mathcal{D}'^{+}_{p,p}(X), \qquad \text{resp. } \mathcal{D}'^{\oplus}_{p,p}(X).$$

It is clear that (strong) positivity is a local property and that the sets $\mathcal{D}'^{\oplus}_{p,p}(X) \subset \mathcal{D}'^{+}_{p,p}(X)$ are closed convex cones with respect to the weak topology. Another way of stating Def. 1.13 is:

T is positive (strongly positive) if and only if $T \wedge u \in \mathcal{D}'_{0,0}(X)$ is a positive measure for all strongly positive (positive) forms $u \in C^\infty_{p,p}(X)$.

This is so because a distribution $S \in \mathcal{D}'(X)$ such that $S(f) \ge 0$ for every non-negative function $f \in \mathcal{D}(X)$ is a positive measure.

(1.14) Proposition. *Every positive current $T = i^{(n-p)^2} \sum T_{I,J}\, dz_I \wedge d\overline{z}_J$ in $\mathcal{D}'^{+}_{p,p}(X)$ is real and of order 0, i.e. its coefficients $T_{I,J}$ are complex measures and satisfy $\overline{T_{I,J}} = T_{J,I}$ for all multi-indices $|I| = |J| = n-p$. Moreover $T_{I,I} \ge 0$, and the absolute values $|T_{I,J}|$ of the measures $T_{I,J}$ satisfy the inequality*

$$\lambda_I\lambda_J\,|T_{I,J}| \le 2^p \sum_M \lambda_M^2\, T_{M,M}, \qquad I \cap J \subset M \subset I \cup J$$

where $\lambda_k \ge 0$ are arbitrary coefficients and $\lambda_I = \prod_{k\in I} \lambda_k$.

Proof. Since positive forms are real, positive currents have to be real by duality. Let us denote by $K = \complement I$ and $L = \complement J$ the ordered complementary multi-indices of I, J in $\{1, 2, \ldots, n\}$. The distribution $T_{I,I}$ is a positive measure because

$$T_{I,I}\ \tau = T \wedge i^{p^2} dz_K \wedge d\overline{z}_K \geq 0.$$

On the other hand, the proof of Lemma 1.4 yields

$$T_{I,J}\ \tau = \pm T \wedge i^{p^2} dz_K \wedge d\overline{z}_L = \sum_{a\in(\mathbb{Z}/4\mathbb{Z})^p} \varepsilon_a T \wedge \gamma_a \quad \text{where}$$

$$\gamma_a = \bigwedge_{1\leq s\leq p} \frac{\mathrm{i}}{4}(dz_{k_s} + i^{a_s} dz_{l_s}) \wedge (\overline{dz_{k_s} + i^{a_s} dz_{l_s}}), \qquad \varepsilon_a = \pm 1, \pm i.$$

Now, each $T \wedge \gamma_a$ is a positive measure, hence $T_{I,J}$ is a complex measure and

$$\begin{aligned} |T_{I,J}|\,\tau &\leq \sum_a T \wedge \gamma_a = T \wedge \sum_a \gamma_a \\ &= T \wedge \bigwedge_{1\leq s\leq p} \Big(\sum_{a_s\in\mathbb{Z}/4\mathbb{Z}} \frac{\mathrm{i}}{4}(dz_{k_s} + i^{a_s} dz_{l_s}) \wedge (\overline{dz_{k_s} + i^{a_s} dz_{l_s}})\Big) \\ &= T \wedge \bigwedge_{1\leq s\leq p} \big(\mathrm{i}dz_{k_s} \wedge d\overline{z}_{k_s} + \mathrm{i}dz_{l_s} \wedge d\overline{z}_{l_s}\big). \end{aligned}$$

The last wedge product is a sum of at most 2^p terms, each of which is of the type $i^{p^2} dz_M \wedge d\overline{z}_M$ with $|M| = p$ and $M \subset K \cup L$. Since $T \wedge i^{p^2} dz_M \wedge d\overline{z}_M = T_{\complement M, \complement M}\ \tau$ and $\complement M \supset \complement K \cap \complement L = I \cap J$, we find

$$|T_{I,J}| \leq 2^p \sum_{M\supset I\cap J} T_{M,M}.$$

Now, consider a change of coordinates $(z_1, \ldots, z_n) = \Lambda w = (\lambda_1 w_1, \ldots, \lambda_n w_n)$ with $\lambda_1, \ldots, \lambda_n > 0$. In the new coordinates, the current T becomes $\Lambda^\star T$ and its coefficients become $\lambda_I \lambda_J T_{I,J}(\Lambda w)$. Hence, the above inequality implies

$$\lambda_I \lambda_J\, |T_{I,J}| \leq 2^p \sum_{M\supset I\cap J} \lambda_M^2\, T_{M,M}.$$

This inequality is still true for $\lambda_k \geq 0$ by passing to the limit. The inequality of Prop. 1.14 follows when all coefficients λ_k, $k \notin I \cup J$, are replaced by 0, so that $\lambda_M = 0$ for $M \not\subset I \cup J$. $\square$

(1.15) Remark. If T is of order 0, we define the *mass measure* of T by $\|T\| = \sum |T_{I,J}|$ (of course $\|T\|$ depends on the choice of coordinates). By the Radon-Nikodym theorem, we can write $T_{I,J} = f_{I,J} \|T\|$ with a Borel function $f_{I,J}$ such that $\sum |f_{I,J}| = 1$. Hence

$$T = \|T\|\, f, \qquad \text{where} \ \ f = i^{(n-p)^2} \sum f_{I,J}\, dz_I \wedge d\overline{z}_J.$$

Then T is (strongly) positive if and only if the form $f(x) \in \Lambda^{n-p,n-p}T_x^\star X$ is (strongly) positive at $\|T\|$-almost all points $x \in X$. Indeed, this condition is clearly sufficient. On the other hand, if T is (strongly) positive and u_j is a sequence of forms with constant coefficients in $\Lambda^{p,p}T^\star X$ which is dense in the set of strongly positive (positive) forms, then $T \wedge u_j = \|T\|\, f \wedge u_j$, so $f(x) \wedge u_j$ has to be a positive (n,n)-form except perhaps for x in a set $N(u_j)$ of $\|T\|$-measure 0. By a simple density argument, we see that $f(x)$ is (strongly) positive outside the $\|T\|$-negligible set $N = \bigcup N(u_j)$.

As a consequence of this proof, T is positive (strongly positive) if and only if $T \wedge u$ is a positive measure for all strongly positive (positive) forms u of bidegree (p,p) with *constant coefficients* in the given coordinates $(z_1, \ldots, z_n)$. It follows that if T is (strongly) positive in a coordinate patch Ω, then the convolution $T \star \rho_\varepsilon$ is (strongly) positive in $\Omega_\varepsilon = \{x \in \Omega\,;\, d(x, \partial\Omega) > \varepsilon\}$. □

(1.16) Corollary. *If $T \in \mathcal{D}'_{p,p}(X)$ and $v \in C^0_{s,s}(X)$ are positive, one of them (resp. both of them) strongly positive, then the wedge product $T \wedge v$ is a positive (resp. strongly positive) current.*

This follows immediately from Remark 1.15 and Prop. 1.11 for forms. Similarly, Prop. 1.12 on pull-backs of positive forms easily shows that positivity properties of currents are preserved under direct or inverse images by holomorphic maps.

(1.17) Proposition. *Let $\Phi : X \longrightarrow Y$ be a holomorphic map between complex analytic manifolds.*

a) *If $T \in \mathcal{D}'^{+}_{p,p}(X)$ and $\Phi_{\restriction \mathrm{Supp}\, T}$ is proper, then $\Phi_\star T \in \mathcal{D}'^{+}_{p,p}(Y)$.*

b) *If $T \in \mathcal{D}'^{+}_{p,p}(Y)$ and if Φ is a submersion with m-dimensional fibers, then $\Phi^\star T \in \mathcal{D}'^{+}_{p+m,p+m}(X)$.*

Similar properties hold for strongly positive currents. □

1.C. Basic Examples of Positive Currents

We present here two fundamental examples which will be of interest in many circumstances.

(1.18) Current Associated to a Plurisubharmonic Function Let X be a complex manifold and $u \in \mathrm{Psh}(X) \cap L^1_{\mathrm{loc}}(X)$ a plurisubharmonic function. Then

$$T = \mathrm{i}d'd''u = \mathrm{i} \sum_{1 \le j,k \le n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\, dz_j \wedge d\overline{z}_k$$

is a positive current of bidegree $(1,1)$. Moreover T is closed (we always mean here d-closed, that is, $dT = 0$). Assume conversely that Θ is a closed real

$(1,1)$-current on X. Poincaré's lemma implies that every point $x_0 \in X$ has a neighborhood Ω_0 such that $\Theta = dS$ with $S \in \mathcal{D}'_1(\Omega_0, \mathbb{R})$. Write $S = S^{1,0} + S^{0,1}$, where $S^{0,1} = \overline{S^{1,0}}$. Then $d''S = \Theta^{0,2} = 0$, and the Dolbeault-Grothendieck lemma shows that $S^{0,1} = d''v$ on some neighborhood $\Omega \subset \Omega_0$, with $v \in \mathcal{D}'(\Omega, \mathbb{C})$. Thus

$$\begin{aligned} S &= \overline{d''v} + d''v = d'\overline{v} + d''v, \\ \Theta &= dS = d'd''(v - \overline{v}) = \mathrm{i}d'd''u, \end{aligned}$$

where $u = 2\,\mathrm{Re}\,v \in \mathcal{D}'(\Omega, \mathbb{R})$. If $\Theta \in C^\infty_{1,1}(X)$, the hypoellipticity of d'' in bidegree $(p,0)$ shows that $d'u$ is of class C^∞, so $u \in C^\infty(\Omega)$. When Θ is positive, the distribution u is a plurisubharmonic function (Th. I.3.31). We have thus proved:

(1.19) Proposition. *If $\Theta \in \mathcal{D}'^{+}_{n-1,n-1}(X)$ is a closed positive current of bidegree $(1,1)$, then for every point $x_0 \in X$ there exists a neighborhood Ω of x_0 and $u \in \mathrm{Psh}(\Omega)$ such that $\Theta = \mathrm{i}d'd''u$.* □

(1.20) Current of Integration on a Complex Submanifold Let $Z \subset X$ be a closed p-dimensional complex submanifold with its canonical orientation and $T = [Z]$. Then $T \in \mathcal{D}'^{\oplus}_{p,p}(X)$. Indeed, every (r,s)-form of total degree $r + s = 2p$ has zero restriction to Z unless $(r,s) = (p,p)$, therefore we have $[Z] \in \mathcal{D}'_{p,p}(X)$. Now, if $u \in \mathcal{D}_{p,p}(X)$ is a positive test form, then $u_{\restriction Z}$ is a positive volume form on Z by Criterion 1.6, therefore

$$\langle [Z], u \rangle = \int_Z u_{\restriction Z} \geq 0.$$

In this example the current $[Z]$ is also closed, because $d[Z] = \pm[\partial Z] = 0$ by Stokes' theorem. □

1.D. Trace Measure and Wirtinger's Inequality

We discuss now some questions related to the concept of area on complex submanifolds. Assume that X is equipped with a hermitian metric h, i.e. a positive definite hermitian form $h = \sum h_{jk} dz_j \otimes d\overline{z}_k$ of class C^∞ ; we denote by $\omega = \mathrm{i} \sum h_{jk} dz_j \wedge d\overline{z}_k \in C^\infty_{1,1}(X)$ the associated positive $(1,1)$-form.

(1.21) Definition. *For every $T \in \mathcal{D}'^{+}_{p,p}(X)$, the trace measure of T with respect to ω is the positive measure*

$$\sigma_T = \frac{1}{2^p p!} T \wedge \omega^p.$$

If $(\zeta_1, \ldots, \zeta_n)$ is an orthonormal frame of $T^\star X$ with respect to h on an open subset $U \subset X$, we may write

$$\omega = \mathrm{i} \sum_{1 \le j \le n} \zeta_j \wedge \overline{\zeta}_j, \qquad \omega^p = i^{p^2} p! \sum_{|K|=p} \zeta_K \wedge \overline{\zeta}_K,$$
$$T = i^{(n-p)^2} \sum_{|I|=|J|=n-p} T_{I,J}\, \zeta_I \wedge \overline{\zeta}_J, \qquad T_{I,J} \in \mathcal{D}'(U),$$

where $\zeta_I = \zeta_{i_1} \wedge \ldots \wedge \zeta_{i_{n-p}}$. An easy computation yields

$$(1.22) \quad \sigma_T = 2^{-p} \Big(\sum_{|I|=n-p} T_{I,I} \Big)\, \mathrm{i}\zeta_1 \wedge \overline{\zeta}_1 \wedge \ldots \wedge \mathrm{i}\zeta_n \wedge \overline{\zeta}_n.$$

For $X = \mathbb{C}^n$ with the standard hermitian metric $h = \sum dz_j \otimes d\overline{z}_j$, we get in particular

$$(1.22') \quad \sigma_T = 2^{-p} \Big(\sum_{|I|=n-p} T_{I,I} \Big)\, \mathrm{i}dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}dz_n \wedge d\overline{z}_n.$$

Proposition 1.14 shows that the mass measure $||T|| = \sum |T_{I,J}|$ of a positive current T is always dominated by $C\sigma_T$ where $C > 0$ is a constant. It follows easily that the weak topology of $\mathcal{D}'_p(X)$ and of $\mathcal{D}^0_p{}'(X)$ coincide on $\mathcal{D}'^+_p(X)$, which is moreover a metrizable subspace: its weak topology is in fact defined by the collection of semi-norms $T \longmapsto |\langle T, f_\nu \rangle|$ where (f_ν) is an arbitrary dense sequence in $\mathcal{D}_p(X)$. By the Banach-Alaoglu theorem, the unit ball in the dual of a Banach space is weakly compact, thus:

(1.23) Proposition. *Let δ be a positive continuous function on X. Then the set of currents $T \in \mathcal{D}'^+_p(X)$ such that $\int_X \delta\, T \wedge \omega^p \le 1$ is weakly compact.*

Proof. Note that our set is weakly closed, since a weak limit of positive currents is positive and $\int_X \delta\, T \wedge \omega^p = \sup \langle T, \theta\delta\omega^p \rangle$ when θ runs over all elements of $\mathcal{D}(X)$ such that $0 \le \theta \le 1$. $\square$

Now, let Z be a p-dimensional complex analytic submanifold of X. We claim that

$$(1.24) \quad \sigma_{[Z]} = \frac{1}{2^p p!} [Z] \wedge \omega^p = \text{Riemannian volume measure on } Z.$$

This result is in fact a special case of the following important inequality.

(1.25) Wirtinger's inequality. *Let Y be an oriented real submanifold of class C^1 and dimension $2p$ in X, and let dV_Y be the Riemannian volume form on Y associated with the metric $h_{\restriction Y}$. Set*

$$\frac{1}{2^p p!} \omega^p_{\restriction Y} = \alpha\, dV_Y, \qquad \alpha \in C^0(Y).$$

Then $|\alpha| \le 1$ and the equality holds if and only if Y is a complex analytic submanifold of X. In that case $\alpha = 1$ if the orientation of Y is the canonical one, $\alpha = -1$ otherwise.

Proof. The restriction $\omega_{\restriction Y}$ is a real antisymmetric 2-form on TY. At any point $z \in Y$, we can thus find an oriented orthonormal $\mathbb{R}$-basis $(e_1, e_2, \ldots, e_{2p})$ of T_zY such that

$$\frac{1}{2}\omega = \sum_{1\le k\le p} \alpha_k\, e^\star_{2k-1} \wedge e^\star_{2k} \quad \text{on} \quad T_zY, \qquad \text{where}$$
$$\alpha_k = \frac{1}{2}\omega(e_{2k-1}, e_{2k}) = -\operatorname{Im} h(e_{2k-1}, e_{2k}).$$

We have $dV_Y = e^\star_1 \wedge \ldots \wedge e^\star_{2p}$ by definition of the Riemannian volume form. By taking the p-th power of ω, we get

$$\frac{1}{2^p p!}\omega^p_{\restriction T_zY} = \alpha_1 \ldots \alpha_p\, e^\star_1 \wedge \ldots \wedge e^\star_{2p} = \alpha_1 \ldots \alpha_p\, dV_Y.$$

Since (e_k) is an orthonormal $\mathbb{R}$-basis, we have $\operatorname{Re} h(e_{2k-1}, e_{2k}) = 0$, thus $h(e_{2k-1}, e_{2k}) = -\mathrm{i}\alpha_k$. As $|e_{2k-1}| = |e_{2k}| = 1$, we get

$$0 \le |e_{2k} \pm Je_{2k-1}|^2 = 2\big(1 \pm \operatorname{Re} h(Je_{2k-1}, e_{2k})\big) = 2(1 \pm \alpha_k).$$

Therefore

$$|\alpha_k| \le 1, \qquad |\alpha| = |\alpha_1 \ldots \alpha_p| \le 1,$$

with equality if and only if $\alpha_k = \pm 1$ for all k, i.e. $e_{2k} = \pm Je_{2k-1}$. In this case $T_zY \subset T_zX$ is a complex vector subspace at every point $z \in Y$, thus Y is complex analytic by Lemma I.4.23. Conversely, assume that Y is a $\mathbb{C}$-analytic submanifold and that $(e_1, e_3, \ldots, e_{2p-1})$ is an orthonormal complex basis of T_zY. If $e_{2k} := Je_{2k-1}$, then $(e_1, \ldots, e_{2p})$ is an orthonormal $\mathbb{R}$-basis corresponding to the canonical orientation and

$$\frac{1}{2}\omega_{\restriction Y} = \sum_{1\le k\le p} e^\star_{2k-1} \wedge e^\star_{2k}, \qquad \frac{1}{2^p p!}\omega^p_{\restriction Y} = e^\star_1 \wedge \ldots \wedge e^\star_{2p} = dV_Y. \qquad \square$$

Note that in the case of the standard hermitian metric ω on $X = \mathbb{C}^n$, the form $\omega = \mathrm{i}\sum dz_j \wedge d\overline{z}_j = d\big(\mathrm{i}\sum z_j\, d\overline{z}_j\big)$ is globally exact. Under this hypothesis, we are going to see that $\mathbb{C}$-analytic submanifolds are always *minimal surfaces* for the Plateau problem, which consists in finding a compact subvariety Y of minimal area with prescribed boundary ∂Y.

(1.26) Theorem. *Assume that the $(1,1)$-form ω is exact, say $\omega = d\gamma$ with $\gamma \in C^\infty_1(X, \mathbb{R})$, and let $Y, Z \subset X$ be $(2p)$-dimensional oriented compact real* *submanifolds of class C^1 with boundary. If $\partial Y = \partial Z$ and Z is complex analytic, then*

$$\mathrm{Vol}(Y) \geq \mathrm{Vol}(\mathrm{Z}).$$

Proof. Write $\omega = d\gamma$. Wirtinger's inequality and Stokes' theorem imply

$$\begin{aligned}\mathrm{Vol}(Y) &\geq \frac{1}{2^p p!}\Big|\int_Y \omega^p\Big| = \frac{1}{2^p p!}\Big|\int_Y d(\omega^{p-1}\wedge\gamma)\Big| = \frac{1}{2^p p!}\Big|\int_{\partial Y}\omega^{p-1}\wedge\gamma\Big|,\\ \mathrm{Vol}(Z) &= \frac{1}{2^p p!}\int_Z \omega^p = \frac{1}{2^p p!}\int_{\partial Z}\omega^{p-1}\wedge\gamma = \pm\frac{1}{2^p p!}\int_{\partial Y}\omega^{p-1}\wedge\gamma. \qquad \square\end{aligned}$$

2. Closed Positive Currents

2.A. The Skoda-El Mir Extension Theorem

We first prove the Skoda-El Mir extension theorem (Skoda 1982, El Mir 1984), which shows in particular that a closed positive current defined in the complement of an analytic set E can be extended through E if (and only if) the mass of the current is locally finite near E. El Mir simplified Skoda's argument and showed that it is enough to assume E complete pluripolar. We follow here the exposition of Sibony's survey article (Sibony 1985).

(2.1) Definition. *A subset $E \subset X$ is said to be complete pluripolar in X if for every point $x_0 \in X$ there exist a neighborhood Ω of x_0 and a function $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ such that $E \cap \Omega = \{z \in \Omega\ ;\ u(z) = -\infty\}$.*

Note that any closed analytic subset $A \subset X$ is complete pluripolar: if $g_1 = \ldots = g_N = 0$ are holomorphic equations of A on an open set $\Omega \subset X$, we can take $u = \log(|g_1|^2 + \ldots + |g_N|^2)$.

(2.2) Lemma. *Let $E \subset X$ be a closed complete pluripolar set. If $x_0 \in X$ and Ω is a sufficiently small neighborhood of x_0, there exists:*

a) *a function $v \in \mathrm{Psh}(\Omega) \cap C^\infty(\Omega \smallsetminus E)$ such that $v = -\infty$ on $E \cap \Omega$;*

b) *an increasing sequence $v_k \in \mathrm{Psh}(\Omega) \cap C^\infty(\Omega)$, $0 \leq v_k \leq 1$, converging uniformly to 1 on every compact subset of $\Omega \smallsetminus E$, such that $v_k = 0$ on a neighborhood of $E \cap \Omega$.*

Proof. Assume that $\Omega_0 \subset\subset X$ is a coordinate patch of X containing x_0 and that $E \cap \Omega_0 = \{z \in \Omega_0\ ;\ u(z) = -\infty\}$, $u \in \mathrm{Psh}(\Omega_0)$. In addition, we can achieve $u \leq 0$ by shrinking Ω_0 and subtracting a constant to u. Select a convex increasing function $\chi \in C^\infty([0,1],\mathbb{R})$ such that $\chi(t) = 0$ on $[0,1/2]$ and $\chi(1) = 1$. We set

$$u_k = \chi\big(\exp(u/k)\big).$$

Then $0 \le u_k \le 1$, u_k is plurisubharmonic on Ω_0, $u_k = 0$ in a neighborhood ω_k of $E \cap \Omega_0$ and $\lim u_k = 1$ on $\Omega_0 \smallsetminus E$. Let $\Omega \subset\subset \Omega_0$ be a neighborhood of x_0, let $\delta_0 = d(\Omega, \complement\Omega_0)$ and $\varepsilon_k \in\,]0,\delta_0[$ be such that $\varepsilon_k < d(E \cap \overline{\Omega}, \overline{\Omega} \smallsetminus \omega_k)$. Then

$$w_k := \max_{j \le k}\{u_j \star \rho_{\varepsilon_j}\} \in \mathrm{Psh}(\Omega) \cap C^0(\Omega),$$

$0 \le w_k \le 1$, $w_k = 0$ on a neighborhood of $E \cap \Omega$ and w_k is an increasing sequence converging to 1 on $\Omega \smallsetminus E$ (note that $w_k \ge u_k$). Hence, the convergence is uniform on every compact subset of $\Omega \smallsetminus E$ by Dini's lemma. We may therefore choose a subsequence w_{k_s} such that $w_{k_s}(z) \ge 1 - 2^{-s}$ on an increasing sequence of open sets Ω'_s with $\bigcup \Omega'_s = \Omega \smallsetminus E$. Then

$$w(z) := |z|^2 + \sum_{s=0}^{+\infty}(w_{k_s}(z) - 1)$$

is a strictly plurisubharmonic function on Ω that is continuous on $\Omega \smallsetminus E$, and $w = -\infty$ on $E \cap \Omega$. Richberg's theorem I.3.40 applied on $\Omega \smallsetminus E$ produces $v \in \mathrm{Psh}(\Omega \smallsetminus E) \cap C^\infty(\Omega \smallsetminus E)$ such that $w \le v \le w + 1$. If we set $v = -\infty$ on $E \cap \Omega$, then v is plurisubharmonic on Ω and has the properties required in a). After subtraction of a constant, we may assume $v \le 0$ on Ω. Then the sequence (v_k) of statement b) is obtained by letting $v_k = \chi\big(\exp(v/k)\big)$. □

(2.3) Theorem (El Mir). *Let $E \subset X$ be a closed complete pluripolar set and $T \in \mathcal{D}'^{+}_{p,p}(X \smallsetminus E)$ a closed positive current. Assume that T has finite mass in a neighborhood of every point of E. Then the trivial extension $\tilde{T} \in \mathcal{D}'^{+}_{p,p}(X)$ obtained by extending the measures $T_{I,J}$ by 0 on E is closed on X.*

Proof. The statement is local on X, so we may work on a small open set Ω such that there exists a sequence $v_k \in \mathrm{Psh}(\Omega) \cap C^\infty(\Omega)$ as in 2.2 b). Let $\theta \in C^\infty([0,1])$ be a function such that $\theta = 0$ on $[0,1/3]$, $\theta = 1$ on $[2/3,1]$ and $0 \le \theta \le 1$. Then $\theta \circ v_k = 0$ near $E \cap \Omega$ and $\theta \circ v_k = 1$ for k large on every fixed compact subset of $\Omega \smallsetminus E$. Therefore $\tilde{T} = \lim_{k\to+\infty}(\theta \circ v_k)T$ and

$$d'\tilde{T} = \lim_{k\to+\infty} T \wedge d'(\theta \circ v_k)$$

in the weak topology of currents. It is therefore sufficient to check that $T \wedge d'(\theta \circ v_k)$ converges weakly to 0 in $\mathcal{D}'_{p-1,p}(\Omega)$ (note that $d''\tilde{T}$ is conjugate to $d'\tilde{T}$, thus $d''\tilde{T}$ will also vanish).

Assume first that $p = 1$. Then $T \wedge d'(\theta \circ v_k) \in \mathcal{D}'_{0,1}(\Omega)$, and we have to show that

$$\langle T \wedge d'(\theta \circ v_k), \overline{\alpha} \rangle = \langle T, \theta'(v_k) d'v_k \wedge \overline{\alpha} \rangle \longrightarrow 0, \qquad \forall \alpha \in \mathcal{D}_{1,0}(\Omega).$$

As $\gamma \longmapsto \langle T, \mathrm{i}\gamma \wedge \overline{\gamma} \rangle$ is a non-negative hermitian form on $\mathcal{D}_{1,0}(\Omega)$, the Cauchy-Schwarz inequality yields

$$\big|\langle T, \mathrm{i}\beta \wedge \overline{\gamma} \rangle\big|^2 \le \langle T, \mathrm{i}\beta \wedge \overline{\beta} \rangle \; \langle T, \mathrm{i}\gamma \wedge \overline{\gamma} \rangle, \qquad \forall \beta, \gamma \in \mathcal{D}_{1,0}(\Omega).$$

Let $\psi \in \mathcal{D}(\Omega)$, $0 \le \psi \le 1$, be equal to 1 in a neighborhood of $\operatorname{Supp} \alpha$. We find

$$\big|\langle T, \theta'(v_k) d'v_k \wedge \overline{\alpha} \rangle\big|^2 \le \langle T, \psi \mathrm{i} d'v_k \wedge d''v_k \rangle \; \langle T, \theta'(v_k)^2 \mathrm{i}\alpha \wedge \overline{\alpha} \rangle.$$

By hypothesis $\int_{\Omega \smallsetminus E} T \wedge \mathrm{i}\alpha \wedge \overline{\alpha} < +\infty$ and $\theta'(v_k)$ converges everywhere to 0 on Ω, thus $\langle T, \theta'(v_k)^2 \mathrm{i}\alpha \wedge \overline{\alpha} \rangle$ converges to 0 by Lebesgue's dominated convergence theorem. On the other hand

$$\mathrm{i}d'd''v_k^2 = 2v_k \, \mathrm{i}d'd''v_k + 2\mathrm{i}d'v_k \wedge d''v_k \ge 2\mathrm{i}d'v_k \wedge d''v_k,$$
$$2\langle T, \psi \mathrm{i} d'v_k \wedge d''v_k \rangle \le \langle T, \psi \mathrm{i} d'd''v_k^2 \rangle.$$

As $\psi \in \mathcal{D}(\Omega)$, $v_k = 0$ near E and $d'T = d''T = 0$ on $\Omega \smallsetminus E$, an integration by parts yields

$$\langle T, \psi \mathrm{i} d'd''v_k^2 \rangle = \langle T, v_k^2 \mathrm{i} d'd''\psi \rangle \le C \int_{\Omega \smallsetminus E} \|T\| < +\infty$$

where C is a bound for the coefficients of ψ. Thus $\langle T, \psi \mathrm{i} d'v_k \wedge d''v_k \rangle$ is bounded, and the proof is complete when $p = 1$.

In the general case, let $\beta_s = \mathrm{i}\beta_{s,1} \wedge \overline{\beta}_{s,1} \wedge \ldots \wedge \mathrm{i}\beta_{s,p-1} \wedge \overline{\beta}_{s,p-1}$ be a basis of forms of bidegree $(p-1, p-1)$ with constant coefficients (Lemma 1.4). Then $T \wedge \beta_s \in \mathcal{D}'^{+}_{1,1}(\Omega \smallsetminus E)$ has finite mass near E and is closed on $\Omega \smallsetminus E$. Therefore $d(\tilde{T} \wedge \beta_s) = (d\tilde{T}) \wedge \beta_s = 0$ on Ω for all s, and we conclude that $d\tilde{T} = 0$. □

(2.4) Corollary. *If $T \in \mathcal{D}'^{+}_{p,p}(X)$ is closed, if $E \subset X$ is a closed complete pluripolar set and $\mathbb{1}_E$ is its characteristic function, then $\mathbb{1}_E T$ and $\mathbb{1}_{X \smallsetminus E} T$ are closed (and, of course, positive).*

Proof. If we set $\Theta = T_{\restriction X \smallsetminus E}$, then Θ has finite mass near E and we have $\mathbb{1}_{X \smallsetminus E} T = \tilde{\Theta}$ and $\mathbb{1}_E T = T - \tilde{\Theta}$. □

2.B. Current of Integration over an Analytic Set

Let A be a pure p-dimensional analytic subset of a complex manifold X. We would like to generalize Example 1.20 and to define a current of integration $[A]$ by letting

$$(2.5) \quad \langle [A], v \rangle = \int_{A_{\mathrm{reg}}} v, \qquad v \in \mathcal{D}_{p,p}(X).$$

One difficulty is of course to verify that the integral converges near $A_{\rm sing}$. This follows from the following lemma, due to (Lelong 1957).

(2.6) Lemma. *The current* $[A_{\rm reg}] \in \mathcal{D}'^{+}_{p,p}(X \smallsetminus A_{\rm sing})$ *has finite mass in a neighborhood of every point* $z_0 \in A_{\rm sing}$.

Proof. Set $T = [A_{\rm reg}]$ and let $\Omega \ni z_0$ be a coordinate open set. If we write the monomials $dz_K \wedge d\overline{z}_L$ in terms of an arbitrary basis of $\Lambda^{p,p}T^\star\Omega$ consisting of decomposable forms $\beta_s = \mathrm{i}\beta_{s,1} \wedge \overline{\beta}_{s,1} \wedge \ldots \wedge \beta_{s,p} \wedge \overline{\beta}_{s,p}$ (cf. Lemma 1.4), we see that the measures $T_{I,J} \,.\, \tau$ are linear combinations of the positive measures $T \wedge \beta_s$. It is thus sufficient to prove that all $T \wedge \beta_s$ have finite mass near $A_{\rm sing}$. Without loss of generality, we may assume that (A, z_0) is irreducible. Take new coordinates $w = (w_1, \ldots, w_n)$ such that $w_j = \beta_{s,j}(z - z_0)$, $1 \le j \le p$. After a slight perturbation of the $\beta_{s,j}$, we may assume that each projection

$$\pi_s : A \cap (\Delta' \times \Delta''), \qquad w \longmapsto w' = (w_1, \ldots, w_p)$$

defines a ramified covering of A (cf. Prop. II.3.8 and Th. II.3.19), and that (β_s) remains a basis of $\Lambda^{p,p}T^\star\Omega$. Let S be the ramification locus of π_s and $A_S = A \cap \big((\Delta' \smallsetminus S) \times \Delta''\big) \subset A_{\rm reg}$. The restriction of π_s: $A_S \longrightarrow \Delta' \smallsetminus S$ is then a covering with finite sheet number q_s and we find

$$\begin{aligned}\int_{\Delta' \times \Delta''} [A_{\rm reg}] \wedge \beta_s &= \int_{A_{\rm reg} \cap (\Delta' \times \Delta'')} \mathrm{i}dw_1 \wedge d\overline{w}_1 \wedge \ldots \wedge \mathrm{i}dw_p \wedge d\overline{w}_p \\ &= \int_{A_S} \mathrm{i}dw_1 \wedge d\overline{w}_1 \ldots \wedge d\overline{w}_p = q_s \int_{\Delta' \smallsetminus S} \mathrm{i}dw_1 \wedge d\overline{w}_1 \ldots \wedge d\overline{w}_p < +\infty.\end{aligned}$$

The second equality holds because A_S is the complement in $A_{\rm reg} \cap (\Delta' \times \Delta'')$ of an analytic subset (such a set is of zero Lebesgue measure in $A_{\rm reg}$). □

(2.7) Theorem (Lelong, 1957)**.** *For every pure p-dimensional analytic subset $A \subset X$, the current of integration* $[A] \in \mathcal{D}'^{+}_{p,p}(X)$ *is a closed positive current on X.*

Proof. Indeed, $[A_{\rm reg}]$ has finite mass near $A_{\rm sing}$ and $[A]$ is the trivial extension of $[A_{\rm reg}]$ to X through the complete pluripolar set $E = A_{\rm sing}$. Theorem 2.7 is then a consequence of El Mir's theorem. □

2.C. Support Theorems and Lelong-Poincaré Equation

Let $M \subset X$ be a closed C^1 real submanifold of X. The *holomorphic tangent space* at a point $x \in M$ is

(2.8) $\quad {}^hT_xM = T_xM \cap JT_xM,$

that is, the largest complex subspace of T_xX contained in T_xM. We define the *Cauchy-Riemann dimension* of M at x by $\mathrm{CRdim}_x M = \dim_{\mathbb{C}} {}^hT_xM$ and say that M is a *CR submanifold* of X if $\mathrm{CRdim}_x M$ is a constant. In general, we set

$$\text{(2.9)}\quad \mathrm{CRdim}\, M = \max_{x\in M}\ \mathrm{CRdim}_x M = \max_{x\in M}\ \dim_{\mathbb{C}} {}^hT_x M.$$

A current Θ is said to be *normal* if Θ and $d\Theta$ are currents of order 0. For instance, every closed positive current is normal. We are going to prove two important theorems describing the structure of normal currents with support in CR submanifolds.

(2.10) First theorem of support. *Let $\Theta \in \mathcal{D}'_{p,p}(X)$ be a normal current. If* $\mathrm{Supp}\,\Theta$ *is contained in a real submanifold M of CR dimension $< p$, then $\Theta = 0$.*

Proof. Let $x_0 \in M$ and let $g_1, \ldots, g_m$ be real C^1 functions in a neighborhood Ω of x_0 such that $M = \{z \in \Omega\ ;\ g_1(z) = \ldots = g_m(z) = 0\}$ and $dg_1 \wedge \ldots \wedge dg_m \neq 0$ on Ω. Then

$${}^hTM = TM \cap JTM = \bigcap_{1\le k\le m} \ker dg_k \cap \ker(dg_k \circ J) = \bigcap_{1\le k\le m} \ker d'g_k$$

because $d'g_k = \frac{1}{2}\big(dg_k - i(dg_k)\circ J\big)$. As $\dim_{\mathbb{C}} {}^hTM < p$, the rank of the system of $(1,0)$-forms $(d'g_k)$ must be $> n-p$ at every point of $M \cap \Omega$. After a change of the ordering, we may assume for example that $\zeta_1 = d'g_1$, $\zeta_2 = d'g_2$, ..., $\zeta_{n-p+1} = d'g_{n-p+1}$ are linearly independent on Ω (shrink Ω if necessary). Complete $(\zeta_1, \ldots, \zeta_{n-p+1})$ into a continuous frame $(\zeta_1, \ldots, \zeta_n)$ of $T^\star X_{\restriction\Omega}$ and set

$$\Theta = \sum_{|I|=|J|=n-p} \Theta_{I,J}\, \zeta_I \wedge \overline{\zeta}_J \quad \text{on} \quad \Omega.$$

As Θ and $d'\Theta$ have measure coefficients supported on M and $g_k = 0$ on M, we get $g_k\Theta = g_k d'\Theta = 0$, thus

$$d'g_k \wedge \Theta = d'(g_k\Theta) - g_k d'\Theta = 0, \qquad 1 \le k \le m,$$

in particular $\zeta_k \wedge \Theta = 0$ for all $1 \le k \le n-p+1$. When $|I| = n-p$, the multi-index $\complement I$ contains at least one of the elements $1, \ldots, n-p+1$, hence $\Theta \wedge \zeta_{\complement I} \wedge \overline{\zeta}_{\complement J} = 0$ and $\Theta_{I,J} = 0$. □

(2.11) Corollary. *Let $\Theta \in \mathcal{D}'_{p,p}(X)$ be a normal current. If* $\mathrm{Supp}\,\Theta$ *is contained in an analytic subset A of dimension $< p$, then $\Theta = 0$.*

Proof. As A_{reg} is a submanifold of CRdim $< p$ in $X \smallsetminus A_{\mathrm{sing}}$, Theorem 2.9 shows that $\mathrm{Supp}\,\Theta \subset A_{\mathrm{sing}}$ and we conclude by induction on $\dim A$. □

Now, assume that $M \subset X$ is a CR submanifold of class C^1 with $\mathrm{CRdim}\, M = p$ and that ${}^hTM$ is an integrable subbundle of TM ; this means that the Lie bracket of two vector fields in ${}^hTM$ is in ${}^hTM$. The Frobenius integrability theorem then shows that M is locally fibered by complex analytic p-dimensional submanifolds. More precisely, in a neighborhood of every point of M, there is a submersion $\sigma : M \longrightarrow Y$ onto a real C^1 manifold Y such that the tangent space to each fiber $F_t = \sigma^{-1}(t)$, $t \in Y$, is the holomorphic tangent space ${}^hTM$; moreover, the fibers F_t are necessarily complex analytic in view of Lemma 1.7.18. Under these assumptions, with any complex measure μ on Y we associate a current Θ with support in M by

$$(2.12) \qquad \Theta = \int_{t \in Y} [F_t]\, d\mu(t), \qquad \text{i.e. } \langle \Theta, u \rangle = \int_{t \in Y} \Big(\int_{F_t} u \Big)\, d\mu(t)$$

for all $u \in \mathcal{D}'_{p,p}(X)$. Then clearly $\Theta \in \mathcal{D}'_{p,p}(X)$ is a closed current of order 0, for all fibers $[F_t]$ have the same properties. When the fibers F_t are connected, the following converse statement holds:

(2.13) Second theorem of support. *Let $M \subset X$ be a CR submanifold of CR dimension p such that there is a submersion $\sigma : M \longrightarrow Y$ of class C^1 whose fibers $F_t = \sigma^{-1}(t)$ are connected and are the integral manifolds of the holomorphic tangent space ${}^hTM$. Then any closed current $\Theta \in \mathcal{D}'_{p,p}(X)$ of order 0 with support in M can be written $\Theta = \int_Y [F_t]\, d\mu(t)$ with a unique complex measure μ on Y. Moreover Θ is (strongly) positive if and only if the measure μ is positive.*

Proof. Fix a compact set $K \subset Y$ and a C^1 retraction ρ from a neighborhood V of M onto M. By means of a partition of unity, it is easy to construct a positive form $\alpha \in \mathcal{D}^0_{p,p}(V)$ such that $\int_{F_t} \alpha = 1$ for each fiber F_t with $t \in K$. Then the uniqueness and positivity statements for μ follow from the obvious formula

$$\int_Y f(t)\, d\mu(t) = \langle \Theta, (f \circ \rho)\, \alpha \rangle, \qquad \forall f \in C^0(Y), \quad \mathrm{Supp}\, f \subset K.$$

Now, let us prove the existence of μ. Let $x_0 \in M$. There is a small neighborhood Ω of x_0 and real coordinates $(x_1, y_1, \ldots, x_p, y_p, t_1, \ldots, t_q, g_1, \ldots, g_m)$ such that

- $z_j = x_j + \mathrm{i} y_j$, $1 \le j \le p$, are holomorphic functions on Ω that define complex coordinates on all fibers $F_t \cap \Omega$.
- $t_1, \ldots, t_q$ restricted to $M \cap \Omega$ are pull-backs by $\sigma : M \to Y$ of local coordinates on an open set $U \subset Y$ such that $\sigma_{\restriction \Omega} : M \cap \Omega \longrightarrow U$ is proper and surjective.
- $g_1 = \ldots = g_m = 0$ are equations of M in Ω.

Then $TF_t = \{dt_j = dg_k = 0\}$ equals ${}^hTM = \{d' g_k = 0\}$ and the rank of $(d' g_1, \ldots, d' g_m)$ is equal to $n - p$ at every point of $M \cap \Omega$. After a change of the ordering we may suppose that $\zeta_1 = d'g_1, \ldots, \zeta_{n-p} = d'g_{n-p}$ are linearly independent on Ω. As in Prop. 2.10, we get $\zeta_k \wedge \Theta = \overline{\zeta}_k \wedge \Theta = 0$ for $1 \le k \le n-p$ and infer that $\Theta \wedge \zeta_{\complement I} \wedge \overline{\zeta}_{\complement J} = 0$ unless $I = J = L$ where $L = \{1, 2, \ldots, n-p\}$. Hence

$$\Theta = \Theta_{L,L}\ \zeta_1 \wedge \ldots \wedge \zeta_{n-p} \wedge \overline{\zeta}_1 \wedge \ldots \wedge \overline{\zeta}_{n-p} \quad \text{on} \quad \Omega.$$

Now $\zeta_1 \wedge \ldots \wedge \overline{\zeta}_{n-p}$ is proportional to $dt_1 \wedge \ldots dt_q \wedge dg_1 \wedge \ldots \wedge dg_m$ because both induce a volume form on the quotient space $TX_{\restriction M}/{}^hTM$. Therefore, there is a complex measure ν supported on $M \cap \Omega$ such that

$$\Theta = \nu\ dt_1 \wedge \ldots dt_q \wedge dg_1 \wedge \ldots \wedge dg_m \quad \text{on} \quad \Omega.$$

As Θ is supposed to be closed, we have $\partial\nu/\partial x_j = \partial\nu/\partial y_j = 0$. Hence ν is a measure depending only on (t, g), with support in $g = 0$. We may write $\nu = d\mu_U(t) \otimes \delta_0(g)$ where μ_U is a measure on $U = \sigma(M \cap \Omega)$ and δ_0 is the Dirac measure at 0. If $j : M \longrightarrow X$ is the injection, this means precisely that $\Theta = j_\star \sigma^\star \mu_U$ on Ω, i.e.

$$\Theta = \int_{t \in U} [F_t]\, d\mu_U(t) \quad \text{on} \quad \Omega.$$

The uniqueness statement shows that for two open sets Ω_1, Ω_2 as above, the associated measures μ_{U_1} and μ_{U_2} coincide on $\sigma(M \cap \Omega_1 \cap \Omega_2)$. Since the fibers F_t are connected, there is a unique measure μ which coincides with all measures μ_U. □

(2.14) Corollary. *Let A be an analytic subset of X with global irreducible components A_j of pure dimension p. Then any closed current $\Theta \in \mathcal{D}'_{p,p}(X)$ of order 0 with support in A is of the form $\Theta = \sum \lambda_j [A_j]$ where $\lambda_j \in \mathbb{C}$. Moreover, Θ is (strongly) positive if and only if all coefficients λ_j are ≥ 0.*

Proof. The regular part $M = A_{\rm reg}$ is a complex submanifold of $X \smallsetminus A_{\rm sing}$ and its connected components are $A_j \cap A_{\rm reg}$. Thus, we may apply Th. 2.13 in the case where Y is discrete to see that $\Theta = \sum \lambda_j [A_j]$ on $X \smallsetminus A_{\rm sing}$. Now $\dim A_{\rm sing} < p$ and the difference $\Theta - \sum \lambda_j [A_j] \in \mathcal{D}'_{p,p}(X)$ is a closed current of order 0 with support in $A_{\rm sing}$, so this current must vanish by Cor. 2.11. □

(2.15) Lelong-Poincaré equation. *Let $f \in \mathcal{M}(X)$ be a meromorphic function which does not vanish identically on any connected component of X and let $\sum m_j Z_j$ be the divisor of f. Then the function $\log|f|$ is locally integrable on X and satisfies the equation*

$$\frac{\mathrm{i}}{\pi} d'd'' \log|f| = \sum\ m_j\, [Z_j]$$

in the space $\mathcal{D}'_{n-1,n-1}(X)$ of currents of bidimension $(n-1, n-1)$.

We refer to Sect. 2.6 for the definition of divisors, and especially to (2.6.14). Observe that if f is holomorphic, then $\log|f| \in \mathrm{Psh}(X)$, the coefficients m_j are positive integers and the right hand side is a positive current in $\mathcal{D}'^{+}_{n-1,n-1}(X)$.

Proof. Let $Z = \bigcup Z_j$ be the support of $\mathrm{div}(f)$. Observe that the sum in the right hand side is locally finite and that $d'd'' \log|f|$ is supported on Z, since

$$d' \log|f|^2 = d' \log(f\overline{f}) = \frac{\overline{f}\,df}{f\overline{f}} = \frac{df}{f} \quad \text{on} \quad X \smallsetminus Z.$$

In a neighborhood Ω of a point $a \in Z_j \cap Z_{\mathrm{reg}}$, we can find local coordinates $(w_1, \ldots, w_n)$ such that $Z_j \cap \Omega$ is given by the equation $w_1 = 0$. Then Th. 2.6.6 shows that f can be written $f(w) = u(w)w_1^{m_j}$ with an invertible holomorphic function u on a smaller neighborhood $\Omega' \subset \Omega$. Then we have

$$\mathrm{i}d'd'' \log|f| = \mathrm{i}d'd''\big(\log|u| + m_j \log|w_1|\big) = m_j\,\mathrm{i}d'd'' \log|w_1|.$$

For $z \in \mathbb{C}$, Cor. I.3.4 implies

$$\mathrm{i}d'd'' \log|z|^2 = -\mathrm{i}d''\Big(\frac{dz}{z}\Big) = -\mathrm{i}\pi\delta_0\, d\overline{z} \wedge dz = 2\pi\,[0].$$

If $\Phi : \mathbb{C}^n \longrightarrow \mathbb{C}$ is the projection $z \longmapsto z_1$ and $H \subset \mathbb{C}^n$ the hyperplane $\{z_1 = 0\}$, formula (1.2.19) shows that

$$\mathrm{i}d'd'' \log|z_1| = \mathrm{i}d'd'' \log|\Phi(z)| = \Phi^\star(\mathrm{i}d'd'' \log|z|) = \pi\Phi^\star([0]) = \pi\,[H],$$

because Φ is a submersion. We get therefore $\frac{\mathrm{i}}{\pi}d'd'' \log|f| = m_j[Z_j]$ in Ω'. This implies that the Lelong-Poincaré equation is valid at least on $X \smallsetminus Z_{\mathrm{sing}}$. As $\dim Z_{\mathrm{sing}} < n-1$, Cor. 2.11 shows that the equation holds everywhere on X. □

3. Definition of Monge-Ampère Operators

Let X be a n-dimensional complex manifold. We denote by $d = d' + d''$ the usual decomposition of the exterior derivative in terms of its $(1,0)$ and $(0,1)$ parts, and we set

$$d^c = \frac{1}{2\mathrm{i}\pi}(d' - d'').$$

It follows in particular that d^c is a real operator, i.e. $\overline{d^c u} = d^c\overline{u}$, and that $dd^c = \frac{\mathrm{i}}{\pi}d'd''$. Although not quite standard, the $1/2\mathrm{i}\pi$ normalization is very convenient for many purposes, since we may then forget the factor 2π almost everywhere (e.g. in the Lelong-Poincaré equation (2.15)). In this context, we have the following integration by part formula.

(3.1) Formula. *Let $\Omega \subset\subset X$ be a smoothly bounded open set in X and let f, g be forms of class C^2 on $\overline{\Omega}$ of pure bidegrees (p,p) and (q,q) with $p+q=n-1$. Then*

$$\int_\Omega f \wedge dd^c g - dd^c f \wedge g = \int_{\partial\Omega} f \wedge d^c g - d^c f \wedge g.$$

Proof. By Stokes' theorem the right hand side is the integral over Ω of

$$d(f \wedge d^c g - d^c f \wedge g) = f \wedge dd^c g - dd^c f \wedge g + (df \wedge d^c g + d^c f \wedge dg).$$

As all forms of total degree $2n$ and bidegree $\neq (n,n)$ are zero, we get

$$df \wedge d^c g = \frac{1}{2i\pi}(d'' f \wedge d' g - d' f \wedge d'' g) = -d^c f \wedge dg. \qquad \square$$

Let u be a plurisubharmonic function on X and let T be a closed positive current of bidimension (p,p), i.e. of bidegree $(n-p, n-p)$. Our desire is to define the wedge product $dd^c u \wedge T$ even when neither u nor T are smooth. A priori, this product does not make sense because $dd^c u$ and T have measure coefficients and measures cannot be multiplied; see (Kiselman 1983) for interesting counterexamples. Assume however that u is a *locally bounded* plurisubharmonic function. Then the current uT is well defined since u is a locally bounded Borel function and T has measure coefficients. According to (Bedford-Taylor 1982) we define

$$dd^c u \wedge T = dd^c(uT)$$

where $dd^c(\ \ )$ is taken in the sense of distribution (or current) theory.

(3.2) Proposition. *The wedge product $dd^c u \wedge T$ is again a closed positive current.*

Proof. The result is local. In an open set $\Omega \subset \mathbb{C}^n$, we can use convolution with a family of regularizing kernels to find a decreasing sequence of smooth plurisubharmonic functions $u_k = u \star \rho_{1/k}$ converging pointwise to u. Then $u \le u_k \le u_1$ and Lebesgue's dominated convergence theorem shows that $u_k T$ converges weakly to uT; thus $dd^c(u_k T)$ converges weakly to $dd^c(uT)$ by the weak continuity of differentiations. However, since u_k is smooth, $dd^c(u_k T)$ coincides with the product $dd^c u_k \wedge T$ in its usual sense. As $T \ge 0$ and as $dd^c u_k$ is a positive $(1,1)$-form, we have $dd^c u_k \wedge T \ge 0$, hence the weak limit $dd^c u \wedge T$ is ≥ 0 (and obviously closed). $\square$

Given locally bounded plurisubharmonic functions $u_1, \ldots, u_q$, we define inductively

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T = dd^c(u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T).$$

By (3.2) the product is a closed positive current. In particular, when u is a locally bounded plurisubharmonic function, the bidegree (n,n) current $(dd^c u)^n$ is well defined and is a positive measure. If u is of class C^2, a computation in local coordinates gives

$$(dd^c u)^n = \det\Big(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\Big) \cdot \frac{n!}{\pi^n}\, \mathrm{i}dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge \mathrm{i}dz_n \wedge d\overline{z}_n.$$

The expression "Monge-Ampère operator" classically refers to the non-linear partial differential operator $u \longmapsto \det(\partial^2 u/\partial z_j \partial \overline{z}_k)$. By extension, all operators $(dd^c)^q$ defined above are also called Monge-Ampère operators.

Now, let Θ be a current of order 0. When $K \subset\subset X$ is an arbitrary compact subset, we define a *mass* semi-norm

$$||\Theta||_K = \sum_j \int_{K_j} \sum_{I,J} |\Theta_{I,J}|$$

by taking a partition $K = \bigcup K_j$ where each $\overline{K_j}$ is contained in a coordinate patch and where $\Theta_{I,J}$ are the corresponding measure coefficients. Up to constants, the semi-norm $||\Theta||_K$ does not depend on the choice of the coordinate systems involved. When K itself is contained in a coordinate patch, we set $\beta = dd^c|z|^2$ over K ; then, if $\Theta \geq 0$, there are constants $C_1, C_2 > 0$ such that

$$C_1||\Theta||_K \leq \int_K \Theta \wedge \beta^p \leq C_2||\Theta||_K.$$

We denote by $L^1(K)$, resp. by $L^\infty(K)$, the space of integrable (resp. bounded measurable) functions on K with respect to any smooth positive density on X.

(3.3) Chern-Levine-Nirenberg inequalities (1969). *For all compact subsets K, L of X with $L \subset K^\circ$, there exists a constant $C_{K,L} \geq 0$ such that*

$$||dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T||_L \leq C_{K,L}\ ||u_1||_{L^\infty(K)} \ldots ||u_q||_{L^\infty(K)}\, ||T||_K.$$

Proof. By induction, it is sufficient to prove the result for $q = 1$ and $u_1 = u$. There is a covering of L by a family of balls $B'_j \subset\subset B_j \subset K$ contained in coordinate patches of X. Let $\chi \in \mathcal{D}(B_j)$ be equal to 1 on $\overline{B}'_j$. Then

$$||dd^c u \wedge T||_{L\cap\overline{B}'_j} \leq C \int_{\overline{B}'_j} dd^c u \wedge T \wedge \beta^{p-1} \leq C \int_{B_j} \chi\, dd^c u \wedge T \wedge \beta^{p-1}.$$

As T and β are closed, an integration by parts yields

$$||dd^c u \wedge T||_{L\cap\overline{B}'_j} \leq C \int_{B_j} u\, T \wedge dd^c\chi \wedge \beta^{p-1} \leq C'||u||_{L^\infty(K)}||T||_K$$

where C' is equal to C multiplied by a bound for the coefficients of the smooth form $dd^c\chi \wedge \beta^{p-1}$. □

(3.4) Remark. With the same notations as above, any plurisubharmonic function V on X satisfies inequalities of the type

a) $||dd^cV||_L \le C_{K,L}\,||V||_{L^1(K)}$.

b) $\sup_L V_+ \le C_{K,L}\,||V||_{L^1(K)}$.

In fact the inequality

$$\int_{L\cap\overline{B}'_j} dd^cV \wedge \beta^{n-1} \le \int_{B_j} \chi dd^cV \wedge \beta^{n-1} = \int_{B_j} V dd^c\chi \wedge \beta^{n-1}$$

implies a), and b) follows from the mean value inequality.

(3.5) Remark. Products of the form $\Theta = \gamma_1 \wedge \ldots \wedge \gamma_q \wedge T$ with mixed $(1,1)$-forms $\gamma_j = dd^cu_j$ or $\gamma_j = dv_j \wedge d^cw_j + dw_j \wedge d^cv_j$ are also well defined whenever u_j, v_j, w_j are locally bounded plurisubharmonic functions. Moreover, for $L \subset K^\circ$, we have

$$||\Theta||_L \le C_{K,L}||T||_K \prod ||u_j||_{L^\infty(K)} \prod ||v_j||_{L^\infty(K)} \prod ||w_j||_{L^\infty(K)}.$$

To check this, we may suppose $v_j, w_j \ge 0$ and $||v_j|| = ||w_j|| = 1$ in $L^\infty(K)$. Then the inequality follows from (3.3) by the polarization identity

$$2(dv_j \wedge d^cw_j + dw_j \wedge d^cv_j) = dd^c(v_j + w_j)^2 - dd^cv_j^2 - dd^cw_j^2 - v_j dd^cw_j - w_j dd^cv_j$$

in which all dd^c operators act on plurisubharmonic functions.

(3.6) Corollary. *Let $u_1, \ldots, u_q$ be continuous (finite) plurisubharmonic functions and let $u_1^k, \ldots, u_q^k$ be sequences of plurisubharmonic functions converging locally uniformly to $u_1, \ldots, u_q$. If T_k is a sequence of closed positive currents converging weakly to T, then*

a) $u_1^k dd^cu_2^k \wedge \ldots \wedge dd^cu_q^k \wedge T_k \longrightarrow u_1 dd^cu_2 \wedge \ldots \wedge dd^cu_q \wedge T$ *weakly.*

b) $dd^cu_1^k \wedge \ldots \wedge dd^cu_q^k \wedge T_k \longrightarrow dd^cu_1 \wedge \ldots \wedge dd^cu_q \wedge T$ *weakly.*

Proof. We observe that b) is an immediate consequence of a) by the weak continuity of dd^c. By using induction on q, it is enough to prove result a) when $q = 1$. If (u^k) converges locally uniformly to a finite continuous plurisubharmonic function u, we introduce local regularizations $u_\varepsilon = u \star \rho_\varepsilon$ and write

$$u^kT_k - uT = (u^k - u)T_k + (u - u_\varepsilon)T_k + u_\varepsilon(T_k - T).$$

As the sequence T_k is weakly convergent, it is locally uniformly bounded in mass, thus $||(u^k - u)T_k||_K \le ||u^k - u||_{L^\infty(K)}||T_k||_K$ converges to 0 on every compact set K. The same argument shows that $||(u-u_\varepsilon)T_k||_K$ can be made arbitrarily small by choosing ε small enough. Finally u_ε is smooth, so $u_\varepsilon(T_k-T)$ converges weakly to 0. $\square$

Now, we prove a deeper monotone continuity theorem due to (Bedford-Taylor 1982) according to which the continuity and uniform convergence assumptions can be dropped if each sequence (u_j^k) is decreasing and T_k is a constant sequence.

(3.7) Theorem. *Let $u_1,\ldots,u_q$ be locally bounded plurisubharmonic functions and let $u_1^k,\ldots,u_q^k$ be decreasing sequences of plurisubharmonic functions converging pointwise to $u_1,\ldots,u_q$. Then*

a) $u_1^k dd^c u_2^k\wedge\ldots\wedge dd^c u_q^k\wedge T\longrightarrow u_1 dd^c u_2\wedge\ldots\wedge dd^c u_q\wedge T$ *weakly.*

b) $dd^c u_1^k\wedge\ldots\wedge dd^c u_q^k\wedge T\longrightarrow dd^c u_1\wedge\ldots\wedge dd^c u_q\wedge T$ *weakly.*

Proof. Again by induction, observing that a) $\Longrightarrow$ b) and that a) is obvious for $q=1$ thanks to Lebesgue's bounded convergence theorem. To proceed with the induction step, we first have to make some slight modifications of our functions u_j and u_j^k.

As the sequence (u_j^k) is decreasing and as u_j is locally bounded, the family $(u_j^k)_{k\in\mathbb{N}}$ is locally uniformly bounded. The results are local, so we can work on a Stein open set $\Omega\subset\subset X$ with strongly pseudoconvex boundary. We use the following notations:

(3.8) let ψ be a strongly plurisubharmonic function of class C^∞ near $\overline{\Omega}$ with $\psi<0$ on Ω and $\psi=0$, $d\psi\neq 0$ on $\partial\Omega$;

(3.8′) we set $\Omega_\delta=\{z\in\Omega\,;\,\psi(z)<-\delta\}$ for all $\delta>0$.

After addition of a constant we can assume that $-M\le u_j^k\le -1$ near $\overline{\Omega}$. Let us denote by $(u_j^{k,\varepsilon})$, $\varepsilon\in\,]0,\varepsilon_0]$, an increasing family of regularizations converging to u_j^k as $\varepsilon\to 0$ and such that $-M\le u_j^{k,\varepsilon}\le -1$ on $\overline{\Omega}$. Set $A=M/\delta$ with $\delta>0$ small and replace u_j^k by $v_j^k=\max\{A\psi,u_j^k\}$ and $u_j^{k,\varepsilon}$ by $v_j^{k,\varepsilon}=\max_\varepsilon\{A\psi,u_j^{k,\varepsilon}\}$ where $\max_\varepsilon=\max\star\rho_\varepsilon$ is a regularized max function. Then v_j^k coincides with u_j^k on Ω_δ since $A\psi<-A\delta=-M$ on Ω_δ, and v_j^k is equal to $A\psi$ on the corona $\Omega\setminus\Omega_{\delta/M}$. Without loss of generality, we can therefore assume that all u_j^k (and similarly all $u_j^{k,\varepsilon}$) coincide with $A\psi$ on a fixed neighborhood of $\partial\Omega$. We need a lemma.

(3.9) Lemma. *Let f_k be a decreasing sequence of upper semi-continuous functions converging to f on some separable locally compact space X and μ_k a sequence of positive measures converging weakly to μ on X. Then every weak limit ν of $f_k\mu_k$ satisfies $\nu\le f\mu$.*

Indeed if (g_p) is a decreasing sequence of continuous functions converging to f_{k_0} for some k_0, then $f_k\mu_k\le f_{k_0}\mu_k\le g_p\mu_k$ for $k\ge k_0$, thus $\nu\le g_p\mu$

Fig. 1 Construction of v_j^k

as $k \to +\infty$. The monotone convergence theorem then gives $\nu \le f_{k_0}\mu$ as $p \to +\infty$ and $\nu \le f\mu$ as $k_0 \to +\infty$. $\square$

Proof of Theorem 3.7 (end). Assume that a) has been proved for $q-1$. Then

$$S^k = dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \longrightarrow S = dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T.$$

By 3.3 the sequence $(u_1^k S^k)$ has locally bounded mass, hence is relatively compact for the weak topology. In order to prove a), we only have to show that every weak limit Θ of $u_1^k S^k$ is equal to $u_1 S$. Let (m,m) be the bidimension of S and let γ be an arbitrary smooth and strongly positive form of bidegree (m,m). Then the positive measures $S^k \wedge \gamma$ converge weakly to $S \wedge \gamma$ and Lemma 3.9 shows that $\Theta \wedge \gamma \le u_1 S \wedge \gamma$, hence $\Theta \le u_1 S$. To get the equality, we set $\beta = dd^c\psi > 0$ and show that $\int_\Omega u_1 S \wedge \beta^m \le \int_\Omega \Theta \wedge \beta^m$, i.e.

$$\int_\Omega u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \le \liminf_{k\to+\infty} \int_\Omega u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \wedge \beta^m.$$

As $u_1 \le u_1^k \le u_1^{k,\varepsilon_1}$ for every $\varepsilon_1 > 0$, we get

$$\begin{aligned}
&\int_\Omega u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \\
&\qquad \le \int_\Omega u_1^{k,\varepsilon_1} dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \\
&\qquad = \int_\Omega dd^c u_1^{k,\varepsilon_1} \wedge u_2 dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m
\end{aligned}$$

after an integration by parts (there is no boundary term because u_1^{k,ε_1} and u_2 both vanish on $\partial\Omega$). Repeating this argument with $u_2, \ldots, u_q$, we obtain

$$\begin{aligned}\int_\Omega & u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \\ &\le \int_\Omega dd^c u_1^{k,\varepsilon_1} \wedge \ldots \wedge dd^c u_{q-1}^{k,\varepsilon_{q-1}} \wedge u_q T \wedge \beta^m \\ &\le \int_\Omega u_1^{k,\varepsilon_1} dd^c u_2^{k,\varepsilon_2} \wedge \ldots \wedge dd^c u_q^{k,\varepsilon_q} \wedge T \wedge \beta^m.\end{aligned}$$

Now let $\varepsilon_q \to 0, \ldots, \varepsilon_1 \to 0$ in this order. We have weak convergence at each step and $u_1^{k,\varepsilon_1} = 0$ on the boundary; therefore the integral in the last line converges and we get the desired inequality

$$\int_\Omega u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \beta^m \le \int_\Omega u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \wedge \beta^m. \square$$

(3.10) Corollary. *The product $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ is symmetric with respect to $u_1, \ldots, u_q$.*

Proof. Observe that the definition was unsymmetric. The result is true when $u_1, \ldots, u_q$ are smooth and follows in general from Th. 3.7 applied to the sequences $u_j^k = u_j \star \rho_{1/k}$, $1 \le j \le q$. $\square$

(3.11) Proposition. *Let K, L be compact subsets of X such that $L \subset K^\circ$. For any plurisubharmonic functions $V, u_1, \ldots, u_q$ on X such that $u_1, \ldots, u_q$ are locally bounded, there is an inequality*

$$||V dd^c u_1 \wedge \ldots \wedge dd^c u_q||_L \le C_{K,L}\, ||V||_{L^1(K)} ||u_1||_{L^\infty(K)} \ldots ||u_q||_{L^\infty(K)}.$$

Proof. We may assume that L is contained in a strongly pseudoconvex open set $\Omega = \{\psi < 0\} \subset K$ (otherwise we cover L by small balls contained in K). A suitable normalization gives $-2 \le u_j \le -1$ on K ; then we can modify u_j on $\Omega \setminus L$ so that $u_j = A\psi$ on $\Omega \setminus \Omega_\delta$ with a fixed constant A and $\delta > 0$ such that $L \subset \Omega_\delta$. Let $\chi \ge 0$ be a smooth function equal to $-\psi$ on Ω_δ with compact support in Ω. If we take $||V||_{L^1(K)} = 1$, we see that V_+ is uniformly bounded on Ω_δ by 3.4 b); after subtraction of a fixed constant we can assume $V \le 0$ on Ω_δ. First suppose $q \le n-1$. As $u_j = A\psi$ on $\Omega \setminus \Omega_\delta$, we find

$$\begin{aligned}\int_{\Omega_\delta} & -V\, dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q} \\ &= \int_\Omega V\, dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q-1} \wedge dd^c\chi - A^q \int_{\Omega\setminus\Omega_\delta} V\, \beta^{n-1} \wedge dd^c\chi \\ &= \int_\Omega \chi\, dd^c V \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge \beta^{n-q-1} - A^q \int_{\Omega\setminus\Omega_\delta} V\, \beta^{n-1} \wedge dd^c\chi.\end{aligned}$$

The first integral of the last line is uniformly bounded thanks to 3.3 and 3.4 a), and the second one is bounded by $||V||_{L^1(\Omega)} \le$ constant. Inequality

3.11 follows for $q \le n-1$. If $q = n$, we can work instead on $X \times \mathbb{C}$ and consider $V, u_1, \ldots, u_q$ as functions on $X \times \mathbb{C}$ independent of the extra variable

4. Case of Unbounded Plurisubharmonic Functions

We would like to define $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ also in some cases when $u_1, \ldots, u_q$ are not bounded below everywhere, especially when the u_j have logarithmic poles. Consider first the case $q = 1$ and let u be a plurisubharmonic function on X. The *pole set* of u is by definition $P(u) = u^{-1}(-\infty)$. We define the *unbounded locus* $L(u)$ to be the set of points $x \in X$ such that u is unbounded in every neighborhood of x. Clearly $L(u)$ is closed and we have $L(u) \supset \overline{P(u)}$ but in general these sets are different: in fact, $u(z) = \sum k^{-2} \log(|z - 1/k| + e^{-k^3})$ is everywhere finite in $\mathbb{C}$ but $L(u) = \{0\}$.

(4.1) Proposition. *We make two additional assumptions:*

a) *T has non zero bidimension (p, p) (i.e. degree of $T < 2n$).*

b) *X is covered by a family of Stein open sets $\Omega \subset\subset X$ whose boundaries $\partial\Omega$ do not intersect $L(u) \cap \operatorname{Supp} T$.*

Then the current uT has locally finite mass in X.

For any current T, hypothesis 4.1 b) is clearly satisfied when u has a discrete unbounded locus $L(u)$; an interesting example is $u = \log |F|$ where $F = (F_1, \ldots, F_N)$ are holomorphic functions having a discrete set of common zeros. Observe that the current uT need not have locally finite mass when T has degree $2n$ (i.e. T is a measure); example: $T = \delta_0$ and $u(z) = \log |z|$ in $\mathbb{C}^n$. The result also fails when the sets Ω are not assumed to be Stein; example: $X =$ blow-up of $\mathbb{C}^n$ at 0, $T = [E] =$ current of integration on the exceptional divisor and $u(z) = \log |z|$ (see § 7.12 for the definition of blow-ups).

Proof. By shrinking Ω slightly, we may assume that Ω has a smooth strongly pseudoconvex boundary. Let ψ be a defining function of Ω as in (3.8). By subtracting a constant to u, we may assume $u \le -\varepsilon$ on $\overline{\Omega}$. We fix δ so small that $\overline{\Omega} \smallsetminus \Omega_\delta$ does not intersect $L(u) \cap \operatorname{Supp} T$ and we select a neighborhood ω of $(\overline{\Omega} \smallsetminus \Omega_\delta) \cap \operatorname{Supp} T$ such that $\overline{\omega} \cap L(u) = \emptyset$. Then we define

$$u_s(z) = \begin{cases} \max\{u(z), A\psi(z)\} & \text{on } \omega, \\ \max\{u(z), s\} & \text{on } \Omega_\delta = \{\psi < -\delta\}. \end{cases}$$

By construction $u \ge -M$ on ω for some constant $M > 0$. We fix $A \ge M/\delta$ and take $s \le -M$, so

$$\max\{u(z), A\psi(z)\} = \max\{u(z), s\} = u(z) \quad \text{on } \omega \cap \Omega_\delta$$

and our definition of u_s is coherent. Observe that u_s is defined on $\omega \cup \Omega_\delta$, which is a neighborhood of $\overline{\Omega} \cap \operatorname{Supp} T$. Now, $u_s = A\psi$ on $\omega \cap (\Omega \smallsetminus \Omega_{\varepsilon/A})$, hence Stokes' theorem implies

$$\int_\Omega dd^c u_s \wedge T \wedge (dd^c\psi)^{p-1} - \int_\Omega A dd^c\psi \wedge T \wedge (dd^c\psi)^{p-1} = \int_\Omega dd^c\big[(u_s - A\psi) T \wedge (dd^c\psi)^{p-1}\big] = 0$$

because the current $[\ldots]$ has a compact support contained in $\overline{\Omega}_{\varepsilon/A}$. Since u_s and ψ both vanish on $\partial\Omega$, an integration by parts gives

$$\begin{aligned}\int_\Omega u_s T \wedge (dd^c\psi)^p &= \int_\Omega \psi dd^c u_s \wedge T \wedge (dd^c\psi)^{p-1}\\ &\geq -||\psi||_{L^\infty(\Omega)} \int_\Omega T \wedge dd^c u_s \wedge (dd^c\psi)^{p-1}\\ &= -||\psi||_{L^\infty(\Omega)} A \int_\Omega T \wedge (dd^c\psi)^p.\end{aligned}$$

Finally, take $A = M/\delta$, let s tend to $-\infty$ and use the inequality $u \geq -M$ on ω. We obtain

$$\begin{aligned}\int_\Omega u T \wedge (dd^c\psi)^p &\geq -M \int_\omega T \wedge (dd^c\psi)^p + \lim_{s\to-\infty} \int_{\Omega_\delta} u_s T \wedge (dd^c\psi)^p\\ &\geq -\big(M + ||\psi||_{L^\infty(\Omega)} M/\delta\big) \int_\Omega T \wedge (dd^c\psi)^p.\end{aligned}$$

The last integral is finite. This concludes the proof. □

(4.2) Remark. If Ω is smooth and strongly pseudoconvex, the above proof shows in fact that

$$||uT||_{\overline{\Omega}} \leq \frac{C}{\delta} ||u||_{L^\infty((\overline{\Omega} \smallsetminus \Omega_\delta) \cap \operatorname{Supp} T)} ||T||_{\overline{\Omega}}$$

when $L(u) \cap \operatorname{Supp} T \subset \Omega_\delta$. In fact, if u is continuous and if ω is chosen sufficiently small, the constant M can be taken arbitrarily close to $||u||_{L^\infty((\overline{\Omega} \smallsetminus \Omega_\delta) \cap \operatorname{Supp} T)}$. Moreover, the maximum principle implies

$$||u_+||_{L^\infty(\overline{\Omega} \cap \operatorname{Supp} T)} = ||u_+||_{L^\infty(\partial\Omega \cap \operatorname{Supp} T)},$$

so we can achieve $u < -\varepsilon$ on a neighborhood of $\overline{\Omega} \cap \operatorname{Supp} T$ by subtracting $||u||_{L^\infty((\overline{\Omega} \smallsetminus \Omega_\delta) \cap \operatorname{Supp} T)} + 2\varepsilon$ [Proof of maximum principle: if $u(z_0) > 0$ at $z_0 \in \Omega \cap \operatorname{Supp} T$ and $u \leq 0$ near $\partial\Omega \cap \operatorname{Supp} T$, then

$$\int_\Omega u_+ T \wedge (dd^c\psi)^p = \int_\Omega \psi dd^c u_+ \wedge T \wedge (dd^c\psi)^{p-1} \leq 0,$$

a contradiction]. □

(4.3) Corollary. *Let $u_1, \ldots, u_q$ be plurisubharmonic functions on X such that X is covered by Stein open sets Ω with $\partial\Omega \cap L(u_j) \cap \operatorname{Supp} T = \emptyset$. We use again induction to define*

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T = dd^c(u_1 dd^c u_2 \ldots \wedge dd^c u_q \wedge T).$$

Then, if $u_1^k, \ldots, u_q^k$ are decreasing sequences of plurisubharmonic functions converging pointwise to $u_1, \ldots, u_q$, $q \le p$, properties (3.7 a, b) *hold.*

Fig. 2 Modified construction of v_j^k

Proof. Same proof as for Th. 3.7, with the following minor modification: the max procedure $v_j^k := \max\{u_j^k, A\psi\}$ is applied only on a neighborhood ω of $\operatorname{Supp} T \cap (\overline{\Omega} \smallsetminus \Omega_\delta)$ with $\delta > 0$ small, and u_j^k is left unchanged near $\operatorname{Supp} T \cap \overline{\Omega}_\delta$. Observe that the integration by part process requires the functions u_j^k and $u_j^{k,\varepsilon}$ to be defined only near $\overline{\Omega} \cap \operatorname{Supp} T$. □

(4.4) Proposition. *Let $\Omega \subset\subset X$ be a Stein open subset. If V is a plurisubharmonic function on X and $u_1, \ldots, u_q$, $1 \le q \le n-1$, are plurisubharmonic functions such that $\partial\Omega \cap L(u_j) = \emptyset$, then $V dd^c u_1 \wedge \ldots \wedge dd^c u_q$ has locally finite mass in Ω.*

Proof. Same proof as for 3.11, when $\delta > 0$ is taken so small that $\Omega_\delta \supset L(u_j)$ for all $1 \le j \le q$. □

Finally, we show that Monge-Ampère operators can also be defined in the case of plurisubharmonic functions with non compact pole sets, provided that the mutual intersections of the pole sets are of sufficiently small Hausdorff dimension with respect to the dimension p of T.

(4.5) Theorem. *Let $u_1, \ldots, u_q$ be plurisubharmonic functions on X. The currents $u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$ and $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ are well defined and have locally finite mass in X as soon as $q \le p$ and*

$$\mathcal{H}_{2p-2m+1}\big(L(u_{j_1}) \cap \ldots \cap L(u_{j_m}) \cap \operatorname{Supp} T\big) = 0$$

for all choices of indices $j_1 < \ldots < j_m$ in $\{1, \ldots, q\}$.

The proof is an easy induction on q, thanks to the following improved version of the Chern-Levine-Nirenberg inequalities.

(4.6) Proposition. *Let $A_1, \ldots, A_q \subset X$ be closed sets such that*

$$\mathcal{H}_{2p-2m+1}\big(A_{j_1} \cap \ldots \cap A_{j_m} \cap \operatorname{Supp} T\big) = 0$$

for all choices of $j_1 < \ldots < j_m$ in $\{1, \ldots, q\}$. Then for all compact sets K, L of X with $L \subset K^\circ$, there exist neighborhoods V_j of $K \cap A_j$ and a constant $C = C(K, L, A_j)$ such that the conditions $u_j \le 0$ on K and $L(u_j) \subset A_j$ imply

a) $||u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T||_L \le C ||u_1||_{L^\infty(K \smallsetminus V_1)} \cdots ||u_q||_{L^\infty(K \smallsetminus V_q)} ||T||_K$

b) $||dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T||_L \le C ||u_1||_{L^\infty(K \smallsetminus V_1)} \cdots ||u_q||_{L^\infty(K \smallsetminus V_q)} ||T||_K$.

Proof. We need only show that every point $x_0 \in K^\circ$ has a neighborhood L such that a), b) hold. Hence it is enough to work in a coordinate open set. We may thus assume that $X \subset \mathbb{C}^n$ is open, and after a regularization process $u_j = \lim u_j \star \rho_\varepsilon$ for $j = q, q-1, \ldots, 1$ in this order, that $u_1, \ldots, u_q$ are smooth. We proceed by induction on q in two steps:

Step 1. $(\mathrm{b}_{q-1}) \Longrightarrow (\mathrm{b}_q)$,

Step 2. (a_{q-1}) and $(\mathrm{b}_q) \Longrightarrow (\mathrm{a}_q)$,

where (b_0) is the trivial statement $||T||_L \le ||T||_K$ and (a_0) is void. Observe that we have $(\mathrm{a}_q) \Longrightarrow (\mathrm{a}_\ell)$ and $(\mathrm{b}_q) \Longrightarrow (\mathrm{b}_\ell)$ for $\ell \le q \le p$ by taking $u_{\ell+1}(z) = \ldots = u_q(z) = |z|^2$. We need the following elementary fact.

(4.7) Lemma. *Let $F \subset \mathbb{C}^n$ be a closed set such that $\mathcal{H}_{2s+1}(F) = 0$ for some integer $0 \le s < n$. Then for almost all choices of unitary coordinates $(z_1, \ldots, z_n) = (z', z'')$ with $z' = (z_1, \ldots, z_s)$, $z'' = (z_{s+1}, \ldots, z_n)$ and almost all radii of balls $B'' = B(0, r'') \subset \mathbb{C}^{n-s}$, the set $\{0\} \times \partial B''$ does not intersect F.*

Proof. The unitary group $U(n)$ has real dimension n^2. There is a proper submersion

$$\Phi : U(n) \times \big(\mathbb{C}^{n-s} \smallsetminus \{0\}\big) \longrightarrow \mathbb{C}^n \smallsetminus \{0\}, \qquad (g, z'') \longmapsto g(0, z''),$$

whose fibers have real dimension $N = n^2 - 2s$. It follows that the inverse image $\Phi^{-1}(F)$ has zero Hausdorff measure $\mathcal{H}_{N+2s+1} = \mathcal{H}_{n^2+1}$. The set of pairs $(g, r'') \in U(n) \times \mathbb{R}_+^\star$ such that $g(\{0\} \times \partial B'')$ intersects F is precisely the image of $\Phi^{-1}(F)$ in $U(n) \times \mathbb{R}_+^\star$ by the Lipschitz map $(g, z'') \mapsto (g, |z''|)$. Hence this set has zero $\mathcal{H}_{n^2+1}$-measure. □

Proof of step 1. Take $x_0 = 0 \in K^\circ$. Suppose first $0 \in A_1 \cap \ldots \cap A_q$ and set $F = A_1 \cap \ldots \cap A_q \cap \operatorname{Supp} T$. Since $\mathcal{H}_{2p-2q+1}(F) = 0$, Lemma 4.7 implies that there are coordinates $z' = (z_1, \ldots, z_s)$, $z'' = (z_{s+1}, \ldots, z_n)$ with $s = p - q$ and a ball $\overline{B}''$ such that $F \cap \big(\{0\} \times \partial B''\big) = \emptyset$ and $\{0\} \times \overline{B}'' \subset K^\circ$. By compactness of K, we can find neighborhoods W_j of $K \cap A_j$ and a ball $B' = B(0, r') \subset \mathbb{C}^s$ such that $\overline{B}' \times \overline{B}'' \subset K^\circ$ and

$$(4.8) \quad \overline{W}_1 \cap \ldots \cap \overline{W}_q \cap \operatorname{Supp} T \cap \Big(\overline{B}' \times \big(\overline{B}'' \smallsetminus (1-\delta)B''\big)\Big) = \emptyset$$

for $\delta > 0$ small. If $0 \notin A_j$ for some j, we choose instead W_j to be a small neighborhood of 0 such that $\overline{W}_j \subset (\overline{B}' \times (1-\delta)B'') \smallsetminus A_j$; property (4.8) is then automatically satisfied. Let $\chi_j \geq 0$ be a function with compact support in W_j, equal to 1 near $K \cap A_j$ if $A_j \ni 0$ (resp. equal to 1 near 0 if $A_j \not\ni 0$) and let $\chi(z') \geq 0$ be a function equal to 1 on $1/2\, B'$ with compact support in B'. Then

$$\int_{B' \times B''} dd^c(\chi_1 u_1) \wedge \ldots \wedge dd^c(\chi_q u_q) \wedge T \wedge \chi(z')\,(dd^c|z'|^2)^s = 0$$

because the integrand is dd^c exact and has compact support in $B' \times B''$ thanks to (4.8). If we expand all factors $dd^c(\chi_j u_j)$, we find a term

$$\chi_1 \ldots \chi_q \chi(z') dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T \geq 0$$

which coincides with $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ on a small neighborhood of 0 where $\chi_j = \chi = 1$. The other terms involve

$$d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j + u_j dd^c \chi_j$$

for at least one index j. However $d\chi_j$ and $dd^c\chi_j$ vanish on some neighborhood V_j' of $K \cap A_j$ and therefore u_j is bounded on $\overline{B}' \times \overline{B}'' \smallsetminus V_j'$. We then apply the induction hypothesis (b_{q-1}) to the current

$$\Theta = dd^c u_1 \wedge \ldots \wedge \widehat{dd^c u_j} \wedge \ldots \wedge dd^c u_q \wedge T$$

and the usual Chern-Levine-Nirenberg inequality to the product of Θ with the mixed term $d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j$. Remark 3.5 can be applied because χ_j is smooth and is therefore a difference $\chi_j^{(1)} - \chi_j^{(2)}$ of locally bounded plurisubharmonic functions in $\mathbb{C}^n$. Let K' be a compact neighborhood of

$\overline{B}' \times \overline{B}''$ with $K' \subset K^\circ$, and let V_j be a neighborhood of $K \cap A_j$ with $\overline{V}_j \subset V_j'$. Then with $L' := (\overline{B}' \times \overline{B}'') \smallsetminus V_j' \subset (K' \smallsetminus V_j)^\circ$ we obtain

$$\begin{aligned}||(d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j) \wedge \Theta||_{\overline{B}' \times \overline{B}''} &= ||(d\chi_j \wedge d^c u_j + du_j \wedge d^c \chi_j) \wedge \Theta||_{L'} \\ &\le C_1 ||u_j||_{L^\infty(K' \smallsetminus V_j)} ||\Theta||_{K' \smallsetminus V_j}, \\ ||\Theta||_{K' \smallsetminus V_j} \le ||\Theta||_{K'} \le C_2 ||u_1||_{L^\infty(K \smallsetminus V_1)} &\ldots \widehat{||u_j||} \ldots ||u_q||_{L^\infty(K \smallsetminus V_q)} ||T||_K.\end{aligned}$$

Now, we may slightly move the unitary basis in $\mathbb{C}^n$ and get coordinate systems $z^m = (z_1^m, \ldots, z_n^m)$ with the same properties as above, such that the forms

$$(dd^c|z^{m\prime}|^2)^s = \frac{s!}{\pi^s} \mathrm{i}\, dz_1^m \wedge d\overline{z}_1^m \wedge \ldots \wedge \mathrm{i}\, dz_s^m \wedge d\overline{z}_s^m, \qquad 1 \le m \le N$$

define a basis of $\bigwedge^{s,s}(\mathbb{C}^n)^\star$. It follows that all measures

$$dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \mathrm{i}\, dz_1^m \wedge d\overline{z}_1^m \wedge \ldots \wedge \mathrm{i}\, dz_s^m \wedge d\overline{z}_s^m$$

satisfy estimate (b_q) on a small neighborhood L of 0.

Proof of Step 2. We argue in a similar way with the integrals

$$\begin{aligned}&\int_{B' \times B''} \chi_1 u_1 dd^c(\chi_2 u_2) \wedge \ldots dd^c(\chi_q u_q) \wedge T \wedge \chi(z')(dd^c|z'|^2)^s \wedge dd^c|z_{s+1}|^2 \\ &\qquad = \int_{B' \times B''} |z_{s+1}|^2 dd^c(\chi_1 u_1) \wedge \ldots dd^c(\chi_q u_q) \wedge T \wedge \chi(z')(dd^c|z'|^2)^s.\end{aligned}$$

We already know by (b_q) and Remark 3.5 that all terms in the right hand integral admit the desired bound. For $q = 1$, this shows that $(\mathrm{b}_1) \Longrightarrow (\mathrm{a}_1)$. Except for $\chi_1 \ldots \chi_q \chi(z')\, u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T$, all terms in the left hand integral involve derivatives of χ_j. By construction, the support of these derivatives is disjoint from A_j, thus we only have to obtain a bound for

$$\int_L u_1 dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha$$

when $L = \overline{B}(x_0, r)$ is disjoint from A_j for some $j \ge 2$, say $L \cap A_2 = \emptyset$, and α is a constant positive form of type $(p-q, p-q)$. Then $\overline{B}(x_0, r+\varepsilon) \subset K^\circ \smallsetminus \overline{V}_2$ for some $\varepsilon > 0$ and some neighborhood V_2 of $K \cap A_2$. By the max construction used e.g. in Prop. 4.1, we can replace u_2 by a plurisubharmonic function $\widetilde{u}_2$ equal to u_2 in L and to $A(|z - x_0|^2 - r^2) - M$ in $\overline{B}(x_0, r+\varepsilon) \smallsetminus B(x_0, r+\varepsilon/2)$, with $M = ||u_2||_{L^\infty(K \smallsetminus V_2)}$ and $A = M/\varepsilon r$. Let $\chi \ge 0$ be a smooth function equal to 1 on $B(x_0, r+\varepsilon/2)$ with support in $B(x_0, r)$. Then

$$\begin{aligned}&\int_{B(x_0, r+\varepsilon)} u_1 dd^c(\chi \widetilde{u}_2) \wedge dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha \\ &\qquad = \int_{B(x_0, r+\varepsilon)} \chi \widetilde{u}_2 dd^c u_1 \wedge dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha \\ &\qquad \le O(1)\ ||u_1||_{L^\infty(K \smallsetminus V_1)} \ldots ||u_q||_{L^\infty(K \smallsetminus V_q)} ||T||_K\end{aligned}$$

where the last estimate is obtained by the induction hypothesis (b_{q-1}) applied to $dd^c u_1 \wedge dd^c u_3 \wedge \ldots \wedge dd^c u_q \wedge T$. By construction

$$dd^c(\chi \widetilde{u}_2) = \chi\, dd^c \widetilde{u}_2 + (\text{smooth terms involving } d\chi)$$

coincides with $dd^c u_2$ in L, and (a_{q-1}) implies the required estimate for the other terms in the left hand integral. □

(4.9) Proposition. *With the assumptions of Th.* 4.5, *the analogue of the monotone convergence Theorem* 3.7 (a,b) *holds.*

Proof. By the arguments already used in the proof of Th. 3.7 (e.g. by Lemma 3.9), it is enough to show that

$$\begin{aligned}\int_{B'\times B''} &\chi_1 \ldots \chi_q\, u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_q \wedge T \wedge \alpha \\ &\leq \liminf_{k\to+\infty} \int_{B'\times B''} \chi_1 \ldots \chi_q\, u_1^k dd^c u_2^k \wedge \ldots \wedge dd^c u_q^k \wedge T \wedge \alpha\end{aligned}$$

where $\alpha = \chi(z')(dd^c|z'|^2)^s$ is closed. Here the functions χ_j, χ are chosen as in the proof of Step 1 in 4.7, especially their product has compact support in $B' \times B''$ and $\chi_j = \chi = 1$ in a neighborhood of the given point x_0. We argue by induction on q and also on the number m of functions $(u_j)_{j\geq 1}$ which are unbounded near x_0. If u_j is bounded near x_0, we take $W_j'' \subset\subset W_j' \subset\subset W_j$ to be small balls of center x_0 on which u_j is bounded and we modify the sequence u_j^k on the corona $W_j \smallsetminus W_j''$ so as to make it constant and equal to a smooth function $A|z-x_0|^2 + B$ on the smaller corona $W_j \smallsetminus W_j'$. In that case, we take χ_j equal to 1 near $\overline{W}_j'$ and $\operatorname{Supp} \chi_j \subset W_j$. For every $\ell = 1, \ldots, q$, we are going to check that

$$\begin{aligned}&\liminf_{k\to+\infty} \int_{B'\times B''} \chi_1 u_1^k dd^c(\chi_2 u_2^k) \wedge \ldots \\ &\qquad dd^c(\chi_{\ell-1} u_{\ell-1}^k) \wedge dd^c(\chi_\ell u_\ell) \wedge dd^c(\chi_{\ell+1} u_{\ell+1}) \ldots dd^c(\chi_q u_q) \wedge T \wedge \alpha \\ &\leq \liminf_{k\to+\infty} \int_{B'\times B''} \chi_1 u_1^k dd^c(\chi_2 u_2^k) \wedge \ldots \\ &\qquad dd^c(\chi_{\ell-1} u_{\ell-1}^k) \wedge dd^c(\chi_\ell u_\ell^k) \wedge dd^c(\chi_{\ell+1} u_{\ell+1}) \ldots dd^c(\chi_q u_q) \wedge T \wedge \alpha.\end{aligned}$$

In order to do this, we integrate by parts $\chi_1 u_1^k dd^c(\chi_\ell u_\ell)$ into $\chi_\ell u_\ell dd^c(\chi_1 u_1^k)$ for $\ell \geq 2$, and we use the inequality $u_\ell \leq u_\ell^k$. Of course, the derivatives $d\chi_j$, $d^c\chi_j$, $dd^c\chi_j$ produce terms which are no longer positive and we have to take care of these. However, $\operatorname{Supp} d\chi_j$ is disjoint from the unbounded locus of u_j when u_j is unbounded, and contained in $W_j \smallsetminus \overline{W}_j'$ when u_j is bounded. The number m of unbounded functions is therefore replaced by $m-1$ in the first case, whereas in the second case $u_j^k = u_j$ is constant and smooth on $\operatorname{Supp} d\chi_j$, so q can be replaced by $q-1$. By induction on $q+m$ (and thanks to the polarization technique 3.5), the limit of the terms involving derivatives of χ_j is equal on both sides to the corresponding terms obtained by suppressing all indices k. Hence these terms do not give any contribution in the inequalities. $\square$

We finally quote the following simple consequences of Th. 4.5 when T is arbitrary and $q=1$, resp. when $T=1$ has bidegree $(0,0)$ and q is arbitrary.

(4.10) Corollary. *Let T be a closed positive current of bidimension (p,p) and let u be a plurisubharmonic function on X such that $L(u)\cap \operatorname{Supp} T$ is contained in an analytic set of dimension at most $p-1$. Then uT and $dd^cu\wedge T$ are well defined and have locally finite mass in X.* $\square$

(4.11) Corollary. *Let $u_1,\ldots,u_q$ be plurisubharmonic functions on X such that $L(u_j)$ is contained in an analytic set $A_j\subset X$ for every j. Then $dd^cu_1\wedge\ldots\wedge dd^cu_q$ is well defined as soon as $A_{j_1}\cap\ldots\cap A_{j_m}$ has codimension at least m for all choices of indices $j_1<\ldots<j_m$ in $\{1,\ldots,q\}$.* $\square$

In the particular case when $u_j=\log|f_j|$ for some non zero holomorphic function f_j on X, we see that the intersection product of the associated zero divisors $[Z_j]=dd^cu_j$ is well defined as soon as the supports $|Z_j|$ satisfy $\operatorname{codim}|Z_{j_1}|\cap\ldots\cap|Z_{j_m}|=m$ for every m. Similarly, when $T=[A]$ is an analytic p-cycle, Cor. 4.10 shows that $[Z]\wedge[A]$ is well defined for every divisor Z such that $\dim|Z|\cap|A|=p-1$. These observations easily imply the following

(4.12) Proposition. *Suppose that the divisors Z_j satisfy the above codimension condition and let $(C_k)_{k\ge 1}$ be the irreducible components of the point set intersection $|Z_1|\cap\ldots\cap|Z_q|$. Then there exist integers $m_k>0$ such that*

$$[Z_1]\wedge\ldots\wedge[Z_q]=\sum m_k[C_k].$$

The integer m_k is called the multiplicity of intersection of $Z_1,\ldots,Z_q$ along the component C_k.

Proof. The wedge product has bidegree (q,q) and support in $C=\bigcup C_k$ where $\operatorname{codim} C=q$, so it must be a sum as above with $m_k\in\mathbb{R}_+$. We check by induction on q that m_k is a positive integer. If we denote by A some irreducible component of $|Z_1|\cap\ldots\cap|Z_{q-1}|$, we need only check that $[A]\wedge[Z_q]$ is an integral analytic cycle of codimension q with positive coefficients on each component C_k of the intersection. However $[A]\wedge[Z_q]=dd^c(\log|f_q|\,[A])$. First suppose that no component of $A\cap f_q^{-1}(0)$ is contained in the singular part $A_{\rm sing}$. Then the Lelong-Poincaré equation applied on $A_{\rm reg}$ shows that $dd^c(\log|f_q|\,[A])=\sum m_k[C_k]$ on $X\smallsetminus A_{\rm sing}$, where m_k is the vanishing order of f_q along C_k in $A_{\rm reg}$. Since $C\cap A_{\rm sing}$ has codimension $q+1$ at least, the equality must hold on X. In general, we replace f_q by $f_q-\varepsilon$ so that the divisor of $f_q - \varepsilon$ has no component contained in $A_{\rm sing}$. Then $dd^c(\log|f_q-\varepsilon|\,[A])$ is an integral codimension q cycle with positive multiplicities on each component of $A\cap f_q^{-1}(\varepsilon)$ and we conclude by letting ε tend to zero. □

5. Generalized Lelong Numbers

The concepts we are going to study mostly concern the behaviour of currents or plurisubharmonic functions in a neighborhood of a point at which we have for instance a logarithmic pole. Since the interesting applications are local, we assume from now on (unless otherwise stated) that X is a Stein manifold, i.e. that X has a strictly plurisubharmonic exhaustion function. Let $\varphi : X \longrightarrow [-\infty,+\infty[$ be a continuous plurisubharmonic function (in general φ may have $-\infty$ poles, our continuity assumption means that e^φ is continuous). The sets

(5.1) $\quad S(r) = \{x\in X\,;\,\varphi(x)=r\},$

(5.1′) $\quad B(r) = \{x\in X\,;\,\varphi(x)<r\},$

(5.1″) $\quad \overline{B}(r) = \{x\in X\,;\,\varphi(x)\le r\}$

will be called *pseudo-spheres* and *pseudo-balls* associated with φ. Note that $\overline{B}(r)$ is not necessarily equal to the closure of $B(r)$, but this is often true in concrete situations. The most simple example we have in mind is the case of the function $\varphi(z)=\log|z-a|$ on an open subset $X\subset\mathbb{C}^n$; in this case $B(r)$ is the euclidean ball of center a and radius e^r ; moreover, the forms

(5.2) $$\frac{1}{2}dd^ce^{2\varphi} = \frac{\mathrm{i}}{2\pi}d'd''|z|^2,\qquad dd^c\varphi = \frac{\mathrm{i}}{\pi}d'd''\log|z-a|$$

can be interpreted respectively as the flat hermitian metric on $\mathbb{C}^n$ and as the pull-back over $\mathbb{C}^n$ of the Fubini-Study metric of $\mathbb{P}^{n-1}$, translated by a.

(5.3) Definition. *We say that φ is semi-exhaustive if there exists a real number R such that $B(R)\subset\subset X$. Similarly, φ is said to be semi-exhaustive on a closed subset $A\subset X$ if there exists R such that $A\cap B(R)\subset\subset X$.*

We are interested especially in the set of poles $S(-\infty)=\{\varphi=-\infty\}$ and in the behaviour of φ near $S(-\infty)$. Let T be a closed positive current of bidimension (p,p) on X. Assume that φ is semi-exhaustive on Supp T and that $B(R)\cap\operatorname{Supp}T\subset\subset X$. Then $P=S(-\infty)\cap\mathrm{SuppT}$ is compact and the results of §2 show that the measure $T\wedge(dd^c\varphi)^p$ is well defined. Following (Demailly 1982b, 1987a), we introduce:

(5.4) Definition. *If φ is semi-exhaustive on* Supp T *and if R is such that $B(R)\cap\operatorname{Supp}T\subset\subset X$, we set for all $r\in\,]-\infty,R[$*

$$\nu(T,\varphi,r) = \int_{B(r)} T\wedge (dd^c\varphi)^p,$$
$$\nu(T,\varphi) = \int_{S(-\infty)} T\wedge (dd^c\varphi)^p = \lim_{r\to-\infty} \nu(T,\varphi,r).$$

The number $\nu(T,\varphi)$ will be called the (generalized) Lelong number of T with respect to the weight φ.

If we had not required $T\wedge(dd^c\varphi)^p$ to be defined pointwise on $\varphi^{-1}(-\infty)$, the assumption that X is Stein could have been dropped: in fact, the integral over $B(r)$ always makes sense if we define

$$\nu(T,\varphi,r) = \int_{B(r)} T\wedge \big(dd^c\max\{\varphi,s\}\big)^p \quad \text{with} \;\; s<r.$$

Stokes' formula shows that the right hand integral is actually independent of s. The example given after (4.1) shows however that $T\wedge(dd^c\varphi)^p$ need not exist on $\varphi^{-1}(-\infty)$ if $\varphi^{-1}(-\infty)$ contains an exceptional compact analytic subset. We leave the reader consider by himself this more general situation and extend our statements by the $\max\{\varphi,s\}$ technique. Observe that $r\longmapsto \nu(T,\varphi,r)$ is always an increasing function of r. Before giving examples, we need a formula.

(5.5) Formula. *For any convex increasing function $\chi:\mathbb{R}\longrightarrow\mathbb{R}$ we have*

$$\int_{B(r)} T\wedge (dd^c\chi\circ\varphi)^p = \chi'(r-0)^p\,\nu(T,\varphi,r)$$

where $\chi'(r-0)$ denotes the left derivative of χ at r.

Proof. Let χ_ε be the convex function equal to χ on $[r-\varepsilon,+\infty[$ and to a linear function of slope $\chi'(r-\varepsilon-0)$ on $]-\infty,r-\varepsilon]$. We get $dd^c(\chi_\varepsilon\circ\varphi) = \chi'(r-\varepsilon-0)dd^c\varphi$ on $B(r-\varepsilon)$ and Stokes' theorem implies

$$\begin{aligned}\int_{B(r)} T\wedge (dd^c\chi\circ\varphi)^p &= \int_{B(r)} T\wedge (dd^c\chi_\varepsilon\circ\varphi)^p\\ &\ge \int_{B(r-\varepsilon)} T\wedge (dd^c\chi_\varepsilon\circ\varphi)^p\\ &= \chi'(r-\varepsilon-0)^p\nu(T,\varphi,r-\varepsilon).\end{aligned}$$

Similarly, taking $\widetilde\chi_\varepsilon$ equal to χ on $]-\infty,r-\varepsilon]$ and linear on $[r-\varepsilon,r]$, we obtain

$$\int_{B(r-\varepsilon)} T\wedge (dd^c\chi\circ\varphi)^p \le \int_{B(r)} T\wedge (dd^c\widetilde\chi_\varepsilon\circ\varphi)^p = \chi'(r-\varepsilon-0)^p\nu(T,\varphi,r).$$

The expected formula follows when ε tends to 0. $\square$

We get in particular $\int_{B(r)} T \wedge (dd^c e^{2\varphi})^p = (2e^{2r})^p \nu(T, \varphi, r)$, whence the formula

$$(5.6) \quad \nu(T, \varphi, r) = e^{-2pr} \int_{B(r)} T \wedge \Big(\frac{1}{2} dd^c e^{2\varphi}\Big)^p.$$

Now, assume that X is an open subset of $\mathbb{C}^n$ and that $\varphi(z) = \log|z - a|$ for some $a \in X$. Formula (5.6) gives

$$\nu(T, \varphi, \log r) = r^{-2p} \int_{|z-a|<r} T \wedge \Big(\frac{\mathrm{i}}{2\pi} d'd''|z|^2\Big)^p.$$

The positive measure $\sigma_T = \frac{1}{p!} T \wedge (\frac{\mathrm{i}}{2} d'd''|z|^2)^p = 2^{-p} \sum T_{I,I} \,.\, \mathrm{i}^n dz_1 \wedge \ldots \wedge d\overline{z}_n$ is called the *trace measure* of T. We get

$$(5.7) \quad \nu(T, \varphi, \log r) = \frac{\sigma_T\big(B(a, r)\big)}{\pi^p r^{2p}/p!}$$

and $\nu(T, \varphi)$ is the limit of this ratio as $r \to 0$. This limit is called the (*ordinary*) *Lelong number* of T at point a and is denoted $\nu(T, a)$. This was precisely the original definition of Lelong, see (Lelong 1968). Let us mention a simple but important consequence.

(5.8) Consequence. *The ratio $\sigma_T\big(B(a, r)\big)/r^{2p}$ is an increasing function of r. Moreover, for every compact subset $K \subset X$ and every $r_0 < d(K, \partial X)$ we have*

$$\sigma_T\big(B(a, r)\big) \le C r^{2p} \qquad \text{for} \quad a \in K \text{ and } r \le r_0,$$

where $C = \sigma_T\big(K + \overline{B}(0, r_0)\big)/r_0^{2p}$.

All these results are particularly interesting when $T = [A]$ is the current of integration over an analytic subset $A \subset X$ of pure dimension p. Then $\sigma_T\big(B(a, r)\big)$ is the euclidean area of $A \cap B(a, r)$, while $\pi^p r^{2p}/p!$ is the area of a ball of radius r in a p-dimensional subspace of $\mathbb{C}^n$. Thus $\nu(T, \varphi, \log r)$ is the ratio of these areas and the Lelong number $\nu(T, a)$ is the limit ratio.

(5.9) Remark. It is immediate to check that

$$\nu([A], x) = \begin{cases} 0 & \text{for } \ x \notin A, \\ 1 & \text{when } x \in A \text{ is a regular point.} \end{cases}$$

We will see later that $\nu([A], x)$ is always an integer (Thie's theorem 8.7).

(5.10) Remark. When $X = \mathbb{C}^n$, $\varphi(z) = \log|z - a|$ and $A = X$ (i.e. $T = 1$), we obtain in particular $\int_{B(a,r)} (dd^c \log|z - a|)^n = 1$ for all r. This implies

$$(dd^c \log|z - a|)^n = \delta_a.$$

This fundamental formula can be viewed as a higher dimensional analogue of the usual formula $\Delta \log|z-a| = 2\pi\delta_a$ in $\mathbb{C}$. □

We next prove a result which shows in particular that the Lelong numbers of a closed positive current are zero except on a very small set.

(5.11) Proposition. *If T is a closed positive current of bidimension (p,p), then for each $c>0$ the set $E_c = \{x \in X\,;\, \nu(T,x) \ge c$ is a closed set of locally finite $\mathcal{H}_{2p}$ Hausdorff measure in X.*

Proof. By (5.7), we infer $\nu(T,a) = \lim_{r\to 0} \sigma_T\big(\overline{B}(a,r)\big)p!/\pi^p r^{2p}$. The function $a \mapsto \sigma_T\big(\overline{B}(a,r)\big)$ is clearly upper semicontinuous. Hence the decreasing limit $\nu(T,a)$ as r decreases to 0 is also upper semicontinuous in a. This implies that E_c is closed. Now, let K be a compact subset in X and let $\{a_j\}_{1\le j\le N}$, $N = N(\varepsilon)$, be a maximal collection of points in $E_c \cap K$ such that $|a_j - a_k| \ge 2\varepsilon$ for $j \ne k$. The balls $B(a_j, 2\varepsilon)$ cover $E_c \cap K$, whereas the balls $B(a_j,\varepsilon)$ are disjoint. If $K_{c,\varepsilon}$ is the set of points which are at distance $\le \varepsilon$ of $E_c \cap K$, we get

$$\sigma_T(K_{c,\varepsilon}) \ge \sum \sigma_T\big(B(a_j,\varepsilon)\big) \ge N(\varepsilon)\, c\pi^p \varepsilon^{2p}/p!,$$

since $\nu(T,a_j) \ge c$. By the definition of Hausdorff measure, we infer

$$\begin{aligned}\mathcal{H}_{2p}(E_c \cap K) &\le \liminf_{\varepsilon\to 0} \sum \big(\mathrm{diam}\, B(a_j,2\varepsilon)\big)^{2p}\\ &\le \liminf_{\varepsilon\to 0} N(\varepsilon)(4\varepsilon)^{2p} \le \frac{p!4^{2p}}{c\pi^p}\sigma_T(E_c \cap K).\end{aligned}$$ □

Finally, we conclude this section by proving two simple semi-continuity results for Lelong numbers.

(5.12) Proposition. *Let T_k be a sequence of closed positive currents of bidimension (p,p) converging weakly to a limit T. Suppose that there is a closed set A such that* Supp $T_k \subset A$ *for all k and such that φ is semi-exhaustive on A with $A \cap B(R) \subset\subset X$. Then for all $r < R$ we have*

$$\begin{aligned}\int_{B(r)} T \wedge (dd^c\varphi)^p &\le \liminf_{k\to+\infty} \int_{B(r)} T_k \wedge (dd^c\varphi)^p\\ &\le \limsup_{k\to+\infty} \int_{\overline{B}(r)} T_k \wedge (dd^c\varphi)^p \le \int_{\overline{B}(r)} T \wedge (dd^c\varphi)^p.\end{aligned}$$

When r tends to $-\infty$, we find in particular

$$\limsup_{k\to+\infty} \nu(T_k,\varphi) \le \nu(T,\varphi).$$

Proof. Let us prove for instance the third inequality. Let φ_ℓ be a sequence of smooth plurisubharmonic approximations of φ with $\varphi \le \varphi_\ell < \varphi + 1/\ell$ on $\{r-\varepsilon \le \varphi \le r+\varepsilon\}$. We set

$$\psi_\ell = \begin{cases} \varphi & \text{on } \overline{B}(r), \\ \max\{\varphi, (1+\varepsilon)(\varphi_\ell - 1/\ell) - r\varepsilon\} & \text{on } X \smallsetminus B(r). \end{cases}$$

This definition is coherent since $\psi_\ell = \varphi$ near $S(r)$, and we have

$$\psi_\ell = (1+\varepsilon)(\varphi_\ell - 1/\ell) - r\varepsilon \quad \text{near} \quad S(r+\varepsilon/2)$$

as soon as ℓ is large enough, i.e. $(1+\varepsilon)/\ell \le \varepsilon^2/2$. Let χ_ε be a cut-off function equal to 1 in $B(r+\varepsilon/2)$ with support in $B(r+\varepsilon)$. Then

$$\begin{aligned} \int_{\overline{B}(r)} T_k \wedge (dd^c\varphi)^p &\le \int_{B(r+\varepsilon/2)} T_k \wedge (dd^c\psi_\ell)^p \\ &= (1+\varepsilon)^p \int_{B(r+\varepsilon/2)} T_k \wedge (dd^c\varphi_\ell)^p \\ &\le (1+\varepsilon)^p \int_{B(r+\varepsilon)} \chi_\varepsilon T_k \wedge (dd^c\varphi_\ell)^p. \end{aligned}$$

As $\chi_\varepsilon(dd^c\varphi_\ell)^p$ is smooth with compact support and as T_k converges weakly to T, we infer

$$\limsup_{k\to+\infty} \int_{\overline{B}(r)} T_k \wedge (dd^c\varphi)^p \le (1+\varepsilon)^p \int_{B(r+\varepsilon)} \chi_\varepsilon T \wedge (dd^c\varphi_\ell)^p.$$

We then let ℓ tend to $+\infty$ and ε tend to 0 to get the desired inequality. The first inequality is obtained in a similar way, we define ψ_ℓ so that $\psi_\ell = \varphi$ on $X \smallsetminus B(r)$ and $\psi_\ell = \max\{(1-\varepsilon)(\varphi_\ell - 1/\ell) + r\varepsilon\}$ on $\overline{B}(r)$, and we take $\chi_\varepsilon = 1$ on $B(r-\varepsilon)$ with $\operatorname{Supp} \chi_\varepsilon \subset B(r-\varepsilon/2)$. Then for ℓ large

$$\begin{aligned} \int_{B(r)} T_k \wedge (dd^c\varphi)^p &\ge \int_{B(r-\varepsilon/2)} T_k \wedge (dd^c\psi_\ell)^p \\ &\ge (1-\varepsilon)^p \int_{B(r-\varepsilon/2)} \chi_\varepsilon T_k \wedge (dd^c\varphi_\ell)^p. \end{aligned}$$

□

(5.13) Proposition. *Let φ_k be a (non necessarily monotone) sequence of continuous plurisubharmonic functions such that e^{φ_k} converges uniformly to e^φ on every compact subset of X. Suppose that $\{\varphi < R\} \cap \operatorname{Supp} T \subset\subset X$. Then for $r < R$ we have*

$$\limsup_{k\to+\infty} \int_{\{\varphi_k \le r\}\cap\{\varphi<R\}} T \wedge (dd^c\varphi_k)^p \le \int_{\{\varphi\le r\}} T \wedge (dd^c\varphi)^p.$$

In particular $\limsup_{k\to+\infty} \nu(T,\varphi_k) \le \nu(T,\varphi)$.

When we take $\varphi_k(z) = \log|z - a_k|$ with $a_k \to a$, Prop. 5.13 implies the upper semicontinuity of $a \mapsto \nu(T, a)$ which was already noticed in the proof of Prop. 5.11.

Proof. Our assumption is equivalent to saying that $\max\{\varphi_k, t\}$ converges locally uniformly to $\max\{\varphi, t\}$ for every t. Then Cor. 3.6 shows that $T \wedge (dd^c \max\{\varphi_k, t\})^p$ converges weakly to $T \wedge (dd^c \max\{\varphi, t\})^p$. If χ_ε is a cut-off function equal to 1 on $\{\varphi \le r + \varepsilon/2\}$ with support in $\{\varphi < r + \varepsilon\}$, we get

$$\lim_{k\to+\infty} \int_X \chi_\varepsilon T \wedge (dd^c \max\{\varphi_k, t\})^p = \int_X \chi_\varepsilon T \wedge (dd^c \max\{\varphi, t\})^p.$$

For k large, we have $\{\varphi_k \le r\} \cap \{\varphi < R\} \subset \{\varphi < r + \varepsilon/2\}$, thus when ε tends to 0 we infer

$$\limsup_{k\to+\infty} \int_{\{\varphi_k \le r\}\cap\{\varphi<R\}} T \wedge (dd^c \max\{\varphi_k, t\})^p \le \int_{\{\varphi\le r\}} T \wedge (dd^c \max\{\varphi, t\})^p.$$

When we choose $t < r$, this is equivalent to the first inequality in statement (5.13). □

6. The Jensen-Lelong Formula

We assume in this section that X is Stein, that φ is *semi-exhaustive* on X and that $B(R) \subset\subset X$. We set for simplicity $\varphi_{\ge r} = \max\{\varphi, r\}$. For every $r \in\,]-\infty, R[$, the measures $dd^c(\varphi_{\ge r})^n$ are well defined. By Cor. 3.6, the map $r \longmapsto (dd^c\varphi_{\ge r})^n$ is continuous on $]-\infty, R[$ with respect to the weak topology. As $(dd^c\varphi_{\ge r})^n = (dd^c\varphi)^n$ on $X \setminus \overline{B}(r)$ and as $\varphi_{\ge r} \equiv r$, $(dd^c\varphi_{\ge r})^n = 0$ on $B(r)$, the left continuity implies $(dd^c\varphi_{\ge r})^n \ge \mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n$. Here $\mathbb{1}_A$ denotes the characteristic function of any subset $A \subset X$. According to the definition introduced in (Demailly 1985a), the collection of *Monge-Ampère measures* associated with φ is the family of positive measures μ_r such that

$$(6.1) \quad \mu_r = (dd^c\varphi_{\ge r})^n - \mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n, \qquad r \in\,]-\infty, R[.$$

The measure μ_r is supported on $S(r)$ and $r \longmapsto \mu_r$ is weakly continuous on the left by the bounded convergence theorem. Stokes' formula shows that $\int_{B(s)}(dd^c\varphi_{\ge r})^n - (dd^c\varphi)^n = 0$ for $s > r$, hence the total mass $\mu_r(S(r)) = \mu_r(B(s))$ is equal to the difference between the masses of $(dd^c\varphi)^n$ and $\mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n$ over $B(s)$, i.e.

$$(6.2) \quad \mu_r\big(S(r)\big) = \int_{B(r)} (dd^c\varphi)^n.$$

(6.3) Example. When $(dd^c\varphi)^n = 0$ on $X \setminus \varphi^{-1}(-\infty)$, formula (6.1) can be simplified into $\mu_r = (dd^c\varphi_{\ge r})^n$. This is so for $\varphi(z) = \log|z|$. In this case, the invariance of φ under unitary transformations implies that μ_r is also invariant. As the total mass of μ_r is equal to 1 by 5.10 and (6.2), we see that μ_r is the invariant measure of mass 1 on the euclidean sphere of radius e^r.

(6.4) Proposition. *Assume that φ is smooth near $S(r)$ and that $d\varphi \neq 0$ on $S(r)$, i.e. r is a non critical value. Then $S(r) = \partial B(r)$ is a smooth oriented real hypersurface and the measure μ_r is given by the $(2n-1)$-volume form $(dd^c\varphi)^{n-1} \wedge d^c\varphi_{\restriction S(r)}$.*

Proof. Write $\max\{t, r\} = \lim_{k\to+\infty} \chi_k(t)$ where χ is a decreasing sequence of smooth convex functions with $\chi_k(t) = r$ for $t \leq r - 1/k$, $\chi_k(t) = t$ for $t \geq r+1/k$. Theorem 3.6 shows that $(dd^c\chi_k \circ \varphi)^n$ converges weakly to $(dd^c\varphi_{\geq r})^n$. Let h be a smooth function h with compact support near $S(r)$. Let us apply Stokes' theorem with $S(r)$ considered as the boundary of $X \setminus B(r)$:

$$\begin{aligned}\int_X h(dd^c\varphi_{\geq r})^n &= \lim_{k\to+\infty} \int_X h(dd^c\chi_k \circ \varphi)^n \\ &= \lim_{k\to+\infty} \int_X -dh \wedge (dd^c\chi_k \circ \varphi)^{n-1} \wedge d^c(\chi_k \circ \varphi) \\ &= \lim_{k\to+\infty} \int_X -\chi_k'(t)^n\, dh \wedge (dd^c\varphi)^{n-1} \wedge d^c\varphi \\ &= \int_{X\setminus B(r)} -dh \wedge (dd^c\varphi)^{n-1} \wedge d^c\varphi \\ &= \int_{S(r)} h\,(dd^c\varphi)^{n-1} \wedge d^c\varphi + \int_{X\setminus B(r)} h\,(dd^c\varphi)^{n-1} \wedge d^c\varphi.\end{aligned}$$

Near $S(r)$ we thus have an equality of measures

$$(dd^c\varphi_{\geq r})^n = (dd^c\varphi)^{n-1} \wedge d^c\varphi_{\restriction S(r)} + \mathbb{1}_{X\setminus B(r)}(dd^c\varphi)^n. \qquad \square$$

(6.5) Jensen-Lelong formula. *Let V be any plurisubharmonic function on X. Then V is μ_r-integrable for every $r \in\,]-\infty, R[$ and*

$$\mu_r(V) - \int_{B(r)} V(dd^c\varphi)^n = \int_{-\infty}^r \nu(dd^cV, \varphi, t)\, dt.$$

Proof. Proposition 3.11 shows that V is integrable with respect to the measure $(dd^c\varphi_{\geq r})^n$, hence V is μ_r-integrable. By definition

$$\nu(dd^cV, \varphi, t) = \int_{\varphi(z)<t} dd^cV \wedge (dd^c\varphi)^{n-1}$$

and the Fubini theorem gives

$$\int_{-\infty}^{r} \nu(dd^c V, \varphi, t)\, dt = \iint_{\varphi(z)<t<r} dd^c V(z) \wedge (dd^c\varphi(z))^{n-1}\, dt$$

$$(6.6) \qquad = \int_{B(r)} (r-\varphi) dd^c V \wedge (dd^c\varphi)^{n-1}.$$

We first show that Formula 6.5 is true when φ and V are smooth. As both members of the formula are left continuous with respect to r and as almost all values of φ are non critical by Sard's theorem, we may assume r non critical. Formula 3.1 applied with $f = (r-\varphi)(dd^c\varphi)^{n-1}$ and $g = V$ shows that integral (6.6) is equal to

$$\int_{S(r)} V(dd^c\varphi)^{n-1} \wedge d^c\varphi - \int_{B(r)} V\,(dd^c\varphi)^n = \mu_r(V) - \int_{B(r)} V\,(dd^c\varphi)^n.$$

Formula 6.5 is thus proved when φ and V are smooth. If V is smooth and φ merely continuous and finite, one can write $\varphi = \lim \varphi_k$ where φ_k is a decreasing sequence of smooth plurisubharmonic functions (because X is Stein). Then $dd^c V \wedge (dd^c\varphi_k)^{n-1}$ converges weakly to $dd^c V \wedge (dd^c\varphi)^{n-1}$ and (6.6) converges, since $\mathbb{1}_{B(r)}(r-\varphi)$ is continuous with compact support on X. The left hand side of Formula 6.5 also converges because the definition of μ_r implies

$$\mu_{k,r}(V) - \int_{\varphi_k<r} V(dd^c\varphi_k)^n = \int_X V\big((dd^c\varphi_{k,\geq r})^n - (dd^c\varphi_k)^n\big)$$

and we can apply again weak convergence on a neighborhood of $\overline{B}(r)$. If φ takes $-\infty$ values, replace φ by $\varphi_{\geq -k}$ where $k \to +\infty$. Then $\mu_r(V)$ is unchanged, $\int_{B(r)} V(dd^c\varphi_{\geq -k})^n$ converges to $\int_{B(r)} V(dd^c\varphi)^n$ and the right hand side of Formula 6.5 is replaced by $\int_{-k}^{r} \nu(dd^c V, \varphi, t)\, dt$. Finally, for V arbitrary, write $V = \lim \downarrow V_k$ with a sequence of smooth functions V_k. Then $dd^c V_k \wedge (dd^c\varphi)^{n-1}$ converges weakly to $dd^c V \wedge (dd^c\varphi)^{n-1}$ by Prop. 4.4, so the integral (6.6) converges to the expected limit and the same is true for the left hand side of 6.5 by the monotone convergence theorem. □

For $r < r_0 < R$, the Jensen-Lelong formula implies

$$(6.7) \quad \mu_r(V) - \mu_{r_0}(V) + \int_{B(r_0)\setminus B(r)} V(dd^c\varphi)^n = \int_{r_0}^{r} \nu(dd^c V, \varphi, t)\, dt.$$

(6.8) Corollary. *Assume that $(dd^c\varphi)^n = 0$ on $X \setminus S(-\infty)$. Then $r \mapsto \mu_r(V)$ is a convex increasing function of r and the lelong number $\nu(dd^c V, \varphi)$ is given by*

$$\nu(dd^c V, \varphi) = \lim_{r\to-\infty} \frac{\mu_r(V)}{r}.$$

Proof. By (6.7) we have

$$\mu_r(V) = \mu_{r_0}(V) + \int_{r_0}^{r} \nu(dd^cV, \varphi, t)\, dt.$$

As $\nu(dd^cV, \varphi, t)$ is increasing and nonnegative, it follows that $r \longmapsto \mu_r(V)$ is convex and increasing. The formula for $\nu(dd^cV, \varphi) = \lim_{t\to-\infty} \nu(dd^cV, \varphi, t)$ is then obvious. $\square$

(6.9) Example. Let X be an open subset of $\mathbb{C}^n$ equipped with the semi-exhaustive function $\varphi(z) = \log|z-a|$, $a \in X$. Then $(dd^c\varphi)^n = \delta_a$ and the Jensen-Lelong formula becomes

$$\mu_r(V) = V(a) + \int_{-\infty}^{r} \nu(dd^cV, \varphi, t)\, dt.$$

As μ_r is the mean value measure on the sphere $S(a, e^r)$, we make the change of variables $r \mapsto \log r$, $t \mapsto \log t$ and obtain the more familiar formula

$$\text{(6.9 a)} \quad \mu(V, S(a,r)) = V(a) + \int_0^r \nu(dd^cV, a, t)\, \frac{dt}{t}$$

where $\nu(dd^cV, a, t) = \nu(dd^cV, \varphi, \log t)$ is given by (5.7):

$$\text{(6.9 b)} \quad \nu(dd^cV, a, t) = \frac{1}{\pi^{n-1}t^{2n-2}/(n-1)!} \int_{B(a,t)} \frac{1}{2\pi}\Delta V.$$

In this setting, Cor. 6.8 implies

$$\text{(6.9 c)} \quad \nu(dd^cV, a) = \lim_{r\to 0} \frac{\mu\big(V, S(a,r)\big)}{\log r} = \lim_{r\to 0} \frac{\sup_{S(a,r)} V}{\log r}.$$

To prove the last equality, we may assume $V \le 0$ after subtraction of a constant. Inequality $\ge$ follows from the obvious estimate $\mu(V, S(a,r)) \le \sup_{S(a,r)} V$, while inequality $\le$ follows from the standard Harnack estimate

$$\text{(6.9 d)} \quad \sup_{S(a,\varepsilon r)} V \le \frac{1-\varepsilon}{(1+\varepsilon)^{2n-1}}\, \mu\big(V, S(a,r)\big)$$

when ε is small (this estimate follows easily from the Green-Riesz representation formula 1.4.6 and 1.4.7). As $\sup_{S(a,r)} V = \sup_{B(a,r)} V$, Formula (6.9 c) can also be rewritten $\nu(dd^cV, a) = \liminf_{z\to a} V(z)/\log|z-a|$. Since $\sup_{S(a,r)} V$ is a convex (increasing) function of $\log r$, we infer that

$$\text{(6.9 e)} \quad V(z) \le \gamma \log|z-a| + \mathrm{O}(1)$$

with $\gamma = \nu(dd^cV, a)$, and $\nu(dd^cV, a)$ is the largest constant γ which satisfies this inequality. Thus $\nu(dd^cV, a) = \gamma$ is equivalent to V having a logarithmic pole of coefficient γ.

(6.10) Special case Take in particular $V = \log|f|$ where f is a holomorphic function on X. The Lelong-Poincaré formula shows that $dd^c \log|f|$ is equal to the zero divisor $[Z_f] = \sum m_j [H_j]$, where H_j are the irreducible components of $f^{-1}(0)$ and m_j is the multiplicity of f on H_j. The trace $\frac{1}{2\pi}\Delta f$ is then the euclidean area measure of Z_f (with corresponding multiplicities m_j). By Formula (6.9 c), we see that the Lelong number $\nu([Z_f], a)$ is equal to the vanishing order $\mathrm{ord}_a(f)$, that is, the smallest integer m such that $D^\alpha f(a) \neq 0$ for some multiindex α with $|\alpha| = m$. In dimension $n = 1$, we have $\frac{1}{2\pi}\Delta f = \sum m_j \delta_{a_j}$. Then (6.9 a) is the usual Jensen formula

$$\mu\big(\log|f|, S(0,r)\big) - \log|f(0)| = \int_0^r \nu(t)\frac{dt}{t} = \sum m_j \log\frac{r}{|a_j|}$$

where $\nu(t)$ is the number of zeros a_j in the disk $D(0,t)$, counted with multiplicities m_j.

(6.11) Example. Take $\varphi(z) = \log\max|z_j|^{\lambda_j}$ where $\lambda_j > 0$. Then $B(r)$ is the polydisk of radii $(e^{r/\lambda_1}, \ldots, e^{r/\lambda_n})$. If some coordinate z_j is non zero, say z_1, we can write $\varphi(z)$ as $\lambda_1 \log|z_1|$ plus some function depending only on the $(n-1)$ variables $z_j/z_1^{\lambda_1/\lambda_j}$. Hence $(dd^c\varphi)^n = 0$ on $\mathbb{C}^n \setminus \{0\}$. It will be shown later that

(6.11 a) $\quad (dd^c\varphi)^n = \lambda_1 \ldots \lambda_n\, \delta_0.$

We now determine the measures μ_r. At any point z where not all terms $|z_j|^{\lambda_j}$ are equal, the smallest one can be omitted without changing φ in a neighborhood of z. Thus φ depends only on $(n-1)$-variables and $(dd^c\varphi_{\geq r})^n = 0$, $\mu_r = 0$ near z. It follows that μ_r is supported by the distinguished boundary $|z_j| = e^{r/\lambda_j}$ of the polydisk $B(r)$. As φ is invariant by all rotations $z_j \longmapsto e^{\mathrm{i}\theta_j} z_j$, the measure μ_r is also invariant and we see that μ_r is a constant multiple of $d\theta_1 \ldots d\theta_n$. By formula (6.2) and (6.11 a) we get

(6.11 b) $\quad \mu_r = \lambda_1 \ldots \lambda_n\, (2\pi)^{-n} d\theta_1 \ldots d\theta_n.$

In particular, the Lelong number $\nu(dd^cV, \varphi)$ is given by

$$\nu(dd^cV, \varphi) = \lim_{r\to-\infty} \frac{\lambda_1 \ldots \lambda_n}{r} \int_{\theta_j \in [0,2\pi]} V(e^{r/\lambda_1 + \mathrm{i}\theta_1}, \ldots, e^{r/\lambda_n + \mathrm{i}\theta_n})\, \frac{d\theta_1 \ldots d\theta_n}{(2\pi)^n}.$$

These numbers have been introduced and studied by (Kiselman 1986). We call them *directional Lelong numbers* with coefficients $(\lambda_1, \ldots, \lambda_n)$. For an arbitrary current T, we define

(6.11 c) $\quad \nu(T, x, \lambda) = \nu\big(T, \log\max|z_j - x_j|^{\lambda_j}\big).$

The above formula for $\nu(dd^cV, \varphi)$ combined with the analogue of Harnack's inequality (6.9 d) for polydisks gives

$$\nu(dd^cV,x,\lambda) = \lim_{r\to 0}\frac{\lambda_1\ldots\lambda_n}{\log r}\int V(r^{1/\lambda_1}e^{\mathrm{i}\theta_1},\ldots,r^{1/\lambda_n}e^{\mathrm{i}\theta_n})\,\frac{d\theta_1\ldots d\theta_n}{(2\pi)^n}$$

$$= \lim_{r\to 0}\frac{\lambda_1\ldots\lambda_n}{\log r}\sup_{\theta_1,\ldots,\theta_n} V(r^{1/\lambda_1}e^{\mathrm{i}\theta_1},\ldots,r^{1/\lambda_n}e^{\mathrm{i}\theta_n}). \tag{6.11 d}$$

7. Comparison Theorems for Lelong Numbers

Let T be a closed positive current of bidimension (p,p) on a Stein manifold X equipped with a semi-exhaustive plurisubharmonic weight φ. We first show that the Lelong numbers $\nu(T,\varphi)$ only depend on the asymptotic behaviour of φ near the polar set $S(-\infty)$. In a precise way:

(7.1) First comparison theorem. *Let $\varphi,\psi : X \longrightarrow [-\infty,+\infty[$ be continuous plurisubharmonic functions. We assume that φ,ψ are semi-exhaustive on* Supp T *and that*

$$\ell := \limsup\frac{\psi(x)}{\varphi(x)} < +\infty \qquad \textit{as}\quad x\in \operatorname{Supp} T \quad \textit{and}\quad \varphi(x)\to -\infty.$$

Then $\nu(T,\psi)\le \ell^p\nu(T,\varphi)$, and the equality holds if $\ell = \lim\psi/\varphi$.

Proof. Definition 6.4 shows immediately that $\nu(T,\lambda\varphi) = \lambda^p\nu(T,\varphi)$ for every scalar $\lambda > 0$. It is thus sufficient to verify the inequality $\nu(T,\psi)\le\nu(T,\varphi)$ under the hypothesis $\limsup\psi/\varphi < 1$. For all $c > 0$, consider the plurisubharmonic function

$$u_c = \max(\psi - c,\varphi).$$

Let R_φ and R_ψ be such that $B_\varphi(R_\varphi)\cap\operatorname{Supp} T$ and $B_\psi(R_\psi)\cap\operatorname{Supp} T$ be relatively compact in X. Let $r < R_\varphi$ and $a < r$ be fixed. For $c > 0$ large enough, we have $u_c = \varphi$ on $\varphi^{-1}([a,r])$ and Stokes' formula gives

$$\nu(T,\varphi,r) = \nu(T,u_c,r) \ge \nu(T,u_c).$$

The hypothesis $\limsup\psi/\varphi < 1$ implies on the other hand that there exists $t_0 < 0$ such that $u_c = \psi - c$ on $\{u_c < t_0\}\cap\operatorname{Supp} T$. We infer

$$\nu(T,u_c) = \nu(T,\psi - c) = \nu(T,\psi),$$

hence $\nu(T,\psi)\le\nu(T,\varphi)$. The equality case is obtained by reversing the roles of φ and ψ and observing that $\lim\varphi/\psi = 1/l$. □

Assume in particular that $z^k = (z_1^k,\ldots,z_n^k)$, $k = 1,2$, are coordinate systems centered at a point $x\in X$ and let

$$\varphi_k(z) = \log|z^k| = \log\big(|z_1^k|^2+\ldots+|z_n^k|^2\big)^{1/2}.$$

We have $\lim_{z\to x}\varphi_2(z)/\varphi_1(z)=1$, hence $\nu(T,\varphi_1)=\nu(T,\varphi_2)$ by Th. 7.1.

(7.2) Corollary. *The usual Lelong numbers $\nu(T,x)$ are independent of the choice of local coordinates.* □

This result had been originally proved by (Siu 1974) with a much more delicate proof. Another interesting consequence is:

(7.3) Corollary. *On an open subset of $\mathbb{C}^n$, the Lelong numbers and Kiselman numbers are related by*

$$\nu(T,x)=\nu\big(T,x,(1,\ldots,1)\big).$$

Proof. By definition, the Lelong number $\nu(T,x)$ is associated with the weight $\varphi(z)=\log|z-x|$ and the Kiselman number $\nu\big(T,x,(1,\ldots,1)\big)$ to the weight $\psi(z)=\log\max|z_j-x_j|$. It is clear that $\lim_{z\to x}\psi(z)/\varphi(z)=1$, whence the conclusion. □

Another consequence of Th. 7.1 is that $\nu(T,x,\lambda)$ is an increasing function of each variable λ_j. Moreover, if $\lambda_1\le\ldots\le\lambda_n$, we get the inequalities

$$\lambda_1^p\nu(T,x)\le\nu(T,x,\lambda)\le\lambda_n^p\nu(T,x).$$

These inequalities will be improved in section 7 (see Cor. 9.16). For the moment, we just prove the following special case.

(7.4) Corollary. *For all $\lambda_1,\ldots,\lambda_n>0$ we have*

$$\big(dd^c\log\max_{1\le j\le n}|z_j|^{\lambda_j}\big)^n=\Big(dd^c\log\sum_{1\le j\le n}|z_j|^{\lambda_j}\Big)^n=\lambda_1\ldots\lambda_n\,\delta_0.$$

Proof. In fact, our measures vanish on $\mathbb{C}^n\smallsetminus\{0\}$ by the arguments explained in example 6.11. Hence they are equal to $c\,\delta_0$ for some constant $c\ge0$ which is simply the Lelong number of the bidimension (n,n)-current $T=[X]=1$ with the corresponding weight. The comparison theorem shows that the first equality holds and that

$$\Big(dd^c\log\sum_{1\le j\le n}|z_j|^{\lambda_j}\Big)^n=\ell^{-n}\Big(dd^c\log\sum_{1\le j\le n}|z_j|^{\ell\lambda_j}\Big)^n$$

for all $\ell>0$. By taking ℓ large and approximating $\ell\lambda_j$ with $2[\ell\lambda_j/2]$, we may assume that $\lambda_j=2s_j$ is an even integer. Then formula (5.6) gives

$$\int_{\sum |z_j|^{2s_j}<r^2} \left(dd^c \log \sum |z_j|^{2s_j}\right)^n = r^{-2n} \int_{\sum |z_j|^{2s_j}<r^2} \left(dd^c \sum |z_j|^{2s_j}\right)^n$$
$$= s_1 \ldots s_n\, r^{-2n} \int_{\sum |w_j|^2<r^2} 2^n \left(\frac{\mathrm{i}}{2\pi} d'd''|w|^2\right)^n = \lambda_1 \ldots \lambda_n$$

by using the $s_1 \ldots s_n$-sheeted change of variables $w_j = z_j^{s_j}$. $\square$

Now, we assume that $T = [A]$ is the current of integration over an analytic set $A \subset X$ of pure dimension p. The above comparison theorem will enable us to give a simple proof of P. Thie's main result (Thie 1967): the Lelong number $\nu([A], x)$ can be interpreted as the multiplicity of the analytic set A at point x. Our starting point is the following consequence of Th. II.3.19 applied simultaneously to all irreducible components of (A, x).

(7.5) Lemma. *For a generic choice of local coordinates $z' = (z_1, \ldots, z_p)$ and $z'' = (z_{p+1}, \ldots, z_n)$ on (X, x), the germ (A, x) is contained in a cone $|z''| \le C|z'|$. If $B' \subset \mathbb{C}^p$ is a ball of center 0 and radius r' small, and $B'' \subset \mathbb{C}^{n-p}$ is the ball of center 0 and radius $r'' = Cr'$, then the projection*

$$\mathrm{pr} : A \cap (B' \times B'') \longrightarrow B'$$

is a ramified covering with finite sheet number m. $\square$

We use these properties to compute the Lelong number of $[A]$ at point x. When $z \in A$ tends to x, the functions

$$\varphi(z) = \log |z| = \log(|z'|^2 + |z''|^2)^{1/2}, \qquad \psi(z) = \log |z'|.$$

are equivalent. As φ, ψ are semi-exhaustive on A, Th. 7.1 implies

$$\nu([A], x) = \nu([A], \varphi) = \nu([A], \psi).$$

Let us apply formula (5.6) to ψ: for every $t < r'$ we get

$$\begin{aligned}
\nu([A], \psi, \log t) &= t^{-2p} \int_{\{\psi<\log t\}} [A] \wedge \left(\frac{1}{2} dd^c e^{2\psi}\right)^p \\
&= t^{-2p} \int_{A\cap\{|z'|<t\}} \left(\frac{1}{2} \mathrm{pr}^\star dd^c |z'|^2\right)^p \\
&= m\, t^{-2p} \int_{\mathbb{C}^p\cap\{|z'|<t\}} \left(\frac{1}{2} dd^c |z'|^2\right)^p = m,
\end{aligned}$$

hence $\nu([A], \psi) = m$. Here, we have used the fact that pr is an étale covering with m sheets over the complement of the ramification locus $S \subset B'$, and the fact that S is of zero Lebesgue measure in B'. We have thus obtained simultaneously the following two results:

(7.6) Theorem and Definition. *Let A be an analytic set of dimension p in a complex manifold X of dimension n. For a generic choice of local coordinates $z' = (z_1, \ldots, z_p)$, $z'' = (z_{p+1}, \ldots, z_n)$ near a point $x \in A$ such that the germ (A, x) is contained in a cone $|z''| \le C|z'|$, the sheet number m of the projection $(A, x) \to (\mathbb{C}^p, 0)$ onto the first p coordinates is independent of the choice of z', z''. This number m is called the multiplicity of A at x.*

(7.7) Theorem (Thie 1967). *One has $\nu([A], x) = m$.* □

There is another interesting version of the comparison theorem which compares the Lelong numbers of two currents obtained as intersection products (in that case, we take the same weight for both).

(7.8) Second comparison theorem. *Let $u_1, \ldots, u_q$ and $v_1, \ldots, v_q$ be plurisubharmonic functions such that each q-tuple satisfies the hypotheses of Th. 4.5 with respect to T. Suppose moreover that $u_j = -\infty$ on* Supp $T \cap \varphi^{-1}(-\infty)$ *and that*

$$\ell_j := \limsup \frac{v_j(z)}{u_j(z)} < +\infty \qquad \text{when} \quad z \in \operatorname{Supp} T \smallsetminus u_j^{-1}(-\infty), \quad \varphi(z) \to -\infty.$$

Then

$$\nu(dd^c v_1 \wedge \ldots \wedge dd^c v_q \wedge T, \varphi) \le \ell_1 \ldots \ell_q \, \nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T, \varphi).$$

Proof. By homogeneity in each factor v_j, it is enough to prove the inequality with constants $\ell_j = 1$ under the hypothesis $\limsup v_j/u_j < 1$. We set

$$w_{j,c} = \max\{v_j - c, u_j\}.$$

Our assumption implies that $w_{j,c}$ coincides with $v_j - c$ on a neighborhood Supp $T \cap \{\varphi < r_0\}$ of Supp $T \cap \{\varphi < -\infty\}$, thus

$$\nu(dd^c v_1 \wedge \ldots \wedge dd^c v_q \wedge T, \varphi) = \nu(dd^c w_{1,c} \wedge \ldots \wedge dd^c w_{q,c} \wedge T, \varphi)$$

for every c. Now, fix $r < R_\varphi$. Proposition 4.9 shows that the current $dd^c w_{1,c} \wedge \ldots \wedge dd^c w_{q,c} \wedge T$ converges weakly to $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ when c tends to $+\infty$. By Prop. 5.12 we get

$$\limsup_{c \to +\infty} \nu(dd^c w_{1,c} \wedge \ldots \wedge dd^c w_{q,c} \wedge T, \varphi) \le \nu(dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T, \varphi). \square$$

(7.9) Corollary. *If $dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T$ is well defined, then at every point $x \in X$ we have*

$$\nu\big(dd^c u_1 \wedge \ldots \wedge dd^c u_q \wedge T, x\big) \ge \nu(dd^c u_1, x) \ldots \nu(dd^c u_q, x) \, \nu(T, x).$$

Proof. Apply (7.8) with $\varphi(z) = v_1(z) = \ldots = v_q(z) = \log|z-x|$ and observe that $\ell_j := \limsup v_j/u_j = 1/\nu(dd^c u_j, x)$ (there is nothing to prove if $\nu(dd^c u_j, x) = 0$). □

Finally, we present an interesting stability property of Lelong numbers due to (Siu 1974): almost all slices of a closed positive current T along linear subspaces passing through a given point have the same Lelong number as T. Before giving a proof of this, we need a useful formula known as *Crofton's formula.*

(7.10) Lemma. *Let α be a closed positive (p,p)-form on $\mathbb{C}^n \smallsetminus \{0\}$ which is invariant under the unitary group $U(n)$. Then α has the form*

$$\alpha = \big(dd^c\chi(\log|z|)\big)^p$$

where χ is a convex increasing function. Moreover α is invariant by homotheties if and only if χ is an affine function, i.e. $\alpha = \lambda\,(dd^c\log|z|)^p$.

Proof. A radial convolution $\alpha_\varepsilon(z) = \int_{\mathbb{R}} \rho(t/\varepsilon)\,\alpha(e^t z)\,dt$ produces a smooth form with the same properties as α and $\lim_{\varepsilon\to 0}\alpha_\varepsilon = \alpha$. Hence we can suppose that α is smooth on $\mathbb{C}^n \smallsetminus \{0\}$. At a point $z = (0,\ldots,0,z_n)$, the (p,p)-form $\alpha(z) \in \bigwedge^{p,p}(\mathbb{C}^n)^\star$ must be invariant by $U(n-1)$ acting on the first $(n-1)$ coordinates. We claim that the subspace of $U(n-1)$-invariants in $\bigwedge^{p,p}(\mathbb{C}^n)^\star$ is generated by $(dd^c|z|^2)^p$ and $(dd^c|z|^2)^{p-1}\wedge \mathrm{i}dz_n\wedge d\overline{z}_n$. In fact, a form $\beta = \sum \beta_{I,J}dz_I\wedge d\overline{z}_J$ is invariant by $U(1)^{n-1}\subset U(n-1)$ if and only if $\beta_{I,J} = 0$ for $I\neq J$, and invariant by the permutation group $\mathcal{S}_{n-1}\subset U(n-1)$ if and only if all coefficients $\beta_{I,I}$ (resp. $\beta_{Jn,Jn}$) with $I,J\subset\{1,\ldots,n-1\}$ are equal. Hence

$$\beta = \lambda\sum_{|I|=p} dz_I\wedge d\overline{z}_I + \mu\Big(\sum_{|J|=p-1} dz_J\wedge d\overline{z}_J\Big)\wedge dz_n\wedge d\overline{z}_n.$$

This proves our claim. As $d|z|^2\wedge d^c|z|^2 = \frac{\mathrm{i}}{\pi}|z_n|^2 dz_n\wedge d\overline{z}_n$ at $(0,\ldots,0,z_n)$, we conclude that

$$\alpha(z) = f(z)(dd^c|z|^2)^p + g(z)(dd^c|z|^2)^{p-1}\wedge d|z|^2\wedge d^c|z|^2$$

for some smooth functions f,g on $\mathbb{C}^n\smallsetminus\{0\}$. The $U(n)$-invariance of α shows that f and g are radial functions. We may rewrite the last formula as

$$\alpha(z) = u(\log|z|)(dd^c\log|z|)^p + v(\log|z|)(dd^c\log|z|)^{p-1}\wedge d\log|z|\wedge d^c\log|z|.$$

Here $(dd^c\log|z|)^p$ is a positive (p,p)-form coming from $\mathbb{P}^{n-1}$, hence it has zero contraction in the radial direction, while the contraction of the form $(dd^c\log|z|)^{p-1}\wedge d\log|z|\wedge d^c\log|z|$ by the radial vector field is $(dd^c\log|z|)^{p-1}$. This shows easily that $\alpha(z)\geq 0$ if and only if $u,v\geq 0$. Next, the closedness condition $d\alpha = 0$ gives $u' - v = 0$. Thus u is increasing and we define a convex increasing function χ by $\chi' = u^{1/p}$. Then $v = u' = p\chi'^{p-1}\chi''$ and

$$\alpha(z) = \big(dd^c\chi(\log|z|)\big)^p.$$

If α is invariant by homotheties, the functions u and v must be constant, thus $v = 0$ and $\alpha = \lambda(dd^c\log|z|)^p$. □

(7.11) Corollary (Crofton's formula). *Let dv be the unique $U(n)$-invariant measure of mass 1 on the Grassmannian $G(p,n)$ of p-dimensional subspaces in $\mathbb{C}^n$. Then*

$$\int_{S\in G(p,n)} [S]\,dv(S) = (dd^c\log|z|)^{n-p}.$$

Proof. The left hand integral is a closed positive bidegree $(n-p,n-p)$ current which is invariant by $U(n)$ and by homotheties. By Lemma 7.10, this current must coincide with the form $\lambda(dd^c\log|z|)^{n-p}$ for some $\lambda \geq 0$. The coefficient λ is the Lelong number at 0. As $\nu([S],0) = 1$ for every S, we get $\lambda = \int_{G(p,n)} dv(S) = 1$. □

We now recall a few basic facts of slicing theory; see (Federer 1969) for details. Let $\sigma : M \to M'$ be a submersion of smooth differentiable manifolds and let Θ be a *locally flat* current on M, that is a current which can be written locally as $\Theta = U + dV$ where U, V have locally integrable coefficients. It can be shown that every current Θ such that both Θ and $d\Theta$ have measure coefficients is locally flat; in particular, closed positive currents are locally flats. Then, for almost every $x' \in M'$, there is a well defined slice $\Theta_{x'}$, which is the current on the fiber $\sigma^{-1}(x')$ defined by

$$\Theta_{x'} = U_{\restriction\sigma^{-1}(x')} + dV_{\restriction\sigma^{-1}(x')}.$$

The restrictions of U, V to the fibers exist for almost all x' by the Fubini theorem. It is easy to show by a regularization $\Theta_\varepsilon = \Theta \star \rho_\varepsilon$ that the slices of a closed positive current are again closed and positive: in fact $U_{\varepsilon,x'}$ and $V_{\varepsilon,x'}$ converge to $U_{x'}$ and $V_{x'}$ in L^1_{loc}, thus $\Theta_{\varepsilon,x'}$ converges weakly to $\Theta_{x'}$ for almost every x'. This kind of slicing can be referred to as *parallel slicing* (if we think of σ as being a projection map). The kind of slicing we need (where the slices are taken over linear subspaces passing through a given point) is of a slightly different nature and is called *concurrent slicing.*

The possibility of concurrent slicing is proved as follows. Let T be a closed positive current of bidimension (p,p) in the ball $B(0,R) \subset \mathbb{C}^n$. Let

$$Y = \big\{(x,S) \in \mathbb{C}^n \times G(q,n)\,;\, x \in S\big\}$$

be the total space of the tautological rank q vector bundle over the Grassmannian $G(q,n)$, equipped with the obvious projections

$$\sigma : Y \longrightarrow G(q,n), \qquad \pi : Y \longrightarrow \mathbb{C}^n.$$

We set $Y_R = \pi^{-1}(B(0,R))$ and $Y_R^\star = \pi^{-1}(B(0,R) \smallsetminus \{0\})$. The restriction π_0 of π to $Y_R^\star$ is a submersion onto $B(0,R) \smallsetminus \{0\}$, so we have a well defined pull-back $\pi_0^\star T$ over $Y_R^\star$. We would like to extend it as a pull-back $\pi^\star T$ over Y_R, so as to define slices $T_{\restriction S} = (\pi^\star T)_{\restriction \sigma^{-1}(S)}$; of course, these slices can be non zero only if the dimension of S is at least equal to the degree of T, i.e. if $q \geq n-p$. We first claim that $\pi_0^\star T$ has locally finite mass near the zero section $\pi^{-1}(0)$ of σ. In fact let ω_G be a unitary invariant Kähler metric over $G(q,n)$ and let $\beta = dd^c|z|^2$ in $\mathbb{C}^n$. Then we get a Kähler metric on Y defined by $\omega_Y = \sigma^\star\omega_G + \pi^\star\beta$. If $N = (q-1)(n-q)$ is the dimension of the fibers of π, the projection formula $\pi_\star(u \wedge \pi^\star v) = (\pi_\star u) \wedge v$ gives

$$\pi_\star\omega_Y^{N+p} = \sum_{1\leq k\leq p} \binom{N+p}{k} \beta^k \wedge \pi_\star(\sigma^\star\omega_G^{N+p-k}).$$

Here $\pi_\star(\sigma^\star\omega_G^{N+p-k})$ is a unitary and homothety invariant $(p-k, p-k)$ closed positive form on $\mathbb{C}^n \smallsetminus \{0\}$, so $\pi_\star(\sigma^\star\omega_G^{N+p-k})$ is proportional to $(dd^c \log|z|)^{n-k}$. With some constants $\lambda_k > 0$, we thus get

$$\begin{aligned}\int_{Y_r^\star} \pi_0^\star T \wedge \omega_Y^{N+p} &= \sum_{0\leq k\leq p} \lambda_k \int_{B(0,r)\smallsetminus\{0\}} T \wedge \beta^k \wedge (dd^c\log|z|)^{k-p}\\ &= \sum_{0\leq k\leq p} \lambda_k 2^{-(p-k)} r^{-2(p-k)} \int_{B(0,r)\smallsetminus\{0\}} T \wedge \beta^p < +\infty.\end{aligned}$$

The Skoda-El Mir theorem 2.3 shows that the trivial extension $\widetilde{\pi}_0^\star T$ of $\pi_0^\star T$ is a closed positive current on Y_R. Of course, the zero section $\pi^{-1}(0)$ might also carry some extra mass of the desired current $\pi^\star T$. Since $\pi^{-1}(0)$ has codimension q, this extra mass cannot exist when $q > n-p = \operatorname{codim} \pi^\star T$ and we simply set $\pi^\star T = \widetilde{\pi}_0^\star T$. On the other hand, if $q = n-p$, we set

(7.12) $\quad \pi^\star T := \widetilde{\pi}_0^\star T + \nu(T,0)\,[\pi^{-1}(0)].$

We can now apply parallel slicing with respect to $\sigma : Y_R \to G(q,n)$, which is a submersion: for almost all $S \in G(q,n)$, there is a well defined slice $T_{\restriction S} = (\pi^\star T)_{\restriction \sigma^{-1}(S)}$. These slices coincide with the usual restrictions of T to S if T is smooth.

(7.13) Theorem (Siu 1974)**.** *For almost all $S \in G(q,n)$ with $q \geq n-p$, the slice $T_{\restriction S}$ satisfies $\nu(T_{\restriction S}, 0) = \nu(T,0)$.*

Proof. If $q = n-p$, the slice $T_{\restriction S}$ consists of some positive measure with support in $S \smallsetminus \{0\}$ plus a Dirac measure $\nu(T,0)\,\delta_0$ coming from the slice of $\nu(T,0)\,[\pi^{-1}(0)]$. The equality $\nu(T_{\restriction S}, 0) = \nu(T,0)$ thus follows directly from (7.12).

In the general case $q > n-p$, it is clearly sufficient to prove the following two properties:

a) $\nu(T,0,r) = \int_{S\in G(q,n)} \nu(T_{\restriction S},0,r)\,dv(S)$ for all $r \in\,]0,R[\,$;

b) $\nu(T_{\restriction S},0) \ge \nu(T,0)$ for almost all S.

In fact, a) implies that $\nu(T,0)$ is the average of all Lelong numbers $\nu(T_{\restriction S},0)$ and the conjunction with b) implies that these numbers must be equal to $\nu(T,0)$ for almost all S. In order to prove a) and b), we can suppose without loss of generality that T is smooth on $B(0,R)\smallsetminus\{0\}$. Otherwise, we perform a small convolution with respect to the action of $\mathrm{Gl}_n(\mathbb{C})$ on $\mathbb{C}^n$:

$$T_\varepsilon = \int_{g\in \mathrm{Gl}_n(\mathbb{C})} \rho_\varepsilon(g)\, g^\star T\, dv(g)$$

where (ρ_ε) is a regularizing family with support in an ε-neighborhood of the unit element of $\mathrm{Gl}_n(\mathbb{C})$. Then T_ε is smooth in $B(0,(1-\varepsilon)R)\smallsetminus\{0\}$ and converges weakly to T. Moreover, we have $\nu(T_\varepsilon,0) = \nu(T,0)$ by (7.2) and $\nu(T_{\restriction S},0) \ge \limsup_{\varepsilon\to 0} \nu(T_{\varepsilon,\restriction S},0)$ by (5.12), thus a), b) are preserved in the limit. If T is smooth on $B(0,R)\smallsetminus\{0\}$, the slice $T_{\restriction S}$ is defined for all S and is simply the restriction of T to $S\smallsetminus\{0\}$ (carrying no mass at the origin).

a) Here we may even assume that T is smooth at 0 by performing an ordinary convolution. As $T_{\restriction S}$ has bidegree $(n-p,n-p)$, we have

$$\nu(T_{\restriction S},0,r) = \int_{S\cap B(0,r)} T\wedge \alpha_S^{q-(n-p)} = \int_{B(0,r)} T\wedge [S]\wedge \alpha_S^{p+q-n}$$

where $\alpha_S = dd^c\log|w|$ and $w=(w_1,\ldots,w_q)$ are orthonormal coordinates on S. We simply have to check that

$$\int_{S\in G(q,n)} [S]\wedge\alpha_S^{p+q-n}\,dv(S) = (dd^c\log|z|)^p.$$

However, both sides are unitary and homothety invariant (p,p)-forms with Lelong number 1 at the origin, so they must coincide by Lemma 7.11.

b) We prove the inequality when $S=\mathbb{C}^q\times\{0\}$. By the comparison theorem 7.1, for every $r>0$ and $\varepsilon>0$ we have

$$(7.14)\qquad \int_{B(0,r)} T\wedge\gamma_\varepsilon^p \ge \nu(T,0) \quad\text{where}$$
$$\gamma_\varepsilon = \frac{1}{2}dd^c\log(\varepsilon|z_1|^2+\ldots+\varepsilon|z_q|^2+|z_{q+1}|^2+\ldots+|z_n|^2).$$

We claim that the current γ_ε^p converges weakly to

$$[S]\wedge\alpha_S^{p+q-n} = [S]\wedge\Big(\frac{1}{2}dd^c\log(|z_1|^2+\ldots+|z_q|^2)\Big)^{p+q-n}$$

as ε tends to 0. In fact, the Lelong number of γ_ε^p at 0 is 1, hence by homogeneity

$$\int_{B(0,r)} \gamma_\varepsilon^p \wedge (dd^c|z|^2)^{n-p} = (2r^2)^p$$

for all $\varepsilon, r > 0$. Therefore the family (γ_ε^p) is relatively compact in the weak topology. Since $\gamma_0 = \lim \gamma_\varepsilon$ is smooth on $\mathbb{C}^n \smallsetminus S$ and depends only on $n-q$ variables ($n - q \le p$), we have $\lim \gamma_\varepsilon^p = \gamma_0^p = 0$ on $\mathbb{C}^n \smallsetminus S$. This shows that every weak limit of (γ_ε^p) has support in S. Each of these is the direct image by inclusion of a unitary and homothety invariant $(p+q-n, p+q-n)$-form on S with Lelong number equal to 1 at 0. Therefore we must have

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^p = (i_S)_\star(\alpha_S^{p+q-n}) = [S] \wedge \alpha_S^{p+q-n},$$

and our claim is proved (of course, this can also be checked by direct elementary calculations). By taking the limsup in (7.14) we obtain

$$\nu(T_{\restriction S}, 0, r+0) = \int_{\overline{B}(0,r)} T \wedge [S] \wedge \alpha_S^{p+q-n} \ge \nu(T, 0)$$

(the singularity of T at 0 does not create any difficulty because we can modify T by a dd^c-exact form near 0 to make it smooth everywhere). Property b) follows when r tends to 0. □

8. Siu's Semicontinuity Theorem

Let X, Y be complex manifolds of dimension n, m such that X is Stein. Let $\varphi : X \times Y \longrightarrow [-\infty, +\infty[$ be a continuous plurisubharmonic function. We assume that φ is *semi-exhaustive* with respect to Supp T , i.e. that for every compact subset $L \subset Y$ there exists $R = R(L) < 0$ such that

$$\text{(8.1)} \quad \{(x, y) \in \operatorname{Supp} T \times L\,;\, \varphi(x, y) \le R\} \subset\subset X \times Y.$$

Let T be a closed positive current of bidimension (p, p) on X. For every point $y \in Y$, the function $\varphi_y(x) := \varphi(x, y)$ is semi-exhaustive on Supp T; one can therefore associate with y a generalized Lelong number $\nu(T, \varphi_y)$. Proposition 5.13 implies that the map $y \mapsto \nu(T, \varphi_y)$ is upper semi-continuous, hence the upperlevel sets

$$\text{(8.2)} \quad E_c = E_c(T, \varphi) = \{y \in Y; \nu(T, \varphi_y) \ge c\}\ ,\ \ c > 0$$

are closed. Under mild additional hypotheses, we are going to show that the sets E_c are in fact analytic subsets of Y (Demailly 1987a).

(8.3) Definition. *We say that a function $f(x,y)$ is locally Hölder continuous with respect to y on $X \times Y$ if every point of $X \times Y$ has a neighborhood Ω on which*

$$|f(x,y_1) - f(x,y_2)| \le M|y_1 - y_2|^\gamma$$

for all $(x,y_1) \in \Omega$, $(x,y_2) \in \Omega$, with some constants $M > 0$, $\gamma \in \,]0,1]$, and suitable coordinates on Y.

(8.4) Theorem (Demailly 1987a). *Let T be a closed positive current on X and let*

$$\varphi : X \times Y \longrightarrow [-\infty, +\infty[$$

be a continuous plurisubharmonic function. Assume that φ is semi-exhaustive on $\operatorname{Supp} T$ *and that $e^{\varphi(x,y)}$ is locally Hölder continuous with respect to y on $X \times Y$. Then the upperlevel sets*

$$E_c(T,\varphi) = \{y \in Y; \nu(T,\varphi_y) \ge c\}$$

are analytic subsets of Y.

This theorem can be rephrased by saying that $y \longmapsto \nu(T,\varphi_y)$ is upper semi-continuous with respect to the analytic Zariski topology. As a special case, we get the following important result of (Siu 1974):

(8.5) Corollary. *If T is a closed positive current of bidimension (p,p) on a complex manifold X, the upperlevel sets $E_c(T) = \{x \in X\,;\, \nu(T,x) \ge c\}$ of the usual Lelong numbers are analytic subsets of dimension $\le p$.*

Proof. The result is local, so we may assume that $X \subset \mathbb{C}^n$ is an open subset. Theorem 8.4 with $Y = X$ and $\varphi(x,y) = \log|x-y|$ shows that $E_c(T)$ is analytic. Moreover, Prop. 5.11 implies $\dim E_c(T) \le p$. □

(8.6) Generalization. Theorem 8.4 can be applied more generally to weight functions of the type

$$\varphi(x,y) = \max_j \,\log\Big(\sum_k |F_{j,k}(x,y)|^{\lambda_{j,k}}\Big)$$

where $F_{j,k}$ are holomorphic functions on $X \times Y$ and where $\gamma_{j,k}$ are positive real constants; in this case e^φ is Hölder continuous of exponent $\gamma = \min\{\lambda_{j,k},1\}$ and φ is semi-exhaustive with respect to the whole space X as soon as the projection $\mathrm{pr}_2 : \bigcap F_{j,k}^{-1}(0) \longrightarrow Y$ is proper and finite.

For example, when $\varphi(x,y) = \log\max|x_j - y_j|_j^\lambda$ on an open subset X of $\mathbb{C}^n$, we see that the upperlevel sets for Kiselman's numbers $\nu(T,x,\lambda)$ are analytic in X (a result first proved in (Kiselman 1986). More generally, set

$\psi_\lambda(z) = \log\max|z_j|^{\lambda_j}$ and $\varphi(x,y,g) = \psi_\lambda\big(g(x-y)\big)$ where $x, y \in \mathbb{C}^n$ and $g \in \mathrm{Gl}(\mathbb{C}^n)$. Then $\nu(T, \varphi_{y,g})$ is the Kiselman number of T at y when the coordinates have been rotated by g. It is clear that φ is plurisubharmonic in (x,y,g) and semi-exhaustive with respect to x, and that e^φ is locally Hölder continuous with respect to (y,g). Thus the upperlevel sets

$$E_c = \{(y,g) \in X \times \mathrm{Gl}(\mathbb{C}^n)\,;\, \nu(T, \varphi_{y,g}) \geq c\}$$

are analytic in $X \times \mathrm{Gl}(\mathbb{C}^n)$. However this result is not meaningful on a manifold, because it is not invariant under coordinate changes. One can obtain an invariant version as follows. Let X be a manifold and let $J^k\mathcal{O}_X$ be the bundle of k-jets of holomorphic functions on X. We consider the bundle S_k over X whose fiber $S_{k,y}$ is the set of n-tuples of k-jets $u = (u_1, \ldots, u_n) \in (J^k\mathcal{O}_{X,y})^n$ such that $u_j(y) = 0$ and $du_1 \wedge \ldots \wedge du_n(y) \neq 0$. Let (z_j) be local coordinates on an open set $\Omega \subset X$. Modulo $O(|z-y|^{k+1})$, we can write

$$u_j(z) = \sum_{1 \leq |\alpha| \leq k} a_{j,\alpha}(z-y)^\alpha$$

with $\det(a_{j,(0,\ldots,1_k,\ldots,0)}) \neq 0$. The numbers $((y_j), (a_{j,\alpha}))$ define a coordinate system on the total space of $S_{k\restriction\Omega}$. For $(x,(y,u)) \in X \times S_k$, we introduce the function

$$\varphi(x,y,u) = \log\max|u_j(x)|^{\lambda_j} = \log\max\Big|\sum_{1 \leq |\alpha| \leq k} a_{j,\alpha}(x-y)^\alpha\Big|^{\lambda_j}$$

which has all properties required by Th. 8.4 on a neighborhood of the diagonal $x = y$, i.e. a neighborhood of $X \times_X S_k$ in $X \times S_k$. For k large, we claim that Kiselman's directional Lelong numbers

$$\nu(T, y, u, \lambda) := \nu(T, \varphi_{y,u})$$

with respect to the coordinate system (u_j) at y do not depend on the selection of the jet representives and are therefore canonically defined on S_k. In fact, a change of u_j by $O(|z-y|^{k+1})$ adds $O(|z-y|^{(k+1)\lambda_j})$ to e^φ, and we have $e^\varphi \geq O(|z-y|^{\max\lambda_j})$. Hence by (7.1) it is enough to take $k+1 \geq \max\lambda_j/\min\lambda_j$. Theorem 8.4 then shows that the upperlevel sets $E_c(T,\varphi)$ are analytic in S_k. $\square$

Proof of the Semicontinuity Theorem 8.4 As the result is local on Y, we may assume without loss of generality that Y is a ball in $\mathbb{C}^m$. After addition of a constant to φ, we may also assume that there exists a compact subset $K \subset X$ such that

$$\{(x,y) \in X \times Y; \varphi(x,y) \leq 0\} \subset K \times Y.$$

By Th. 7.1, the Lelong numbers depend only on the asymptotic behaviour of φ near the (compact) polar set $\varphi^{-1}(-\infty) \cap (\mathrm{Supp\,T} \times Y)$. We can add a smooth strictly plurisubharmonic function on $X \times Y$ to make φ strictly plurisuharmonic. Then Richberg's approximation theorem for continuous plurisubharmonic functions shows that there exists a smooth plurisubharmonic function $\widetilde{\varphi}$ such that $\varphi \le \widetilde{\varphi} \le \varphi + 1$. We may therefore assume that φ is smooth on $(X \times Y) \setminus \varphi^{-1}(-\infty)$.

• **First step**: *construction of a local plurisubharmonic potential.*

Our goal is to generalize the usual construction of plurisubharmonic potentials associated with a closed positive current (Lelong 1967, Skoda 1972a). We replace here the usual kernel $|z-\zeta|^{-2p}$ arising from the hermitian metric of $\mathbb{C}^n$ by a kernel depending on the weight φ. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be an increasing function such that $\chi(t) = t$ for $t \le -1$ and $\chi(t) = 0$ for $t \ge 0$. We consider the half-plane $H = \{z \in \mathbb{C};\ \mathrm{Re} z < -1\}$ and associate with T the potential function V on $Y \times H$ defined by

$$(8.7)\quad V(y,z) = -\int_{\mathrm{Re} z}^{0} \nu(T, \varphi_y, t)\chi'(t)\,dt.$$

For every $t > \mathrm{Re}\, z$, Stokes' formula gives

$$\nu(T, \varphi_y, t) = \int_{\varphi(x,y)<t} T(x) \wedge (dd^c_x \widetilde{\varphi}(x,y,z))^p$$

with $\widetilde{\varphi}(x,y,z) := \max\{\varphi(x,y), \mathrm{Re} z\}$. The Fubini theorem applied to (8.7) gives

$$\begin{aligned} V(y,z) &= -\int_{\substack{x\in X,\varphi(x,y)<t\\ \mathrm{Re}\, z<t<0}} T(x) \wedge (dd^c_x \widetilde{\varphi}(x,y,z))^p\ \chi'(t)dt \\ &= \int_{x\in X} T(x) \wedge \chi(\widetilde{\varphi}(x,y,z))\,(dd^c_x \widetilde{\varphi}(x,y,z))^p. \end{aligned}$$

For all $(n-1, n-1)$-form h of class C^∞ with compact support in $Y \times H$, we get

$$\begin{aligned} \langle dd^c V, h\rangle &= \langle V, dd^c h\rangle \\ &= \int_{X\times Y\times H} T(x) \wedge \chi(\widetilde{\varphi}(x,y,z))(dd^c \widetilde{\varphi}(x,y,z))^p \wedge dd^c h(y,z). \end{aligned}$$

Observe that the replacement of dd^c_x by the total differentiation $dd^c = dd^c_{x,y,z}$ does not modify the integrand, because the terms in dx, $d\overline{x}$ must have total bidegree (n,n). The current $T(x) \wedge \chi(\widetilde{\varphi}(x,y,z))h(y,z)$ has compact support in $X \times Y \times H$. An integration by parts can thus be performed to obtain

$$\langle dd^c V, h\rangle = \int_{X\times Y\times H} T(x) \wedge dd^c(\chi \circ \widetilde{\varphi}(x,y,z)) \wedge (dd^c \widetilde{\varphi}(x,y,z))^p.h(y,z).$$

On the corona $\{-1 \le \varphi(x,y) \le 0\}$ we have $\widetilde{\varphi}(x,y,z) = \varphi(x,y)$, whereas for $\varphi(x,y) < -1$ we get $\widetilde{\varphi} < -1$ and $\chi \circ \widetilde{\varphi} = \widetilde{\varphi}$. As $\widetilde{\varphi}$ is plurisubharmonic, we see that $dd^c V(y,z)$ is the sum of the positive $(1,1)$-form

$$(y,z) \longmapsto \int_{\{x\in X;\varphi(x,y)<-1\}} T(x)\wedge (dd^c_{x,y,z}\widetilde{\varphi}(x,y,z))^{p+1}$$

and of the $(1,1)$-form independent of z

$$y \longmapsto \int_{\{x\in X;-1\le\varphi(x,y)\le 0\}} T\wedge dd^c_{x,y}(\chi\circ\varphi)\wedge (dd^c_{x,y}\varphi)^p.$$

As φ is smooth outside $\varphi^{-1}(-\infty)$, this last form has locally bounded coefficients. Hence $dd^cV(y,z)$ is ≥ 0 except perhaps for locally bounded terms. In addition, V is continuous on $Y\times H$ because $T\wedge(dd^c\widetilde{\varphi}_{y,z})^p$ is weakly continuous in the variables (y,z) by Th. 3.5. We therefore obtain the following result.

(8.8) Proposition. *There exists a positive plurisubharmonic function ρ in $C^\infty(Y)$ such that $\rho(y)+V(y,z)$ is plurisubharmonic on $Y\times H$.*

If we let $\mathrm{Re}\,z$ tend to $-\infty$, we see that the function

$$U_0(y)=\rho(y)+V(y,-\infty)=\rho(y)-\int_{-\infty}^{0}\nu(T,\varphi_y,t)\chi'(t)dt$$

is locally plurisubharmonic or $\equiv -\infty$ on Y. Furthermore, it is clear that $U_0(y)=-\infty$ at every point y such that $\nu(T,\varphi_y)>0$. If Y is connected and $U_0\not\equiv -\infty$, we already conclude that the density set $\bigcup_{c>0}E_c$ is pluripolar in Y.

• **Second step**: *application of Kiselman's minimum principle.*
Let $a\ge 0$ be arbitrary. The function

$$Y\times H\ni (y,z)\longmapsto \rho(y)+V(y,z)-a\mathrm{Re}z$$

is plurisubharmonic and independent of $\mathrm{Im}\,z$. By Kiselman's theorem 1.7.8, the Legendre transform

$$U_a(y)=\inf_{r<-1}\big\{\rho(y)+V(y,r)-ar\big\}$$

is locally plurisubharmonic or $\equiv -\infty$ on Y.

(8.9) Lemma. *Let $y_0\in Y$ be a given point.*

a) *If $a>\nu(T,\varphi_{y_0})$, then U_a is bounded below on a neighborhood of y_0.*

b) *If $a<\nu(T,\varphi_{y_0})$, then $U_a(y_0)=-\infty$.*

Proof. By definition of V (cf. (8.7)) we have

$$(8.10)\quad V(y,r)\le -\nu(T,\varphi_y,r)\int_r^0\chi'(t)dt = r\nu(T,\varphi_y,r)\le r\nu(T,\varphi_y).$$

Then clearly $U_a(y_0) = -\infty$ if $a < \nu(T, \varphi_{y_0})$. On the other hand, if $\nu(T, \varphi_{y_0}) < a$, there exists $t_0 < 0$ such that $\nu(T, \varphi_{y_0}, t_0) < a$. Fix $r_0 < t_0$. The semi-continuity property (5.13) shows that there exists a neighborhood ω of y_0 such that $\sup_{y\in\omega} \nu(T, \varphi_y, r_0) < a$. For all $y \in \omega$, we get

$$V(y, r) \geq -C - a \int_r^{r_0} \chi'(t)dt = -C + a(r - r_0),$$

and this implies $U_a(y) \geq -C - ar_0$. $\square$

(8.11) Theorem. *If Y is connected and if $E_c \neq Y$, then E_c is a closed complete pluripolar subset of Y, i.e. there exists a continuous plurisubharmonic function $w : Y \longrightarrow [-\infty, +\infty[$ such that $E_c = w^{-1}(-\infty)$.*

Proof. We first observe that the family (U_a) is increasing in a, that $U_a = -\infty$ on E_c for all $a < c$ and that $\sup_{a<c} U_a(y) > -\infty$ if $y \in Y \setminus E_c$ (apply Lemma 8.9). For any integer $k \geq 1$, let $w_k \in C^\infty(Y)$ be a plurisubharmonic regularization of $U_{c-1/k}$ such that $w_k \geq U_{c-1/k}$ on Y and $w_k \leq -2^k$ on $E_c \cap Y_k$ where $Y_k = \{y \in Y\,;\, d(y, \partial Y) \geq 1/k\}$. Then Lemma 8.9 a) shows that the family (w_k) is uniformly bounded below on every compact subset of $Y \setminus E_c$. We can also choose w_k uniformly bounded above on every compact subset of Y because $U_{c-1/k} \leq U_c$. The function

$$w = \sum_{k=1}^{+\infty} 2^{-k} w_k$$

satifies our requirements. $\square$

• **Third step**: *estimation of the singularities of the potentials U_a.*

(8.12) Lemma. *Let $y_0 \in Y$ be a given point, L a compact neighborhood of y_0, $K \subset X$ a compact subset and r_0 a real number < -1 such that*

$$\{(x, y) \in X \times L; \varphi(x, y) \leq r_0\} \subset K \times L.$$

Assume that $e^{\varphi(x,y)}$ is locally Hölder continuous in y and that

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|^\gamma$$

for all $(x, y_1, y_2) \in K \times L \times L$. Then, for all $\varepsilon \in]0, 1[$, there exists a real number $\eta(\varepsilon) > 0$ such that all $y \in Y$ with $|y - y_0| < \eta(\varepsilon)$ satisfy

$$U_a(y) \leq \rho(y) + \big((1-\varepsilon)^p \nu(T, \varphi_{y_0}) - a\big)\Big(\gamma \log|y - y_0| + \log\frac{2eM}{\varepsilon}\Big).$$

Proof. First, we try to estimate $\nu(T, \varphi_y, r)$ when $y \in L$ is near y_0. Set

$$\begin{cases} \psi(x) = (1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 & \text{if} \qquad \varphi_{y_0}(x) \le r-1 \\ \psi(x) = \max\big(\varphi_y(x), (1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2\big) & \text{if } r-1 \le \varphi_{y_0}(x) \le r \\ \psi(x) = \varphi_y(x) & \text{if} \quad r \le \varphi_{y_0}(x) \le r_0 \end{cases}$$

and verify that this definition is coherent when $|y - y_0|$ is small enough. By hypothesis

$$|e^{\varphi_y(x)} - e^{\varphi_{y_0}(x)}| \le M|y-y_0|^\gamma.$$

This inequality implies

$$\varphi_y(x) \le \varphi_{y_0}(x) + \log\big(1 + M|y-y_0|^\gamma e^{-\varphi_{y_0}(x)}\big)$$
$$\varphi_y(x) \ge \varphi_{y_0}(x) + \log\big(1 - M|y-y_0|^\gamma e^{-\varphi_{y_0}(x)}\big).$$

In particular, for $\varphi_{y_0}(x) = r$, we have $(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 = r - \varepsilon/2$, thus

$$\varphi_y(x) \ge r + \log(1 - M|y-y_0|^\gamma e^{-r}).$$

Similarly, for $\varphi_{y_0}(x) = r-1$, we have $(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2 = r - 1 + \varepsilon/2$, thus

$$\varphi_y(x) \le r - 1 + \log(1 + M|y-y_0|^\gamma e^{1-r}).$$

The definition of ψ is thus coherent as soon as $M|y-y_0|^\gamma e^{1-r} \le \varepsilon/2$, i.e.

$$\gamma \log|y-y_0| + \log\frac{2eM}{\varepsilon} \le r.$$

In this case ψ coincides with φ_y on a neighborhood of $\{\psi = r\}$, and with

$$(1-\varepsilon)\varphi_{y_0}(x) + \varepsilon r - \varepsilon/2$$

on a neighborhood of the polar set $\psi^{-1}(-\infty)$. By Stokes' formula applied to $\nu(T, \psi, r)$, we infer

$$\nu(T, \varphi_y, r) = \nu(T, \psi, r) \ge \nu(T, \psi) = (1-\varepsilon)^p \nu(T, \varphi_{y_0}).$$

From (8.10) we get $V(y, r) \le r\nu(T, \varphi_y, r)$, hence

$$U_a(y) \le \rho(y) + V(y,r) - ar \le \rho(y) + r\big(\nu(T, \varphi_y, r) - a\big),$$
$$U_a(y) \le \rho(y) + r\big((1-\varepsilon)^p \nu(T, \varphi_{y_0}) - a\big). \tag{8.13}$$

Suppose $\gamma \log|y-y_0| + \log(2eM/\varepsilon) \le r_0$, i.e. $|y-y_0| \le (\varepsilon/2eM)^{1/\gamma} e^{r_0/\gamma}$; one can then choose $r = \gamma \log|y-y_0| + \log(2eM/\varepsilon)$, and by (8.13) this yields the inequality asserted in Th. 8.12. □

• **Fourth step**: *application of the Hörmander-Bombieri-Skoda theorem.*

The end of the proof relies on the following crucial result, which is a consequence of the Hörmander-Bombieri-Skoda theorem (Bombieri 1970, Skoda 1972a, Skoda 1976); it will be proved in Chapter 8, see Cor. 8.?.?.

(8.14) Proposition. *Let u be a plurisubharmonic function on a complex manifold Y. The set of points in a neighborhood of which e^{-u} is not integrable is an analytic subset of Y.* □

Proof of Theorem 8.4 (end). The main idea in what follows is due to (Kiselman 1979). For $a, b > 0$, we let $Z_{a,b}$ be the set of points in a neighborhood of which $\exp(-U_a/b)$ is not integrable. Then $Z_{a,b}$ is analytic, and as the family (U_a) is increasing in a, we have $Z_{a',b'} \supset Z_{a'',b''}$ if $a' \le a''$, $b' \le b''$.

Let $y_0 \in Y$ be a given point. If $y_0 \notin E_c$, then $\nu(T, \varphi_{y_0}) < c$ by definition of E_c. Choose a such that $\nu(T, \varphi_{y_0}) < a < c$. Lemma 8.9 a) implies that U_a is bounded below in a neighborhood of y_0, thus $\exp(-U_a/b)$ is integrable and $y_0 \notin Z_{a,b}$ for all $b > 0$.

On the other hand, if $y_0 \in E_c$ and if $a < c$, then Lemma 8.12 implies for all $\varepsilon > 0$ that

$$U_a(y) \le (1-\varepsilon)(c-a)\gamma \log|y - y_0| + C(\varepsilon)$$

on a neighborhood of y_0. Hence $\exp(-U_a/b)$ is non integrable at y_0 as soon as $b < (c-a)\gamma/2m$, where $m = \dim Y$. We obtain therefore

$$E_c = \bigcap_{\substack{a<c \\ b<(c-a)\gamma/2m}} Z_{a,b}.$$

This proves that E_c is an analytic subset of Y. □

Finally, we use Cor. 8.5 to derive an important decomposition formula for currents, which is again due to (Siu 1974). We first begin by two simple observations.

(8.15) Lemma. *If T is a closed positive current of bidimension (p,p) and A is an irreducible analytic set in X, we set*

$$m_A = \inf\{\nu(T,x)\,;\, x \in A\}.$$

Then $\nu(T,x) = m_A$ for all $x \in A \smallsetminus \bigcup A'_j$, where (A'_j) is a countable family of proper analytic subsets of A. We say that m_A is the generic Lelong number of T along A.

Proof. By definition of m_A and $E_c(T)$, we have $\nu(T,x) \ge m_A$ for every $x \in A$ and

$$\nu(T,x) = m_A \quad \text{on } A \smallsetminus \bigcup_{c\in\mathbb{Q},\, c>m_A} A \cap E_c(T).$$

However, for $c > m_A$, the intersection $A \cap E_c(T)$ is a proper analytic subset of A. □

(8.16) Proposition. *Let T be a closed positive current of bidimension (p,p) and let A be an irreducible p-dimensional analytic subset of X. Then $\mathbb{1}_A T = m_A[A]$, in particular $T - m_A[A]$ is positive.*

Proof. As the question is local and as a closed positive current of bidimension (p,p) cannot carry any mass on a $(p-1)$-dimensional analytic subset, it is enough to work in a neighborhood of a regular point $x_0 \in A$. Hence, by choosing suitable coordinates, we can suppose that X is an open set in $\mathbb{C}^n$ and that A is the intersection of X with a p-dimensional linear subspace. Then, for every point $a \in A$, the inequality $\nu(T,a) \geq m_A$ implies

$$\sigma_T\big(B(a,r)\big) \geq m_A\, \pi^p r^{2p}/p! = m_A \sigma_{[A]}\big(B(a,r)\big)$$

for all r such that $B(a,r) \subset X$. Now, set $\Theta = T - m_A[A]$ and $\beta = dd^c|z|^2$. Our inequality says that $\int \mathbb{1}_{B(a,r)} \Theta \wedge \beta^p \geq 0$. If we integrate this with respect to some positive continuous function f with compact support in A, we get $\int_X g_r \Theta \wedge \beta^p \geq 0$ where

$$g_r(z) = \int_A \mathbb{1}_{B(a,r)}(z)\, f(a)\, d\lambda(a) = \int_{a \in A \cap B(z,r)} f(a)\, d\lambda(a).$$

Here g_r is continuous on $\mathbb{C}^n$, and as r tends to 0 the function $g_r(z)/(\pi^p r^{2p}/p!)$ converges to f on A and to 0 on $X \smallsetminus A$, with a global uniform bound. Hence we obtain $\int \mathbb{1}_A f\, \Theta \wedge \beta^p \geq 0$. Since this inequality is true for all continuous functions $f \geq 0$ with compact support in A, we conclude that the measure $\mathbb{1}_A \Theta \wedge \beta^p$ is positive. By a linear change of coordinates, we see that

$$\mathbb{1}_A \Theta \wedge \Big(dd^c \sum_{1 \leq j \leq n} \lambda_j |u_j|^2\Big)^n \geq 0$$

for every basis $(u_1, \ldots, u_n)$ of linear forms and for all coefficients $\lambda_j > 0$. Take $\lambda_1 = \ldots = \lambda_p = 1$ and let the other λ_j tend to 0. Then we get $\mathbb{1}_A \Theta \wedge i du_1 \wedge d\overline{u}_1 \wedge \ldots \wedge du_p \wedge d\overline{u}_p \geq 0$. This implies $\mathbb{1}_A \Theta \geq 0$, or equivalently $\mathbb{1}_A T \geq m_A[A]$. By Cor. 2.4 we know that $\mathbb{1}_A T$ is a closed positive current, thus $\mathbb{1}_A T = \lambda[A]$ with $\lambda \geq 0$. We have just seen that $\lambda \geq m_A$. On the other hand, $T \geq \mathbb{1}_A T = \lambda[A]$ clearly implies $m_A \geq \lambda$. □

(8.16) Siu's decomposition formula. *If T is a closed positive current of bidimension (p,p), there is a unique decomposition of T as a (possibly finite) weakly convergent series*

$$T = \sum_{j \geq 1} \lambda_j [A_j] + R, \qquad \lambda_j > 0,$$

where $[A_j]$ *is the current of integration over an irreducible* p*-dimensional analytic set* $A_j \subset X$ *and where* R *is a closed positive current with the property that* $\dim E_c(R) < p$ *for every* $c > 0$.

Proof of uniqueness. If T has such a decomposition, the p-dimensional components of $E_c(T)$ are $(A_j)_{\lambda_j \geq c}$, for $\nu(T,x) = \sum \lambda_j \nu([A_j],x) + \nu(R,x)$ is non zero only on $\bigcup A_j \cup \bigcup E_c(R)$, and is equal to λ_j generically on A_j (more precisely, $\nu(T,x) = \lambda_j$ at every regular point of A_j which does not belong to any intersection $A_j \cup A_k$, $k \neq j$ or to $\bigcup E_c(R)$). In particular A_j and λ_j are unique.

Proof of existence. Let $(A_j)_{j\geq 1}$ be the countable collection of p-dimensional components occurring in one of the sets $E_c(T)$, $c \in \mathbb{Q}_+^\star$, and let $\lambda_j > 0$ be the generic Lelong number of T along A_j. Then Prop. 8.16 shows by induction on N that $R_N = T - \sum_{1\leq j\leq N} \lambda_j [A_j]$ is positive. As R_N is a decreasing sequence, there must be a limit $R = \lim_{N\to+\infty} R_N$ in the weak topology. Thus we have the asserted decomposition. By construction, R has zero generic Lelong number along A_j, so $\dim E_c(R) < p$ for every $c > 0$. □

It is very important to note that some components of lower dimension can actually occur in $E_c(R)$, but they cannot be subtracted because R has bidimension (p,p). A typical case is the case of a bidimension $(n-1,n-1)$ current $T = dd^c u$ with $u = \log(|F_j|^{\gamma_1} + \ldots |F_N|^{\gamma_N})$ and $F_j \in \mathcal{O}(X)$. In general $\bigcup E_c(T) = \bigcap F_j^{-1}(0)$ has dimension $< n-1$. In that case, an important formula due to King plays the role of (8.17). We state it in a somewhat more general form than its original version (King 1970).

(8.18) King's formula. *Let* $F_1,\ldots,F_N$ *be holomorphic functions on a complex manifold* X, *such that the zero variety* $Z = \bigcap F_j^{-1}(0)$ *has codimension* $\geq p$, *and set* $u = \log \sum |F_j|^{\gamma_j}$ *with arbitrary coefficients* $\gamma_j > 0$. *Let* $(Z_k)_{k\geq 1}$ *be the irreducible components of* Z *of codimension* p *exactly. Then there exist multiplicities* $\lambda_k > 0$ *such that*

$$(dd^c u)^p = \sum_{k\geq 1} \lambda_k [Z_k] + R,$$

where R *is a closed positive current such that* $\mathbb{1}_Z R = 0$ *and* $\operatorname{codim} E_c(R) > p$ *for every* $c > 0$. *Moreover the multiplicities* λ_k *are integers if* $\gamma_1,\ldots,\gamma_N$ *are integers, and* $\lambda_k = \gamma_1 \ldots \gamma_p$ *if* $\gamma_1 \leq \ldots \leq \gamma_N$ *and some partial Jacobian determinant of* $(F_1,\ldots,F_p)$ *of order* p *does not vanish identically along* Z_k.

Proof. Observe that $(dd^c u)^p$ is well defined thanks to Cor. 4.11. The comparison theorem 7.8 applied with $\varphi(z) = \log|z-x|$, $v_1 = \ldots = v_p = u$, $u_1 = \ldots = u_p = \varphi$ and $T = 1$ shows that the Lelong number of $(dd^c u)^p$ is equal to 0 at every point of $X \smallsetminus Z$. Hence $E_c((dd^c u)^p)$ is contained in

Z and its $(n-p)$-dimensional components are members of the family (Z_k). The asserted decomposition follows from Siu's formula 8.16. We must have $\mathbb{1}_{Z_k} R = 0$ for all irreducible components of Z: when $\operatorname{codim} Z_k > p$ this is automatically true, and when $\operatorname{codim} Z_k = p$ this follows from (8.16) and the fact that $\operatorname{codim} E_c(R) > p$. If $\det(\partial F_j/\partial z_k)_{1\le j,k\le p} \neq 0$ at some point $x_0 \in Z_k$, then $(Z, x_0) = (Z_k, x_0)$ is a smooth germ defined by the equations $F_1 = \ldots = F_p = 0$. If we denote $v = \log \sum_{j\le p} |F_j|^{\gamma_j}$ with $\gamma_1 \le \ldots \le \gamma_N$, then $u \sim v$ near Z_k and Th. 7.8 implies $\nu((dd^c u)^p, x) = \nu((dd^c v)^p, x)$ for all $x \in Z_k$ near x_0. On the other hand, if $G := (F_1, \ldots, F_p) : X \to \mathbb{C}^p$, Cor. 7.4 gives

$$(dd^c v)^p = G^\star\Big(dd^c \log \sum_{1\le j\le p} |z_j|^{\gamma_j}\Big)^p = \gamma_1 \ldots \gamma_p\, G^\star \delta_0 = \gamma_1 \ldots \gamma_p\, [Z_k]$$

near x_0. This implies that the generic Lelong number of $(dd^c u)^p$ along Z_k is $\lambda_k = \gamma_1 \ldots \gamma_p$. The integrality of λ_k when $\gamma_1, \ldots, \gamma_N$ are integers will be proved in the next section. □

9. Transformation of Lelong Numbers by Direct Images

Let $F : X \to Y$ be a holomorphic map between complex manifolds of respective dimensions $\dim X = n$, $\dim Y = m$, and let T be a closed positive current of bidimension (p,p) on X. If $F_{\restriction \operatorname{Supp} T}$ is proper, the direct image $F_\star T$ is defined by

(9.1) $\langle F_\star T, \alpha\rangle = \langle T, F^\star \alpha\rangle$

for every test form α of bidegree (p,p) on Y. This makes sense because $\operatorname{Supp} T \cap F^{-1}(\operatorname{Supp} \alpha)$ is compact. It is easily seen that $F_\star T$ is a closed positive current of bidimension (p,p) on Y.

(9.2) Example. Let $T = [A]$ where A is a p-dimensional irreducible analytic set in X such that $F_{\restriction A}$ is proper. We know by Remmert's theorem 2.7.8 that $F(A)$ is an analytic set in Y. Two cases may occur. Either $F_{\restriction A}$ is generically finite and F induces an étale covering $A \smallsetminus F^{-1}(Z) \longrightarrow F(A) \smallsetminus Z$ for some nowhere dense analytic subset $Z \subset F(A)$, or $F_{\restriction A}$ has generic fibers of positive dimension and $\dim F(A) < \dim A$. In the first case, let $s < +\infty$ be the covering degree. Then for every test form α of bidegree (p,p) on Y we get

$$\langle F_\star[A], \alpha\rangle = \int_A F^\star \alpha = \int_{A\smallsetminus F^{-1}(Z)} F^\star\alpha = s \int_{F(A)\smallsetminus Z} \alpha = s\,\langle [F(A)], \alpha\rangle$$

because Z and $F^{-1}(Z)$ are negligible sets. Hence $F_\star[A] = s[F(A)]$. On the other hand, if $\dim F(A) < \dim A = p$, the restriction of α to $F(A)_{\mathrm{reg}}$ is zero, and therefore so is this the restriction of $F^\star\alpha$ to A_{reg}. Hence $F_\star[A] = 0$. □

Now, let ψ be a continuous plurisubharmonic function on Y which is semi-exhaustive on $F(\mathrm{Supp}\,T)$ (this set certainly contains $\mathrm{Supp}\,F_\star T$). Since $F_{\restriction \mathrm{Supp}\,T}$ is proper, it follows that $\psi\circ F$ is semi-exhaustive on Supp T, for

$$\mathrm{Supp}\,T\cap\{\psi\circ F<R\}=F^{-1}\big(F(\mathrm{Supp}\,T)\cap\{\psi<R\}\big).$$

(9.3) Proposition. *If $F(\mathrm{Supp}\,T)\cap\{\psi<R\}\subset\subset Y$, we have*

$$\nu(F_\star T,\psi,r)=\nu(T,\psi\circ F,r)\qquad\text{for all } r<R,$$

in particular $\nu(F_\star T,\psi)=\nu(T,\psi\circ F)$.

Here, we do not necessarily assume that X or Y are Stein; we thus replace ψ with $\psi_{\geq s}=\max\{\psi,s\}$, $s<r$, in the definition of $\nu(F_\star T,\psi,r)$ and $\nu(T,\psi\circ F,r)$.

Proof. The first equality can be written

$$\int_Y F_\star T\wedge \mathbb{1}_{\{\psi<r\}}(dd^c\psi_{\geq s})^p=\int_X T\wedge(\mathbb{1}_{\{\psi<r\}}\circ F)(dd^c\psi_{\geq s}\circ F)^p.$$

This follows almost immediately from the adjunction formula (9.1) when ψ is smooth and when we write $\mathbb{1}_{\{\psi<R\}}=\lim\uparrow g_k$ for some sequence of smooth functions g_k. In general, we write $\psi_{\geq s}$ as a decreasing limit of smooth plurisubharmonic functions and we apply our monotone continuity theorems (if Y is not Stein, Richberg's theorem shows that we can obtain a decreasing sequence of almost plurisubharmonic approximations such that the negative part of dd^c converges uniformly to 0; this is good enough to apply the monotone continuity theorem; note that the integration is made on compact subsets, thanks to the semi-exhaustivity assumption on ψ). □

It follows from this that understanding the transformation of Lelong numbers under direct images is equivalent to understanding the effect of F on the weight. We are mostly interested in computing the ordinary Lelong numbers $\nu(F_\star T,y)$ associated with the weight $\psi(w)=\log|w-y|$ in some local coordinates $(w_1,\ldots,w_m)$ on Y near y. Then Prop. 9.3 gives

$$\begin{aligned}&\nu(F_\star T,y)=\nu(T,\log|F-y|)\qquad\text{with}\\&\log|F(z)-y|=\frac12\log\sum|F_j(z)-y_j|^2,\qquad F_j=w_j\circ F.\end{aligned}\tag{9.4}$$

We are going to show that $\nu(T,\log|F-y|)$ is bounded below by a linear combination of the Lelong numbers of T at points x in the fiber $F^{-1}(y)$, with suitable multiplicities attached to F at these points. These multiplicities can be seen as generalizations of the notion of multiplicity of an analytic map introduced by (Stoll 1966).

(9.5) Definition. *Let $x \in X$ and $y = F(x)$. Suppose that the codimension of the fiber $F^{-1}(y)$ at x is $\geq p$. Then we set*

$$\mu_p(F,x) = \nu\big((dd^c \log|F-y|)^p, x\big).$$

Observe that $(dd^c \log|F-y|)^p$ is well defined thanks to Cor. 4.10. The second comparison theorem 7.8 immediately shows that $\mu_p(F,x)$ is independent of the choice of local coordinates on Y (and also on X, since Lelong nombers do not depend on coordinates). By definition, $\mu_p(F,x)$ is the mass carried by $\{x\}$ of the measure

$$(dd^c \log|F(z)-y|)^p \wedge (dd^c \log|z-x|)^{n-p}.$$

We are going to give a more geometric interpretation of this multiplicity, from which it will follow that $\mu_p(F,x)$ is always a positive integer (in particular, the proof of (8.18) will be complete).

(9.6) Example. For $p = n = \dim X$, the assumption $\operatorname{codim}_x F^{-1}(y) \geq p$ means that the germ of map $F : (X,x) \longrightarrow (Y,y)$ is finite. Let U_x be a neighborhood of x such that $\overline{U}_x \cap F^{-1}(y) = \{x\}$, let W_y be a neighborhood of y disjoint from $F(\partial U_x)$ and let $V_x = U_x \cap F^{-1}(W_y)$. Then $F : V_x \to W_y$ is proper and finite, and we have $F_\star[V_x] = s\,[F(V_x)]$ where s is the local covering degree of $F : V_x \to F(V_x)$ at x. Therefore

$$\begin{aligned}\mu_n(F,x) &= \int_{\{x\}} \big(dd^c \log|F-y|\big)^n = \nu\big([V_x], \log|F-y|\big) = \nu\big(F_\star[V_x], y\big)\\ &= s\,\nu\big(F(V_x), y\big).\end{aligned}$$

In the particular case when $\dim Y = \dim X$, we have $(F(V_x), y) = (Y, y)$, so $\mu_n(F,x) = s$. In general, it is a well known fact that the ideal generated by $(F_1 - y_1, \ldots, F_m - y_m)$ in $\mathcal{O}_{X,x}$ has the same integral closure as the ideal generated by n generic linear combinations of the generators, that is, for a generic choice of coordinates $w' = (w_1, \ldots, w_n)$, $w'' = (w_{n+1}, \ldots, w_m)$ on (Y,y), we have $|F(z) - y| \leq C|w' \circ F(z)|$ (this is a simple consequence of Lemma 7.5 applied to $A = F(V_x)$). Hence for $p = n$, the comparison theorem 7.1 gives

$$\mu_n(F,x) = \mu_n(w' \circ F, x) = \text{local covering degree of } w' \circ F \text{ at } x,$$

for a generic choice of coordinates (w', w'') on (Y,y). □

(9.7) Geometric interpretation of $\mu_p(F,x)$. An application of Crofton's formula 7.11 shows, after a translation, that there is a small ball $B(x, r_0)$ on which

$$(dd^c \log |F(z) - y|)^p \wedge (dd^c \log |z - x|)^{n-p} =$$

$$\text{(9.7 a)} \qquad \int_{S \in G(p,n)} (dd^c \log |F(z) - y|)^p \wedge [x + S]\, dv(S).$$

For a rigorous proof of (9.7 a), we replace $\log |F(z)-y|$ by the smooth function $\frac{1}{2}\log(|F(z) - y|^2 + \varepsilon^2)$ and let ε tend to 0 on both sides. By (4.3) (resp. by (4.10)), the wedge product $(dd^c \log |F(z) - y|)^p \wedge [x + S]$ is well defined on a small ball $B(x, r_0)$ as soon as $x + S$ does not intersect $F^{-1}(y) \cap \partial B(x, r_0)$ (resp. intersects $F^{-1}(y) \cap B(x, r_0)$ at finitely many points); thanks to the assumption $\operatorname{codim}(F^{-1}(y), x) \geq p$, Sard's theorem shows that this is the case for all S outside a negligible closed subset E in $G(p, n)$ (resp. by Bertini, an analytic subset A in $G(p, n)$ with $A \subset E$). Fatou's lemma then implies that the inequality $\geq$ holds in (9.7 a). To get equality, we observe that we have bounded convergence on all complements $G(p, n) \smallsetminus V(E)$ of neighborhoods $V(E)$ of E. However the mass of $\int_{V(E)}[x+S]\, dv(S)$ in $B(x, r_0)$ is proportional to $v(V(E))$ and therefore tends to 0 when $V(E)$ is small; this is sufficient to complete the proof, since Prop. 4.6 b) gives

$$\int_{z \in \overline{B}(x,r_0)} \big(dd^c \log(|F(z) - y|^2 + \varepsilon^2)\big)^p \wedge \int_{S \in V(E)} [x + S]\, dv(S) \leq C\, v(V(E))$$

with a constant C independent of ε. By evaluating (9.7 a) on $\{x\}$, we get

$$\text{(9.7 b)} \quad \mu_p(F, x) = \int_{S \in G(p,n) \smallsetminus A} \nu\big((dd^c \log |F_{\restriction x+S} - z|)^p, x\big)\, dv(S).$$

Let us choose a linear parametrization $g_S : \mathbb{C}^p \to S$ depending analytically on local coordinates of S in $G(p, n)$. Then Theorem 8.4 with $T = [\mathbb{C}^p]$ and $\varphi(z, S) = \log |F \circ g_S(z) - y|$ shows that

$$\nu\big((dd^c \log |F_{\restriction x+S} - z|)^p, x\big) = \nu\big([\mathbb{C}^p], \log |F \circ g_S(z) - y|\big)$$

is Zariski upper semicontinuous in S on $G(p, n) \smallsetminus A$. However, (9.6) shows that these numbers are integers, so $S \mapsto \nu\big((dd^c \log |F_{\restriction x+S} - z|)^p, x\big)$ must be constant on a Zariski open subset in $G(p, n)$. By (9.7 b), we obtain

$$\text{(9.7 c)} \quad \mu_p(F, x) = \mu_p(F_{\restriction x+S}, x) = \text{local degree of } w' \circ F_{\restriction x+S} \ \text{ at } x$$

for generic subspaces $S \in G(p, n)$ and generic coordinates $w' = (w_1, \ldots, w_p)$, $w'' = (w_{p+1}, \ldots, w_m)$ on (Y, y). □

(9.8) Example. Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be defined by

$$F(z_1, \ldots, z_n) = (z_1^{s_1}, \ldots, z_n^{s_n}), \qquad s_1 \leq \ldots \leq s_n.$$

We claim that $\mu_p(F, 0) = s_1 \ldots s_p$. In fact, for a generic p-dimensional subspace $S \subset \mathbb{C}^n$ such that $z_1, \ldots, z_p$ are coordinates on S and $z_{p+1}, \ldots, z_n$ are linear forms in $z_1, \ldots, z_p$, and for generic coordinates $w' = (w_1, \ldots, w_p)$,

$w'' = (w_{p+1}, \ldots, w_n)$ on $\mathbb{C}^n$, we can rearrange w' by linear combinations so that $w_j \circ F_{\restriction S}$ is a linear combination of $(z_j^{s_j}, \ldots, z_n^{s_n})$ and has non zero coefficient in $z_j^{s_j}$ as a polynomial in $(z_j, \ldots, z_p)$. It is then an exercise to show that $w' \circ F_{\restriction S}$ has covering degree $s_1 \ldots s_p$ at 0 [compute inductively the roots z_n, $z_{n-1}, \ldots, z_j$ of $w_j \circ F_{\restriction S}(z) = a_j$ and use Lemma II.3.10 to show that the s_j values of z_j lie near 0 when $(a_1, \ldots, a_p)$ are small]. $\square$

We are now ready to prove the main result of this section, which describes the behaviour of Lelong numbers under proper morphisms. A similar weaker result was already proved in (Demailly 1982b) with some other non optimal multiplicities $\mu_p(F, x)$.

(9.9) Theorem. *Let T be a closed positive current of bidimension (p,p) on X and let $F : X \longrightarrow Y$ be an analytic map such that the restriction $F_{\restriction \mathrm{Supp}\, T}$ is proper. Let $I(y)$ be the set of points $x \in \mathrm{Supp}\, T \cap F^{-1}(y)$ such that x is equal to its connected component in* Supp $T \cap F^{-1}(y)$ *and* $\mathrm{codim}(F^{-1}(y), x) \ge p$. *Then we have*

$$\nu(F_\star T, y) \ge \sum_{x \in I(y)} \mu_p(F, x)\, \nu(T, x).$$

In particular, we have $\nu(F_\star T, y) \ge \sum_{x \in I(y)} \nu(T, x)$. This inequality no longer holds if the summation is extended to all points $x \in \mathrm{Supp}\, T \cap F^{-1}(Y)$ and if this set contains positive dimensional connected components: for example, if $F : X \longrightarrow Y$ contracts some exceptional subspace E in X to a point y_0 (e.g. if F is a blow-up map, see § 7.12), then $T = [E]$ has direct image $F_\star[E] = 0$ thanks to (9.2).

Proof. We proceed in three steps.

Step 1. Reduction to the case of a single point x in the fiber. It is sufficient to prove the inequality when the summation is taken over an arbitrary finite subset $\{x_1, \ldots, x_N\}$ of $I(y)$. As x_j is equal to its connected component in Supp $T \cap F^{-1}(y)$, it has a fondamental system of relative open-closed neighborhoods, hence there are disjoint neighborhoods U_j of x_j such that ∂U_j does not intersect Supp $T \cap F^{-1}(y)$. Then the image $F(\partial U_j \cap \mathrm{Supp}\, T)$ is a closed set which does not contain y. Let W be a neighborhood of y disjoint from all sets $F(\partial U_j \cap \mathrm{Supp}\, T)$, and let $V_j = U_j \cap F^{-1}(W)$. It is clear that V_j is a neighborhood of x_j and that $F_{\restriction V_j} : V_j \to W$ has a proper restriction to Supp $T \cap V_j$. Moreover, we obviously have $F_\star T \ge \sum_j (F_{\restriction V_j})_\star T$ on W. Therefore, it is enough to check the inequality $\nu(F_\star T, y) \ge \mu_p(F, x)\, \nu(T, x)$ for a single point $x \in I(y)$, in the case when $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^m$ are open subsets and $x = y = 0$.

Step 2. Reduction to the case when F is finite. By (9.4), we have

$$\nu(F_\star T, 0) = \inf_{V \ni 0} \int_V T \wedge (dd^c \log |F|)^p$$
$$= \inf_{V \ni 0} \lim_{\varepsilon \to 0} \int_V T \wedge \big(dd^c \log(|F| + \varepsilon |z|^N)\big)^p,$$

and the integrals are well defined as soon as ∂V does not intersect the set $\operatorname{Supp} T \cap F^{-1}(0)$ (may be after replacing $\log |F|$ by $\max\{\log |F|, s\}$ with $s \ll 0$). For every V and ε, the last integral is larger than $\nu(G_\star T, 0)$ where G is the finite morphism defined by

$$G : X \longrightarrow Y \times \mathbb{C}^n, \qquad (z_1, \ldots, z_n) \longmapsto (F_1(z), \ldots, F_m(z), z_1^N, \ldots, z_n^N).$$

We claim that for N large enough we have $\mu_p(F, 0) = \mu_p(G, 0)$. In fact, $x \in I(y)$ implies by definition $\operatorname{codim}(F^{-1}(0), 0) \geq p$. Hence, if $S = \{u_1 = \ldots = u_{n-p} = 0\}$ is a generic p-dimensional subspace of $\mathbb{C}^n$, the germ of variety $F^{-1}(0) \cap S$ defined by $(F_1, \ldots, F_m, u_1, \ldots, u_{n-p})$ is $\{0\}$. Hilbert's Nullstellensatz implies that some powers of $z_1, \ldots, z_n$ are in the ideal (F_j, u_k). Therefore $|F(z)| + |u(z)| \geq C|z|^a$ near 0 for some integer a independent of S (to see this, take coefficients of the u_k's as additional variables); in particular $|F(z)| \geq C|z|^a$ for $z \in S$ near 0. The comparison theorem 7.1 then shows that $\mu_p(F, 0) = \mu_p(G, 0)$ for $N \geq a$. If we are able to prove that $\nu(G_\star T, 0) \geq \mu_p(G, 0)\nu(T, 0)$ in case G is finite, the obvious inequality $\nu(F_\star T, 0) \geq \nu(G_\star T, 0)$ concludes the proof.

Step 3. Proof of the inequality $\nu(F_\star T, y) \geq \mu_p(F, x)\, \nu(T, x)$ when F is finite and $F^{-1}(y) = x$. Then $\varphi(z) = \log |F(z) - y|$ has a single isolated pole at x and we have $\mu_p(F, x) = \nu((dd^c\varphi)^p, x)$. It is therefore sufficient to apply to following Proposition.

(9.10) Proposition. *Let φ be a semi-exhaustive continuous plurisubharmonic function on X with a single isolated pole at x. Then*

$$\nu(T, \varphi) \geq \nu(T, x)\, \nu((dd^c\varphi)^p, x).$$

Proof. Since the question is local, we can suppose that X is the ball $B(0, r_0)$ in $\mathbb{C}^n$ and $x = 0$. Set $X' = B(0, r_1)$ with $r_1 < r_0$ and $\Phi(z, g) = \varphi \circ g(z)$ for $g \in \mathrm{Gl}_n(\mathbb{C})$. Then there is a small neighborhood Ω of the unitary group $U(n) \subset \mathrm{Gl}_n(\mathbb{C})$ such that Φ is plurisubharmonic on $X' \times \Omega$ and semi-exhaustive with respect to X'. Theorem 8.4 implies that the map $g \mapsto \nu(T, \varphi \circ g)$ is Zariski upper semi-continuous on Ω. In particular, we must have $\nu(T, \varphi \circ g) \leq \nu(T, \varphi)$ for all $g \in \Omega \smallsetminus A$ in the complement of a complex analytic set A. Since $\mathrm{Gl}_n(\mathbb{C})$ is the complexification of $U(n)$, the intersection $U(n) \cap A$ must be a nowhere dense real analytic subset of $U(n)$. Therefore, if dv is the Haar measure of mass 1 on $U(n)$, we have

$$
\begin{aligned}
\nu(T,\varphi) &\geq \int_{g\in U(n)} \nu(T,\varphi\circ g)\,dv(g) \\
&= \lim_{r\to 0}\int_{g\in U(n)} dv(g)\int_{B(0,r)} T\wedge (dd^c\varphi\circ g)^p. \qquad (9.11)
\end{aligned}
$$

Since $\int_{g\in U(n)}(dd^c\varphi\circ g)^p dv(g)$ is a unitary invariant (p,p)-form on B, Lemma 7.10 implies

$$\int_{g\in U(n)} (dd^c\varphi\circ g)^p dv(g) = \big(dd^c\chi(\log|z|)\big)^p$$

where χ is a convex increasing function. The Lelong number at 0 of the left hand side is equal to $\nu((dd^c\varphi)^p,0)$, and must be equal to the Lelong number of the right hand side, which is $\lim_{t\to-\infty}\chi'(t)^p$ (to see this, use either Formula (5.5) or Th. 7.8). Thanks to the last equality, Formulas (9.11) and (5.5) imply

$$
\begin{aligned}
\nu(T,\varphi) &\geq \lim_{r\to 0}\int_{B(0,r)} T\wedge\big(dd^c\chi(\log|z|)\big)^p \\
&= \lim_{r\to 0}\chi'(\log r-0)^p\nu(T,0,r)\geq \nu((dd^c\varphi)^p,0)\,\nu(T,0). \qquad \square
\end{aligned}
$$

Another interesting question is to know whether it is possible to get inequalities in the opposite direction, i.e. to find upper bounds for $\nu(F_\star T,y)$ in terms of the Lelong numbers $\nu(T,x)$. The example $T=[\Gamma]$ with the curve $\Gamma: t\mapsto (t^a,t^{a+1},t)$ in $\mathbb{C}^3$ and $F:\mathbb{C}^3\to\mathbb{C}^2$, $(z_1,z_2,z_3)\mapsto(z_1,z_2)$, for which $\nu(T,0)=1$ and $\nu(F_\star T,0)=a$, shows that this may be possible only when F is finite. In this case, we have:

(9.12) Theorem. *Let $F:X\to Y$ be a proper and finite analytic map and let T be a closed positive current of bidimension (p,p) on X. Then*

(a) $\displaystyle \nu(F_\star T,y)\leq \sum_{x\in \mathrm{Supp}\ T\cap F^{-1}(y)} \overline{\mu}_p(F,x)\,\nu(T,x)$

where $\overline{\mu}_p(F,x)$ is the multiplicity defined as follows: if $H:(X,x)\to(\mathbb{C}^n,0)$ is a germ of finite map, we set

(b) $\displaystyle \sigma(H,x)=\inf\big\{\alpha>0\,;\,\exists C>0,\ |H(z)|\geq C|z-x|^\alpha \text{ near } x\big\},$

(c) $\displaystyle \overline{\mu}_p(F,x)=\inf_G \frac{\sigma(G\circ F,x)^p}{\mu_p(G,0)},$

where G runs over all germs of maps $(Y,y)\longrightarrow(\mathbb{C}^n,0)$ such that $G\circ F$ is finite.

Proof. If $F^{-1}(y)=\{x_1,\ldots,x_N\}$, there is a neighborhood W of y and disjoint neighborhoods V_j of x_j such that $F^{-1}(W)=\bigcup V_j$. Then $F_\star T=\sum(F_{\restriction V_j})_\star T$

on W, so it is enough to consider the case when $F^{-1}(y)$ consists of a single point x. Therefore, we assume that $F : V \to W$ is proper and finite, where V, W are neighborhoods of 0 in $\mathbb{C}^n$, $\mathbb{C}^m$ and $F^{-1}(0) = \{0\}$. Let $G : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^n, 0)$ be a germ of map such that $G \circ F$ is finite. Hilbert's Nullstellensatz shows that there exists $\alpha > 0$ and $C > 0$ such that $|G \circ F(z)| \geq C|z|^\alpha$ near 0. Then the comparison theorem 7.1 implies

$$\nu(G_\star F_\star T, 0) = \nu(T, \log|G \circ F|) \leq \alpha^p \nu(T, \log|z|) = \alpha^p \nu(T, 0).$$

On the other hand, Th. 9.9 applied to $\Theta = F_\star T$ on W gives

$$\nu(G_\star F_\star T, 0) \geq \mu_p(G, 0)\, \nu(F_\star T, 0).$$

Therefore

$$\nu(F_\star T, 0) \leq \frac{\alpha^p}{\mu_p(G, 0)} \nu(T, 0).$$

The infimum of all possible values of α is by definition $\sigma(G \circ F, 0)$, thus by taking the infimum over G we obtain

$$\nu(F_\star T, 0) \leq \overline{\mu}_p(F, 0)\, \nu(T, 0). \qquad \square$$

(9.13) Example. Let $F(z_1, \ldots, z_n) = (z_1^{s_1}, \ldots, z_n^{s_n})$, $s_1 \leq \ldots \leq s_n$ as in 9.8. Then we have

$$\mu_p(F, 0) = s_1 \ldots s_p, \qquad \overline{\mu}_p(F, 0) = s_{n-p+1} \ldots s_n.$$

To see this, let s be the lowest common multiple of $s_1, \ldots, s_n$ and let $G(z_1, \ldots, z_n) = (z_1^{s/s_1}, \ldots, z_n^{s/s_n})$. Clearly $\mu_p(G, 0) = (s/s_{n-p+1}) \ldots (s/s_n)$ and $\sigma(G \circ F, 0) = s$, so we get by definition $\overline{\mu}_p(F, 0) \leq s_{n-p+1} \ldots s_n$. Finally, if $T = [A]$ is the current of integration over the p-dimensional subspace $A = \{z_1 = \ldots = z_{n-p} = 0\}$, then $F_\star[A] = s_{n-p+1} \ldots s_n\, [A]$ because $F_{\restriction A}$ has covering degree $s_{n-p+1} \ldots s_n$. Theorem 9.12 shows that we must have $s_{n-p+1} \ldots s_n \leq \overline{\mu}_p(F, 0)$, QED. If $\lambda_1 \leq \ldots \leq \lambda_n$ are positive real numbers and s_j is taken to be the integer part of $k\lambda_j$ as k tends to $+\infty$, Theorems 9.9 and 9.12 imply in the limit the following:

(9.14) Corollary. *For $0 < \lambda_1 \leq \ldots \leq \lambda_n$, Kiselman's directional Lelong numbers satisfy the inequalities*

$$\lambda_1 \ldots \lambda_p\, \nu(T, x) \leq \nu(T, x, \lambda) \leq \lambda_{n-p+1} \ldots \lambda_n\, \nu(T, x). \qquad \square$$

(9.15) Remark. It would be interesting to have a direct geometric interpretation of $\overline{\mu}_p(F, x)$. In fact, we do not even know whether $\overline{\mu}_p(F, x)$ is always an integer.

10. A Schwarz Lemma. Application to Number Theory

In this section, we show how Jensen's formula and Lelong numbers can be used to prove a fairly general Schwarz lemma relating growth and zeros of entire functions in $\mathbb{C}^n$. In order to simplify notations, we denote by $|F|_r$ the supremum of the modulus of a function F on the ball of center 0 and radius r. Then, following (Demailly 1982a), we present some applications with a more arithmetical flavour.

(10.1) Schwarz lemma. *Let $P_1, \ldots, P_N \in \mathbb{C}[z_1, \ldots, z_n]$ be polynomials of degree δ, such that their homogeneous parts of degree δ do not vanish simultaneously except at* 0*. Then there is a constant $C \geq 2$ such that for all entire functions $F \in \mathcal{O}(\mathbb{C}^n)$ and all $R \geq r \geq 1$ we have*

$$\log |F|_r \leq \log |F|_R - \delta^{1-n} \nu([Z_F], \log |P|) \log \frac{R}{Cr}$$

where Z_F is the zero divisor of F and $P = (P_1, \ldots, P_N) : \mathbb{C}^n \longrightarrow \mathbb{C}^N$. Moreover

$$\nu([Z_F], \log |P|) \geq \sum_{w \in P^{-1}(0)} \mathrm{ord}(F, w)\, \mu_{n-1}(P, w)$$

where $\mathrm{ord}(F, w)$ *denotes the vanishing order of F at w and $\mu_{n-1}(P, w)$ is the $(n-1)$-multiplicity of P at w, as defined in* (9.5) *and* (9.7).

Proof. Our assumptions imply that P is a proper and finite map. The last inequality is then just a formal consequence of formula (9.4) and Th. 9.9 applied to $T = [Z_F]$. Let Q_j be the homogeneous part of degree δ in P_j. For $z_0 \in B(0, r)$, we introduce the weight functions

$$\varphi(z) = \log |P(z)|, \qquad \psi(z) = \log |Q(z - z_0)|.$$

Since $Q^{-1}(0) = \{0\}$ by hypothesis, the homogeneity of Q shows that there are constants $C_1, C_2 > 0$ such that

$$(10.2) \quad C_1 |z|^\delta \leq |Q(z)| \leq C_2 |z|^\delta \quad \text{on} \quad \mathbb{C}^n.$$

The homogeneity also implies $(dd^c\psi)^n = \delta^n\, \delta_{z_0}$. We apply the Lelong Jensen formula 6.5 to the measures $\mu_{\psi,s}$ associated with ψ and to $V = \log |F|$. This gives

$$(10.3) \quad \mu_{\psi,s}(\log |F|) - \delta^n \log |F(z_0)| = \int_{-\infty}^{s} dt \int_{\{\psi < t\}} [Z_F] \wedge (dd^c\psi)^{n-1}.$$

By (6.2), $\mu_{\psi,s}$ has total mass δ^n and has support in

$$\{\psi(z) = s\} = \{Q(z - z_0) = e^s\} \subset B\big(0, r + (e^s/C_1)^{1/\delta}\big).$$

Note that the inequality in the Schwarz lemma is obvious if $R \le Cr$, so we can assume $R \ge Cr \ge 2r$. We take $s = \delta \log(R/2) + \log C_1$; then

$$\{\psi(z) = s\} \subset B(0, r + R/2) \subset B(0, R).$$

In particular, we get $\mu_{\psi,s}(\log|F|) \le \delta^n \log|F|_R$ and formula (10.3) gives

$$\log|F|_R - \log|F(z_0)| \ge \delta^{-n} \int_{s_0}^{s} dt \int_{\{\psi<t\}} [Z_F] \wedge (dd^c\psi)^{n-1} \tag{10.4}$$

for any real number $s_0 < s$. The proof will be complete if we are able to compare the integral in (10.4) to the corresponding integral with φ in place of ψ. The argument for this is quite similar to the proof of the comparison theorem, if we observe that $\psi \sim \varphi$ at infinity. We introduce the auxiliary function

$$w = \begin{cases} \max\{\psi, (1-\varepsilon)\varphi + \varepsilon t - \varepsilon\} & \text{on } \{\psi \ge t-2\}, \\ (1-\varepsilon)\varphi + \varepsilon t - \varepsilon & \text{on } \{\psi \le t-2\}, \end{cases}$$

with a constant ε to be determined later, such that $(1-\varepsilon)\varphi + \varepsilon t - \varepsilon > \psi$ near $\{\psi = t-2\}$ and $(1-\varepsilon)\varphi + \varepsilon t - \varepsilon < \psi$ near $\{\psi = t\}$. Then Stokes' theorem implies

$$\begin{aligned} \int_{\{\psi<t\}} [Z_F] \wedge (dd^c\psi)^{n-1} &= \int_{\{\psi<t\}} [Z_F] \wedge (dd^c w)^{n-1} \\ &\ge (1-\varepsilon)^{n-1} \int_{\{\psi<t-2\}} [Z_F] \wedge (dd^c\varphi)^{n-1} \ge (1-\varepsilon)^{n-1} \nu([Z_F], \log|P|). \end{aligned} \tag{10.5}$$

By (10.2) and our hypothesis $|z_0| < r$, the condition $\psi(z) = t$ implies

$$\begin{aligned} &|Q(z-z_0)| = e^t \implies e^{t/\delta}/C_1^{1/\delta} \le |z - z_0| \le e^{t/\delta}/C_2^{1/\delta}, \\ &|P(z) - Q(z-z_0)| \le C_3(1+|z_0|)(1+|z|+|z_0|)^{\delta-1} \le C_4 r(r+e^{t/\delta})^{\delta-1}, \\ &\left|\frac{P(z)}{Q(z-z_0)} - 1\right| \le C_4 r e^{-t/\delta}(re^{-t/\delta}+1)^{\delta-1} \le 2^{\delta-1} C_4 r e^{-t/\delta}, \end{aligned}$$

provided that $t \ge \delta \log r$. Hence for $\psi(z) = t \ge s_0 \ge \delta \log(2^\delta C_4 r)$, we get

$$|\varphi(z) - \psi(z)| = \left|\log \frac{|P(z)|}{|Q(z-z_0)|}\right| \le C_5 r e^{-t/\delta}.$$

Now, we have

$$\big[(1-\varepsilon)\varphi + \varepsilon t - \varepsilon\big] - \psi = (1-\varepsilon)(\varphi - \psi) + \varepsilon(t - 1 - \psi),$$

so this difference is $< C_5 r e^{-t/\delta} - \varepsilon$ on $\{\psi = t\}$ and $> -C_5 r e^{(2-t)/\delta} + \varepsilon$ on $\{\psi = t-2\}$. Hence it is sufficient to take $\varepsilon = C_5 r e^{(2-t)/\delta}$. This number has to be < 1, so we take $t \ge s_0 \ge 2 + \delta \log(C_5 r)$. Moreover, (10.5) actually holds only if $P^{-1}(0) \subset \{\psi < t-2\}$, so by (10.2) it is enough to take $t \ge s_0 \ge$

$2 + \log(C_2(r + C_6)^\delta)$ where C_6 is such that $P^{-1}(0) \subset \overline{B}(0, C_6)$. Finally, we see that we can choose

$$s = \delta \log R - C_7, \qquad s_0 = \delta \log r + C_8,$$

and inequalities (10.4), (10.5) together imply

$$\log |F|_R - \log |F(z_0)| \geq \delta^{-n} \Big(\int_{s_0}^{s} (1 - C_5 r e^{(2-t)/\delta})^{n-1}\, dt \Big) \nu([Z_F], \log |P|).$$

The integral is bounded below by

$$\int_{C_8}^{\delta \log(R/r) - C_7} (1 - C_9 e^{-t/\delta})\, dt \geq \delta \log(R/Cr).$$

This concludes the proof, by taking the infimum when z_0 runs over $B(0, r)$. □

(10.6) Corollary. *Let S be a finite subset of $\mathbb{C}^n$ and let δ be the minimal degree of algebraic hypersurfaces containing S. Then there is a constant $C \geq 2$ such that for all $F \in \mathcal{O}(\mathbb{C}^n)$ and all $R \geq r \geq 1$ we have*

$$\log |F|_r \leq \log |F|_R - \operatorname{ord}(F, S) \frac{\delta + n(n-1)/2}{n!} \log \frac{R}{Cr}$$

where $\operatorname{ord}(F, S) = \min_{w \in S} \operatorname{ord}(F, w)$.

Proof. In view of Th. 10.1, we only have to select suitable polynomials $P_1, \ldots, P_N$. The vector space $\mathbb{C}[z_1, \ldots, z_n]_{<\delta}$ of polynomials of degree $< \delta$ in $\mathbb{C}^n$ has dimension

$$m(\delta) = \binom{\delta + n - 1}{n} = \frac{\delta(\delta + 1) \ldots (\delta + n - 1)}{n!}.$$

By definition of δ, the linear forms

$$\mathbb{C}[z_1, \ldots, z_n]_{<\delta} \longrightarrow \mathbb{C}, \qquad P \longmapsto P(w), \quad w \in S$$

vanish simultaneously only when $P = 0$. Hence we can find $m = m(\delta)$ points $w_1, \ldots, w_m \in S$ such that the linear forms $P \mapsto P(w_j)$ define a basis of $\mathbb{C}[z_1, \ldots, z_n]^\star_{<\delta}$. This means that there is a unique polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]_{<\delta}$ which takes given values $P(w_j)$ for $1 \leq j \leq m$. In particular, for every multiindex α, $|\alpha| = \delta$, there is a unique polynomial $R_\alpha \in \mathbb{C}[z_1, \ldots, z_n]_{<\delta}$ such that $R_\alpha(w_j) = w_j^\alpha$. Then the polynomials $P_\alpha(z) = z^\alpha - R_\alpha(z)$ have degree δ, vanish at all points w_j and their homogeneous parts of maximum degree $Q_\alpha(z) = z^\alpha$ do not vanish simultaneously except at 0. We simply use the fact that $\mu_{n-1}(P, w_j) \geq 1$ to get

$$\nu([Z_F], \log |P|) \geq \sum_{w \in P^{-1}(0)} \operatorname{ord}(F, w) \geq m(\delta) \operatorname{ord}(F, S).$$

Theorem 10.1 then gives the desired inequality, because $m(\delta)$ is a polynomial with positive coefficients and with leading terms

$$\frac{1}{n!}\big(\delta^n + n(n-1)/2\,\delta^{n-1} + \ldots\big). \qquad \square$$

Let S be a finite subset of $\mathbb{C}^n$. According to (Waldschmidt 1976), we introduce for every integer $t > 0$ a number $\omega_t(S)$ equal to the minimal degree of polynomials $P \in \mathbb{C}[z_1, \ldots, z_n]$ which vanish at order $\geq t$ at every point of S. The obvious subadditivity property

$$\omega_{t_1+t_2}(S) \leq \omega_{t_1}(S) + \omega_{t_2}(S)$$

easily shows that

$$\Omega(S) := \inf_{t>0} \frac{\omega_t(S)}{t} = \lim_{t\to+\infty} \frac{\omega_t(S)}{t}.$$

We call $\omega_1(S)$ the *degree* of S (minimal degree of algebraic hypersurfaces containing S) and $\Omega(S)$ the *singular degree* of S. If we apply Cor. 10.6 to a polynomial F vanishing at order t on S and fix $r = 1$, we get

$$\log|F|_R \geq t\frac{\delta + n(n-1)/2}{n!}\log\frac{R}{C} + \log|F|_1$$

with $\delta = \omega_1(S)$, in particular

$$\deg F \geq t\frac{\omega_1(S) + n(n-1)/2}{n!}.$$

The minimum of $\deg F$ over all such F is by definition $\omega_t(S)$. If we divide by t and take the infimum over t, we get the interesting inequality

$$\text{(10.7)} \qquad \frac{\omega_t(S)}{t} \geq \Omega(S) \geq \frac{\omega_1(S) + n(n-1)/2}{n!}.$$

(10.8) Remark. The constant $\frac{\omega_1(S)+n(n-1)/2}{n!}$ in (10.6) and (10.7) is optimal for $n = 1, 2$ but not for $n \geq 3$. It can be shown by means of Hörmander's L^2 estimates (Waldschmidt 1978) that for every $\varepsilon > 0$ the Schwarz lemma (10.6) holds with coefficient $\Omega(S) - \varepsilon$:

$$\log|F|_r \leq \log|F|_R - \mathrm{ord}(F, S)(\Omega(S) - \varepsilon)\log\frac{R}{C_\varepsilon r},$$

and that $\Omega(S) \geq (\omega_u(S)+1)/(u+n-1)$ for every $u \geq 1$; this last inequality is due to (Esnault-Viehweg 1983), who used deep tools of algebraic geometry; (Azhari 1990) reproved it recently by means of Hörmander's L^2 estimates. Rather simple examples (Demailly 1982a) lead to the conjecture

$$\Omega(S) \geq \frac{\omega_u(S)+n-1}{u+n-1} \quad \text{for every} \quad u \geq 1.$$

The special case $u = 1$ of the conjecture was first stated by (Chudnovsky 1979).

Finally, let us mention that Cor. 10.6 contains Bombieri's theorem on algebraic values of meromorphic maps satisfying algebraic differential equations (Bombieri 1970). Recall that an entire function $F \in \mathcal{O}(\mathbb{C}^n)$ is said to be of order $\leq \rho$ if for every $\varepsilon > 0$ there is a constant C_ε such that $|F(z)| \leq C_\varepsilon \exp(|z|^{\rho+\varepsilon})$. A meromorphic function is said to be of order $\leq \rho$ if it can be written G/H where G, H are entire functions of order $\leq \rho$.

(10.9) Theorem (Bombieri 1970). *Let $F_1, \ldots, F_N$ be meromorphic functions on $\mathbb{C}^n$, such that $F_1, \ldots, F_d$, $n < d \leq N$, are algebraically independent over $\mathbb{Q}$ and have finite orders $\rho_1, \ldots, \rho_d$. Let K be a number field of degree $[K : \mathbb{Q}]$. Suppose that the ring $K[f_1, \ldots, f_N]$ is stable under all derivations $d/dz_1, \ldots, d/dz_n$. Then the set S of points $z \in \mathbb{C}^n$, distinct from the poles of the F_j's, such that $(F_1(z), \ldots, F_N(z)) \in K^N$ is contained in an algebraic hypersurface whose degree δ satisfies*

$$\frac{\delta + n(n-1)/2}{n!} \leq \frac{\rho_1 + \ldots + \rho_d}{d-n}[K : \mathbb{Q}].$$

Proof. If the set S is not contained in any algebraic hypersurface of degree $< \delta$, the linear algebra argument used in the proof of Cor. 10.6 shows that we can find $m = m(\delta)$ points $w_1, \ldots, w_m \in S$ which are not located on any algebraic hypersurface of degree $< \delta$. Let $H_1, \ldots, H_d$ be the denominators of $F_1, \ldots, F_d$. The standard arithmetical methods of transcendental number theory allow us to construct a sequence of entire functions in the following way: we set

$$G = P(F_1, \ldots, F_d)(H_1 \ldots H_d)^s$$

where P is a polynomial of degree $\leq s$ in each variable with integer coefficients. The polynomials P are chosen so that G vanishes at a very high order at each point w_j. This amounts to solving a linear system whose unknowns are the coefficients of P and whose coefficients are polynomials in the derivatives of the F_j's (hence lying in the number field K). Careful estimates of size and denominators and a use of the Dirichlet-Siegel box principle lead to the following lemma, see e.g. (Waldschmidt 1978).

(10.10) Lemma. *For every $\varepsilon > 0$, there exist constants $C_1, C_2 > 0$, $r \geq 1$ and an infinite sequence G_t of entire functions, $t \in T \subset \mathbb{N}$ (depending on m and on the choice of the points w_j), such that*

a) *G_t vanishes at order $\geq t$ at all points $w_1, \ldots, w_m$;*

b) $|G_t|_r \geq (C_1 t)^{-t\,[K:\mathbb{Q}]}$;

c) $|G_t|_{R(t)} \leq C_2^t$ *where* $R(t) = (t^{d-n}/\log t)^{1/(\rho_1+\ldots+\rho_d+\varepsilon)}$.

An application of Cor. 10.6 to $F = G_t$ and $R = R(t)$ gives the desired bound for the degree δ as t tends to $+\infty$ and ε tends to 0. If δ_0 is the largest integer which satisfies the inequality of Th. 10.9, we get a contradiction if we take $\delta = \delta_0 + 1$. This shows that S must be contained in an algebraic hypersurface of degree $\delta \leq \delta_0$. $\square$