What is a Discrete Painlevé Equation?

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Based on joint work with
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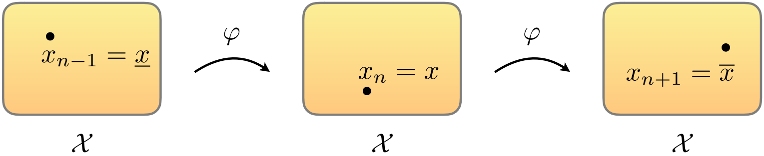

$x_{n-1} = \underline{x}$ $\xrightarrow{\varphi}$ $x_n = x$ $\xrightarrow{\varphi}$ $x_{n+1} = \overline{x}$

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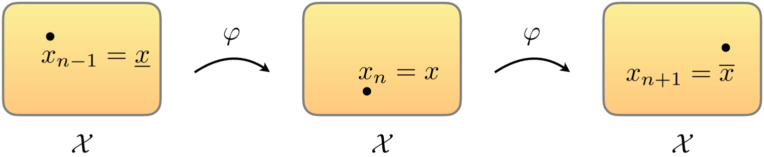

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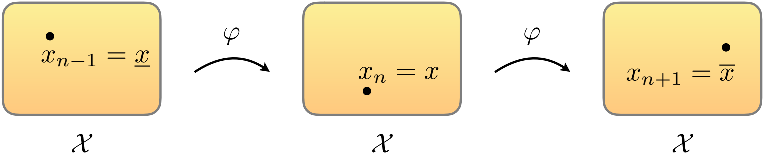

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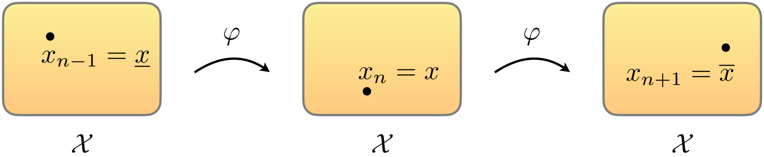

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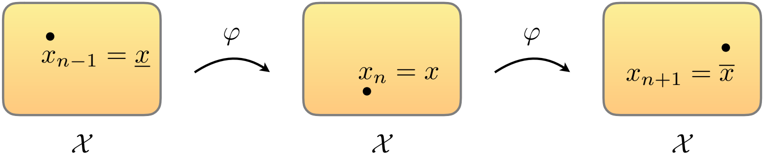

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- The main tools that we use are the *regularization* of the mapping using the *blowup* procedure, *linearization* of the mapping via the induced map on the *Picard lattice* of the resulting algebraic surface, and, in the discrete Painlevé case, *birational representations* of the *(extended) affine Weyl symmetry group* of the equation.

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In certain sense, the Painlevé property is an attempt to single out the equations that have a meaningful notion of a general solution and the associated Riemann surface — *integrability*.

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$n = 2$: P. Painlevé, B. Gambier — Painlevé equations and Painlevé transcendents:

$$\begin{tikzcd} & & P_{\mathrm{IV}} \arrow[dr] & & \\ P_{\mathrm{VI}} \arrow[r] & P_{\mathrm{V}} \arrow[ur] \arrow[dr] & & P_{\mathrm{II}} \arrow[r] & P_{\mathrm{I}} \\ & & P_{\mathrm{III}} \arrow[ur] & & \end{tikzcd}$$

(P-I) $\dfrac{d^2y}{dt^2} = 6y^2 + t;$ Painlevé equations have parameters!

(P-II) $\dfrac{d^2y}{dt^2} = 2y^3 + ty + \alpha;$

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(P-V) $\dfrac{d^2y}{dt^2} = \left(\dfrac{1}{2y} + \dfrac{1}{y-1}\right)\left(\dfrac{dy}{dt}\right)^2 - \dfrac{1}{t}\dfrac{dy}{dt} + \dfrac{(y-1)^2}{t^2}\left(\alpha y + \dfrac{\beta}{y}\right) + \gamma\dfrac{y}{t} + \delta\dfrac{y(y+1)}{y-1};$

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Classification Scheme for Painlevé Equations

$n = 1$: L. Fuchs, H. Poincaré

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$n \geq 3$: Still open.

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As with the differential Painlevé equations, it is not obvious that a given recurrence relation is in the discrete Painlevé class. The naming convention, based on the continuous limit, is also not a very good one – ambiguous and does not cover all the cases. Correct approach is through the algebro-geometric theory due to H. Sakai.
Analogue of the Painlevé property — *singularity confinement*.

Sakai's Classification Scheme for Discrete Painlevé Equations.

- K. Okamoto initiated a geometric approach to differential Painlevé equations. Write them as non-automonous Hamiltonian Systems: $q' = \frac{\partial \mathcal{H}}{\partial p}$, $p' = -\frac{\partial \mathcal{H}}{\partial q}$. Space of initial conditions: $(q(t_0), p(t_0)) \in \mathbb{C}^2$? Not quite — to allow meromorphic solutions, need to compactify $\mathbb{C}^2$ to $\mathbb{P}^1 \times \mathbb{P}^1$, then *blowup* to separate solutions going through the same initial point, and remove some singular leaves. This constructs a *space of initial conditions* for the equation.

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- In 2001 H. Sakai, developing the ideas of K. Okamoto in the differential case, proposed a classification scheme for Painlevé equations based on algebraic geometry. This scheme includes both discrete and differential (as a limit in a particular lattice direction) equations and explains the appearance of discrete Painlevé equations as Bäcklund transformations of the differential equations (commuting flows).

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- To each equation corresponds a pair of orthogonal sub-lattices $(\Pi(R), \Pi(R^{\perp}))$ — the *surface* and the *symmetry* sub-lattice in the $E_8^{(1)}$ lattice (this defines the type of the equation), and a translation element in $\tilde{W}(R^{\perp})$ (this defines the equation itself).

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- Root vectors in the symmetry sub-lattice describe elementary (non-autonomous) symmetries of this family (*Cremona isometries*) acting as reflections on the Picard lattice of the surface (hence the affine Weyl group structure). Elements of infinite order correspond to dynamical systems on this family. Translations on the lattice are called discrete Painlevé equations, elements whose power is a translation are called projective reductions.

$$\begin{tikzcd}
\left(E_8^{(1)}\right)^e \arrow[d] & & & & & & & \left(A_1^{(1)}\right)^q_{|\alpha|^2=8} \arrow[ddr] & & \\
\left(E_8^{(1)}\right)^q \arrow[r] \arrow[dr] & \left(E_7^{(1)}\right)^q \arrow[r] \arrow[dr] & \left(E_6^{(1)}\right)^q \arrow[r] \arrow[dr] & \left(D_5^{(1)}\right)^q \arrow[r] \arrow[dr] & \left(A_4^{(1)}\right)^q \arrow[r] \arrow[dr] & \left((A_2+A_1)^{(1)}\right)^q \arrow[r] \arrow[dr] \arrow[ddr] & \underset{|\alpha|^2=14}{\left((A_1+A_1)^{(1)}\right)^q} \arrow[r] \arrow[ur] \arrow[dr] \arrow[ddr] & \underset{|\alpha|^2=4}{\left(A_1^{(1)}\right)^q} \arrow[r] \arrow[dr] \arrow[ddr] & \left(A_0^{(1)}\right)^q & \\
& \left(E_8^{(1)}\right)^\delta \arrow[r] & \left(E_7^{(1)}\right)^\delta \arrow[r] & \left(E_6^{(1)}\right)^\delta \arrow[r] & \left(D_4^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_3^{(1)}\right)^{c,\delta} \arrow[r] \arrow[dr] & \left(2A_1^{(1)}\right)^{c,\delta} \arrow[r] \arrow[dr] & \left(A_1^{(1)}\right)^{c,\delta} \arrow[r] \arrow[dr] & \left(A_0^{(1)}\right)^c & \\
& & & & & & \left(2A_1^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_1^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_0^{(1)}\right)^c &
\end{tikzcd}$$

Symmetry (above) and surface (below) -type classification schemes for Painlevé equations

$$\begin{tikzcd}
A_0^{(1)} \arrow[dr] & & & & & & & & \overset{\text{q-}P_{\mathrm{I}}}{A_7^{(1)}} \arrow[ddr] & \\
& A_0^{(1)*} \arrow[r] \arrow[dr] & A_1^{(1)} \arrow[r] \arrow[dr] & A_2^{(1)} \arrow[r] \arrow[dr] & \overset{\text{q-}P_{\mathrm{VI}}}{A_3^{(1)}} \arrow[r] \arrow[dr] & \overset{\text{q-}P_{\mathrm{V}}}{A_4^{(1)}} \arrow[r] \arrow[dr] & \overset{\text{q-}P_{\mathrm{IV}},\ \text{q-}P_{\mathrm{III}}}{A_5^{(1)}} \arrow[r] \arrow[dr] \arrow[ddr] & A_6^{(1)} \arrow[r] \arrow[ur] \arrow[dr] \arrow[ddr] & A_7^{(1)} \arrow[r] \arrow[dr] \arrow[ddr] & A_8^{(1)} \\
& & A_0^{(1)**} \arrow[r] & A_1^{(1)*} \arrow[r] & A_2^{(1)*} \arrow[r] & \underset{P_{\mathrm{VI}},\ \text{d-}P_{\mathrm{V}}}{D_4^{(1)}} \arrow[r] & \underset{\substack{P_{\mathrm{V}},\ \text{d-}P_{\mathrm{IV}} \\ \text{d-}P_{\mathrm{III}}}}{D_5^{(1)}} \arrow[r] \arrow[dr] & \underset{\substack{P_{\mathrm{III}} \\ \text{alt. d-}P_{\mathrm{II}}}}{D_6^{(1)}} \arrow[r] \arrow[dr] & \underset{P_{\mathrm{III}}}{D_7^{(1)}} \arrow[r] \arrow[dr] & \underset{P_{\mathrm{III}}}{D_8^{(1)}} \\
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- Consider a *bi-quadractic* curve Γ on $\mathbb{P}^1 \times \mathbb{P}^1$. In an affine $\mathbb{C}^2$-chart Γ is given by a bi-degree $(2,2)$ polynomial equation

$$a_{00}x^2y^2 + a_{01}x^2y + a_{02}x^2 + a_{10}xy^2 + a_{11}xy + a_{12}x + a_{20}y^2 + a_{21}y + a_{22} = 0.$$

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- This equation can be written as

$$\mathbf{x}^T\mathbf{A}\mathbf{y} = \begin{bmatrix} x^2 & x & 1 \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y^2 \\ y \\ 1 \end{bmatrix} = \sum_{i,j=0}^{2} a_{ij}x^{2-i}y^{2-j} = 0,$$

  where

$$\mathbf{x} = \langle x^2, x, 1\rangle, \quad \mathbf{y} = \langle y^2, y, 1\rangle, \qquad \mathbf{A} \in \mathrm{Mat}_{3\times 3}(\mathbb{C}).$$

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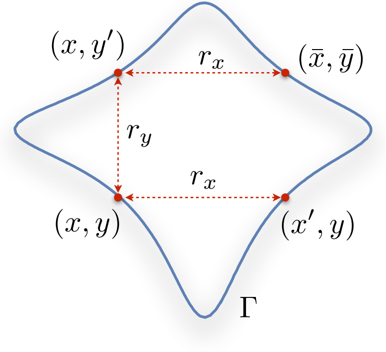
- In general, Γ is an *elliptic curve* that can be rewritten in a Weierstrass normal form $y^2 = 4x^3 - g_2x - g_3$.

Bi-quadratic Curves and Involutions

Since Γ has bi-degree $(2, 2)$, we can define two *involutions*,

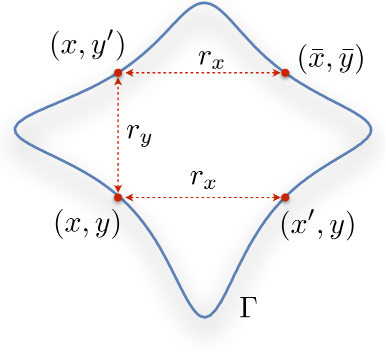
$$r_x : (x, y) \to (x', y) \text{ and } r_y : (x, y) \to (x, y')$$



Bi-quadratic Curves and Involutions

Since Γ has bi-degree $(2, 2)$, we can define two *involutions*,

$$r_x : (x, y) \to (x', y) \text{ and } r_y : (x, y) \to (x, y')$$

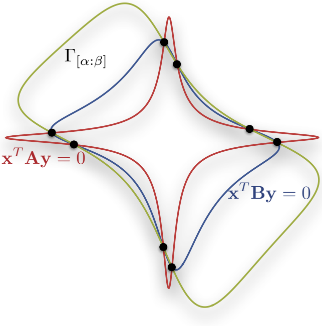


as well as their composition $r_x \circ r_y : (x, y) \to (\bar{x}, \bar{y})$. This composition defines a discrete dynamical system on the curve Γ (which is essentially a shift w.r.t. its Abelian group structure) and the main idea of the QRT map is to extend $r_x \circ r_y$ to all of the $\mathbb{P}^1 \times \mathbb{P}^1$.

The QRT Mapping

For that, take *two* matrices $\mathbf{A}, \mathbf{B} \in \mathrm{Mat}_{3\times 3}(\mathbb{C})$ and consider a *pencil* (i.e., a one-dimensional family) of such curves

$$\Gamma_{[\alpha:\beta]}: \qquad \alpha \mathbf{x}^T \mathbf{A}\mathbf{y} + \beta \mathbf{x}^T \mathbf{B}\mathbf{y} = 0, \quad [\alpha:\beta] \in \mathbb{P}^1$$



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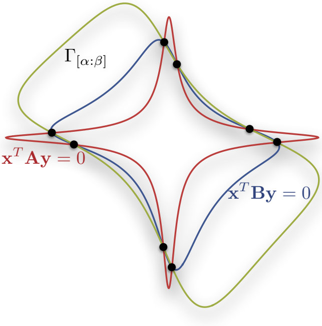

Then, given a point (x_*, y_*), there is only one curve from a family with the parameter $[\alpha : \beta] = [-\mathbf{x}_*^T \mathbf{B} \mathbf{y}_*, \mathbf{x}_*^T \mathbf{A} \mathbf{y}_*]$, *except* for the **eight** base points $\mathbf{x}_*^T \mathbf{A} \mathbf{y}_* = \mathbf{x}_*^T \mathbf{B} \mathbf{y}_* = 0$. Resolving these points using the blowup, we get a rational elliptic surface $\mathcal{X}$ with the *QRT* automorphism $r_x \circ r_y$ preserving the elliptic fibration $\pi : \mathcal{X} \to \mathbb{P}^1$, and $\pi^{-1}([\alpha : \beta])$ is an elliptic curve except for 12 points corresponding to *singular fibers* (classified by K. Kodaira into 22 types).

The Half (or Root) of the QRT Mapping

It is possible to represent the QRT mapping as

$$r_x \circ r_y = \varphi \circ \varphi = \varphi^2, \qquad \varphi = \sigma_{x,y} \circ r_y$$

where $\sigma_{x,y}$ is an involution

$$\sigma_{x,y} : (x, y; \mathbf{A}, \mathbf{B}) \mapsto (y, x; \mathbf{A}^T, \mathbf{B}^T).$$

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solving for $\bar{y}$, and applying $\sigma_{x,y}$, we get the following simple expression for φ:

$$\begin{cases} \bar{x} = \dfrac{f_1(x) - f_2(x)y}{f_2(x) - f_3(x)y} \\ \bar{y} = x \end{cases}, \quad \text{where } \langle f_1(x), f_2(x), f_3(x) \rangle = (\mathbf{x}^T \mathbf{A}) \times (\mathbf{x}^T \mathbf{B}).$$

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This mapping φ is the one that we study and deautonomize for a particular choice of the matrices **A** and **B**.

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We consider the symmetric case and take the matrices **A**, **B** as

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & a + a^{-1} & 1 \\ a + a^{-1} & 0 & -a - a^{-1} \\ 1 & -a - a^{-1} & 1 \end{bmatrix},$$

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$$(0, a),\ (0, a^{-1}),\ (a, 0),\ (a^{-1}, 0),\ (\infty, -a),\ (\infty, -a^{-1}),\ (-a, \infty),\ (-a^{-1}, \infty)$$

We then *blowup* the base points to obtain the QRT surface, or the algebraic surface $\mathcal{X}$ on which the dynamic is *regularized*. This dynamic is further *linearized* on the *Picard lattice* $\mathrm{Pic}(\mathcal{X})$.

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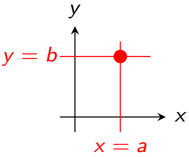

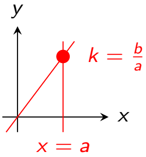

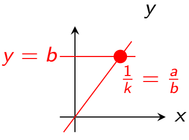

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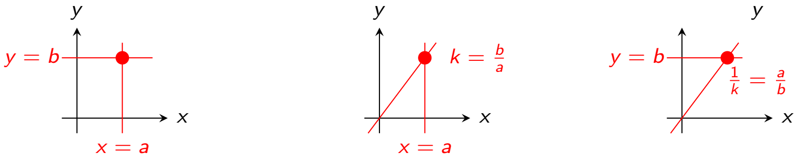

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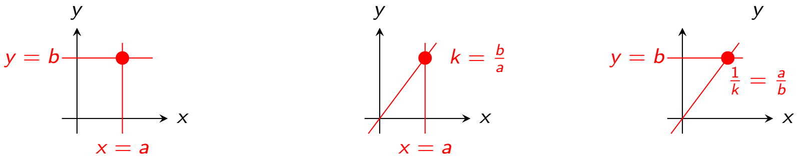

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- Then consider, in the space $\mathbb{C}^2 \times \mathbb{P}^1$ with coordinates $(x, y; [\xi_0 : \xi_1])$, the set $\mathcal{S}$ cut out by the equation $x\xi_0 = y\xi_1$.

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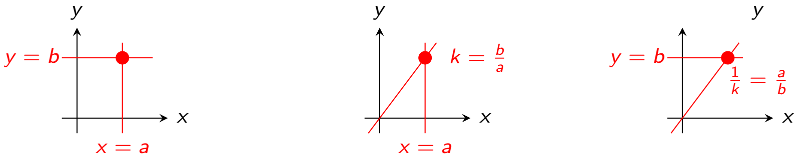

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- In view of the above, for $(x, y) \neq (0, 0)$, the restriction of the projection $\pi : \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ on $\mathcal{S}$ is an isomorphism, but $\pi^{-1}(0, 0) \simeq \mathbb{P}^1$. It is called the *exceptional divisor* and is denoted by E.

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The set $S = V(x\xi_0 - y\xi_1)$ is covered by two charts (u, v) and (U, V). For a blowup with the center at (x_0, y_0) these charts are $(x, y, [\xi_0 : \xi_1]) = (u + x_0, uv + y_0, [u : 1])$ and $(x, y, [\xi_0 : \xi_1]) = (UV + x_0, V + y_0, [1 : V])$.

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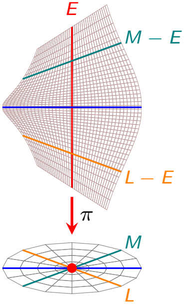
$$L \bullet M = 1$$

$$(L - E) \bullet (M - E) = 0$$

$$E \bullet E = -1$$

If $L^2 = L \bullet L = m$ then

$$(L - E)^2 = m - 1$$



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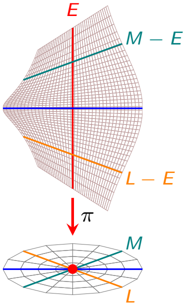

Note that we need to distinguish the *total transform* $\pi^{-1}(L)$ and the *proper transform* $\overline{\pi^{-1}(L - (0,0))}$ that we denote by $L - E$. Exceptional divisor has the self-intersection $E^2 = -1$. Such curves are called -1-curves.

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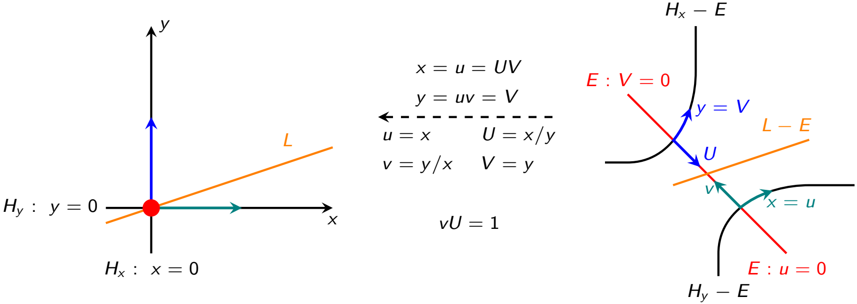

$$x = u = UV$$

$$y = uv = V$$

$$u = x \qquad U = x/y$$

$$v = y/x \qquad V = y$$

$$vU = 1$$

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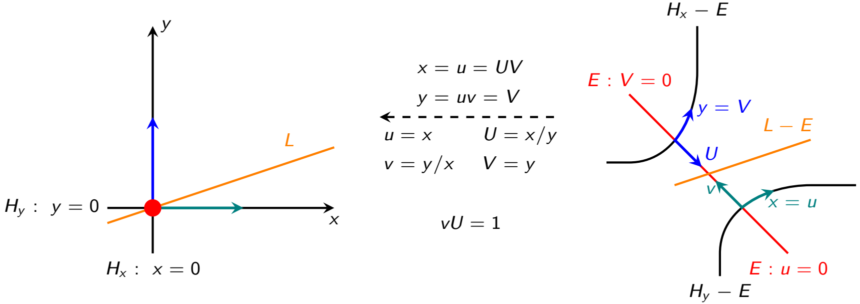

Note the *proper transform* notation and coordinates on E:

- E and $H_x - E$ intersect at $(U = 0, V = 0)$;
- E and $H_y - E$ intersect at $(u = 0, v = 0)$;
- if the line L had a slope $1/3$, E and $L - E$ intersect at $(u = 0, v = 1/3)$ or $(U = 3, V = 0)$.

The QRT Surface

Let us now return to our example. We construct the surface $\mathcal{X} = \mathcal{X}_{\mathbf{b}}$ by successively 8 blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight base points:

$$p_1(0, a),\ p_2(0, a^{-1}),\ p_3(a, 0),\ p_4(a^{-1}, 0),\ p_5(\infty, -a),\ p_6(\infty, -a^{-1}),\ p_7(-a, \infty),\ p_8(-a^{-1}, \infty).$$

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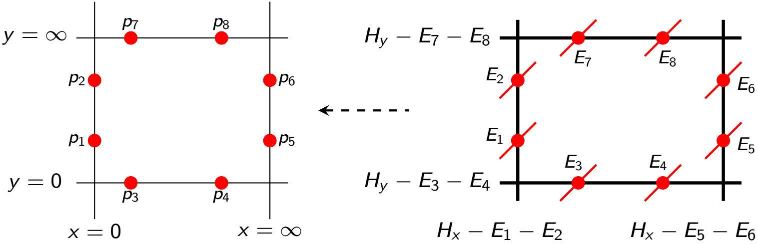

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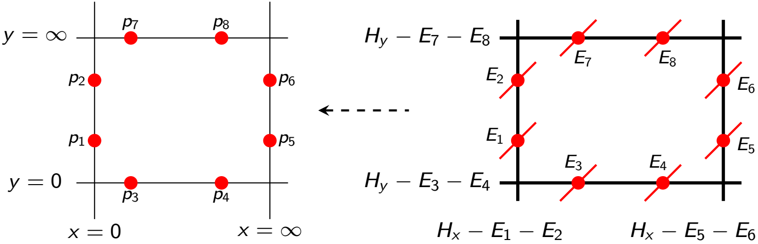

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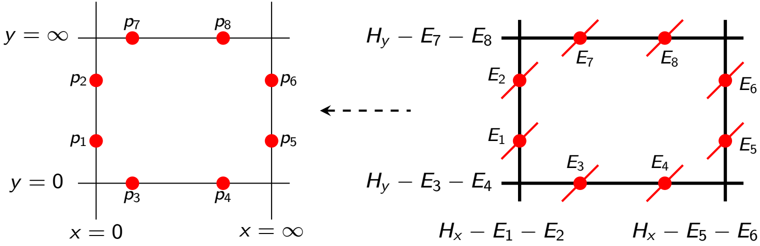

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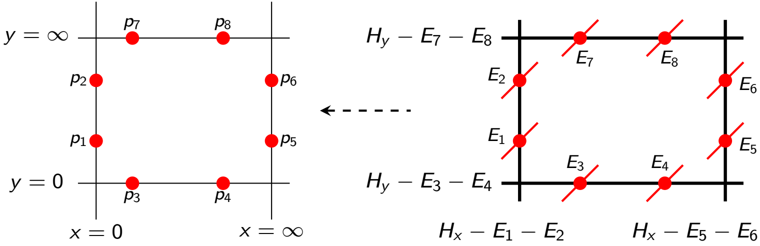
The QRT Surface

Let us now return to our example. We construct the surface $\mathcal{X} = \mathcal{X}_{\mathbf{b}}$ by successively 8 blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight base points:

$$p_1(0,a),\ p_2(0,a^{-1}),\ p_3(a,0),\ p_4(a^{-1},0),\ p_5(\infty,-a),\ p_6(\infty,-a^{-1}),\ p_7(-a,\infty),\ p_8(-a^{-1},\infty).$$

Then φ is lifted to an automorphism on $\mathcal{X}$.



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The QRT dynamics is *integrable* since it preserves the fibration. It is *autonomous* since the points of the blowup (whose coordinates appear as coefficients of the equation) do not evolve.

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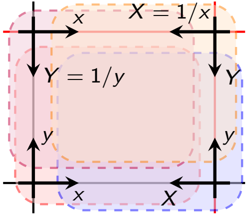

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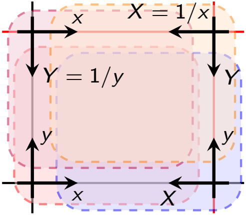

- Thus, we see that all $(2, 2)$-curves in our pencil belong to the anti-canonical divisor class, $\Gamma_k \in \left(-\mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}\right)$. Further, $\left(-\mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}\right)^2 = 8$ gives us the number of base points.

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  Thus, each curve in the fibration is in the *anti-canonical divisor class*;

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- Our mapping then induces a linear mapping on the Picard lattice that preserves the fibration (ad hence both the canonical and the anti-canonical divisor classes). This linear map captures a lot of information about our dynamics (as we will see, essentially everything in the discrete Painlevé case).

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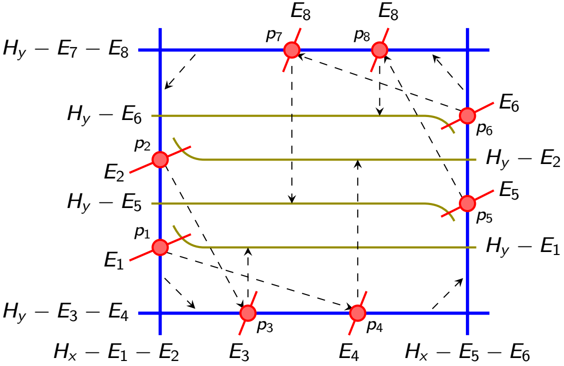

Types of (Singular) Fibers

Putting the curves $\Gamma(x, y; k)$ in the Weierstrass normal form and computing the elliptic discriminant, we can compute the singular fibers of our elliptic fibration. In our example, we see that the singular fibers appear at $k = 0$, $k = \pm 4i(a + a^{-1})$, $k = \pm(a - a^{-1})$, and $k = \infty$ and have the following types:

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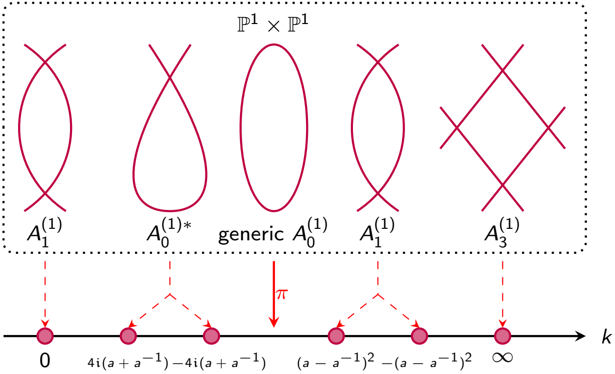

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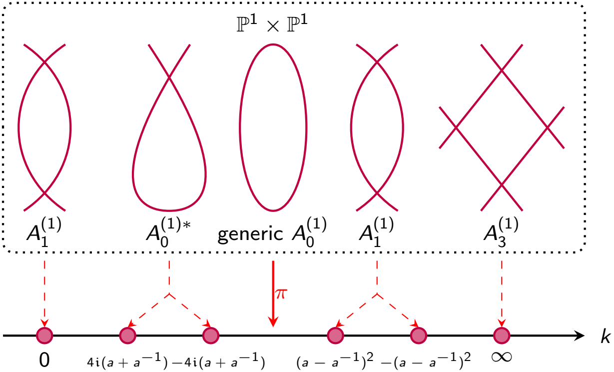

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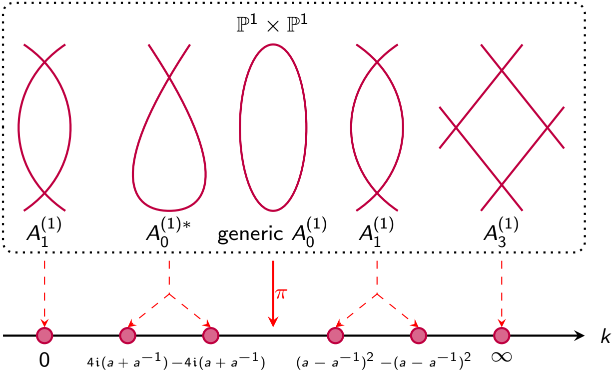

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In particular, $k = \infty$ is $xy = 0$ of $q - P_{\text{VI}}$ of Jimbo-Sakai.

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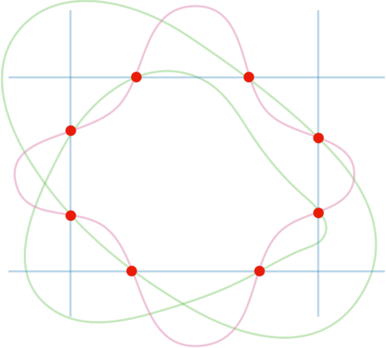

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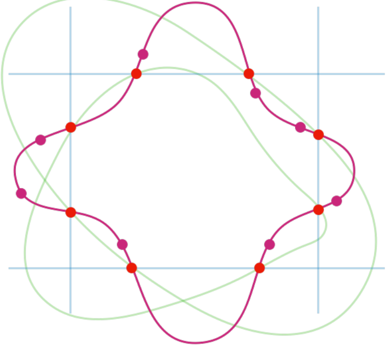

Deautonomization with elliptic $A_0^{(1)}$-fiber.

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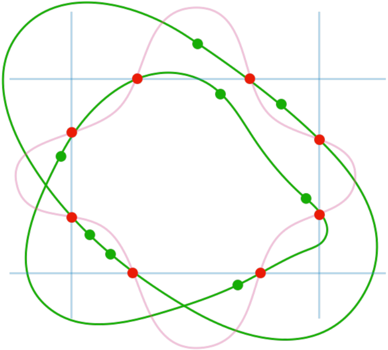

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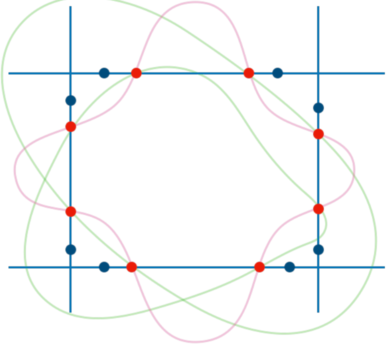

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We again need to resolve the indeterminate points using the *blowup procedure*. This is how the space of initial conditions is constructed.

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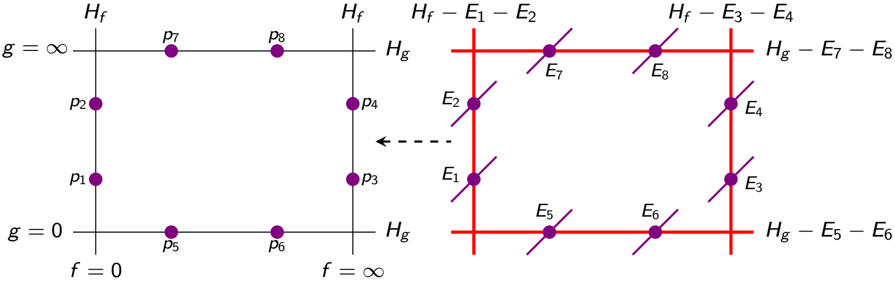
Resolve it using the blowup procedure (turns out there are no more indeterminate points):



Okamoto Space of Initial Conditions $\mathcal{X}_{\mathbf{b}}$ for q-P_{VI}

Generalized Halphen Surfaces

In general, blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight points results in a surface $\mathcal{X}$ with the anti-canonical divisor class

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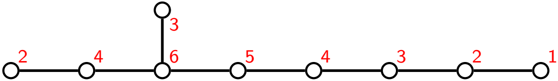
Its orthogonal complement in the Picard lattice Pic($\mathcal{X}$) has the affine type $E_8^{(1)}$



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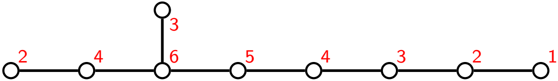

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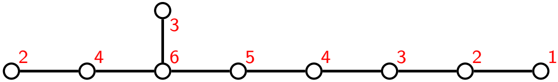

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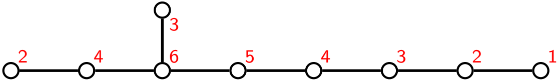

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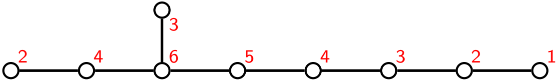

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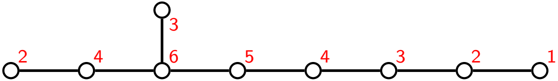

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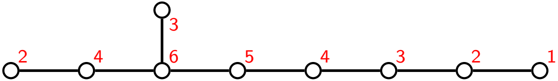

We say that $\mathcal{X}$ is a *generalized Halphen surface of index zero* if it has a unique anti-canonical divisor of canonical type: $-\mathcal{K}_{\mathcal{X}} \bullet \mathcal{D}_i = 0$ for any irreducible component $\mathcal{D}_i$ of $-\mathcal{K}_{\mathcal{X}}$.
The orthogonal complement $(-\mathcal{K}_{\mathcal{X}})^{\perp} \triangleleft \mathrm{Pic}(\mathcal{X})$ has two important complementary sub-lattices:

- The *surface sub-lattice* $\Pi(R) = \mathrm{Span}_{\mathbb{Z}}\{\mathcal{D}_i\}$;
- The *symmetry sub-lattice* $\Pi(R^{\perp}) = \mathrm{Span}_{\mathbb{Z}}\{\alpha_i\}$, which is the orthogonal complement of R (thus, reflections w_i in the roots α_i preserve the type of the surface and the intersection form, i.e., they are *Cremona isometries*).
- Configurations of $\mathcal{D}_i$ and α_j are also described by the affine Dynkin diagrams, automorphisms of these diagrams are also symmetries of the surface (also *Cremona isometries*).
- $W = \langle \alpha_i \mid \cdots \rangle$ and $\widetilde{W} = \langle w_i, \sigma_j \mid \cdots \rangle$ are the *affine Weyl group* and the *extended affine Weyl group* of symmetries of our surface;
- Note that $\Pi(R) \cap \Pi(R^{\perp}) = \mathrm{Span}_{\mathbb{Z}}(-\mathcal{K}_{\mathcal{X}})$, $-K_{\mathcal{X}} = \sum_i m_i \mathcal{D}_i = \sum_j n_j \alpha_j$.

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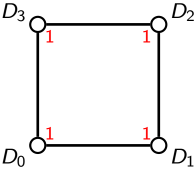

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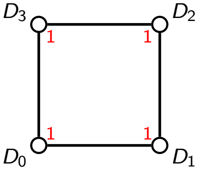

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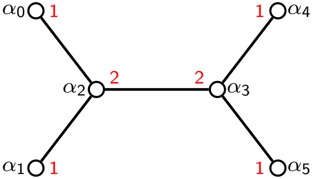

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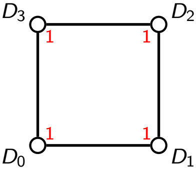

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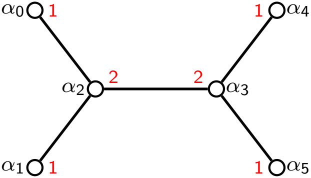

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- The anti-canonical divisor decomposes as

$$-K_{\mathcal{X}} = 2\mathcal{H}_f + 2\mathcal{H}_g - \sum_{i=1}^{8} \mathcal{E}_i = D_0 + D_1 + D_2 + D_3 = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5.$$

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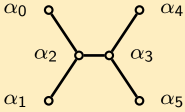
$$W(D_5^{(1)}) = \left\langle w_0,\dots,w_6 \;\middle|\; \begin{array}{cl} w_i^2 = e & \\ w_i\circ w_j = w_j\circ w_i & \text{when } \overset{\circ}{\alpha_i}\;\;\overset{\circ}{\alpha_j} \\ w_i\circ w_j\circ w_i = w_j\circ w_i\circ w_j & \text{when } \overset{\circ\!\!-\!\!-\!\!-\!\!\circ}{\alpha_i\;\;\alpha_j} \end{array}\right\rangle$$



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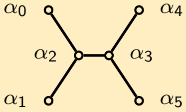
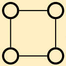


- The finite group of Dynkin diagram automorphisms

$$\mathrm{Aut}\left(D_5^{(1)}\right) \simeq \mathrm{Aut}\left(A_3^{(1)}\right) \simeq \mathbb{D}_4 = \text{The Dihedral Group},$$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The Affine Weyl Group $W(E_6^{(1)})$

Theorem

Reflections w_i on $\operatorname{Pic}(\mathcal{X})$ *are induced by the following elementary birational mappings, also denoted by w_i, on the family $\mathcal{X}_{\mathbf{b}}$. To ensure the group structure, we require that each map fixes b_4 and $\chi(\delta)$.*

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix} \xmapsto{w_0} \left(\begin{matrix} \frac{b_{13}}{b_4} & \frac{b_{23}}{b_4} & b_3 & \frac{b_{33}}{b_4} \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; f, \frac{b_3}{b_4} g \right),$$

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; f, g \right) \xmapsto{w_1} \left(\begin{matrix} b_2 & b_1 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; f, g \right),$$

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; f, g \right) \xmapsto{w_2} \left(\begin{matrix} \frac{b_{33}}{b_1} & \frac{b_{23}}{b_1} & b_3 & \frac{b_{34}}{b_1} \\ \frac{b_{35}}{b_1} & \frac{b_{36}}{b_1} & b_7 & b_8 \end{matrix} ; f \frac{(g - b_3)}{(g - b_1)}, \frac{b_3}{b_1} g \right),$$

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; f, g \right) \xmapsto{w_3} \left(\begin{matrix} \frac{b_{17}}{b_5} & \frac{b_{27}}{b_5} & b_3 & b_4 \\ \frac{b_{77}}{b_5} & \frac{b_{67}}{b_5} & b_7 & \frac{b_{78}}{b_5} \end{matrix} ; \frac{b_7}{b_5} f, g \frac{(f - b_7)}{(f - b_5)} \right).$$

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; f, g \right) \xmapsto{w_4} \left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_6 & b_5 & b_7 & b_8 \end{matrix} ; f, g \right),$$

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Consider now coordinate description of the elementary birational maps. We have the general formula $w_i(\mathcal{C}) = \mathcal{C} + (\alpha_i \bullet \mathcal{C})\alpha_i$. We get:

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  Hence, a projective coordinate on this pencil is $\bar{f} = [-C : A] = [f(g - b_3) : (g - b_1)]$. Recall that the coordinates of our blowup points (parametrization!) points are $p_1(0, b_1)$, $p_2(0, b_2)$, $p_3(\infty, b_3)$, $p_4(\infty, b_4)$, $p_5(b_5, 0)$, $p_6(b_6, 0)$, $p_7(b_7, \infty)$, $p_8(b_8, \infty)$. Since $w_2(\mathcal{E}_2) = \bar{\mathcal{E}}_2$, we want $\bar{f}(0, b_2) = 0$. Thus, $\bar{f} = f(g - b_3)/(g - b_1)$.

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$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix}; f, g \right) \xmapsto{\sigma_1} \left(\begin{matrix} \frac{b_3 b_4}{b_1} & \frac{b_3 b_4}{b_2} & b_4 & b_3 \\ b_7 & b_8 & b_5 & b_6 \end{matrix}; f, \frac{b_3 b_4}{g} \right),$$

Sketch of the Proof

- So we get $\bar{f} = f(g-b_3)/(g-b_1)$, $\bar{g} = g$. Since $\bar{\mathcal{E}}_1 = \mathcal{H}_g - \mathcal{E}_3$, which is given by $g = b_3$, we get $(\bar{f},\bar{g})(\bar{p}_1) = (0,\bar{b}_1) = (0,b_3)$. As another example, since $w_2(\mathcal{E}_5) = \mathcal{E}_5$, $(\bar{f},\bar{g})(\bar{p}_5) = (\bar{f}(b_5,0),\bar{g}(b_5,0)) = (b_5 b_3/b_1, 0)$, and so on. We finally get:

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \Biggr) \xmapsto{w_2} \left(\begin{matrix} b_3 & b_2 & b_1 & b_4 \\ \frac{b_3 b_5}{b_1} & \frac{b_3 b_6}{b_1} & b_7 & b_8 \end{matrix}; f\frac{(g-b_3)}{(g-b_1)}, g\right).$$

  The remaining cases are similar.

- In the similar way, we can compute the birational mappings corresponding to the permutations σ_j:

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix}; f, g\right) \xmapsto{\sigma_1} \left(\begin{matrix} \frac{b_3 b_4}{b_1} & \frac{b_3 b_4}{b_2} & b_4 & b_3 \\ b_7 & b_8 & b_5 & b_6 \end{matrix}; f, \frac{b_3 b_4}{g}\right),$$

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix}; f, g\right) \xmapsto{\sigma_2} \left(\begin{matrix} b_3 & b_4 & b_1 & b_2 \\ \frac{b_7 b_8}{b_5} & \frac{b_7 b_8}{b_6} & b_8 & b_7 \end{matrix}; \frac{b_7 b_8}{f}, g\right).$$

The Dynamic

We can compute the action of ψ_* : Pic$(\mathcal{X}_{\mathbf{b}}) \to$ Pic$(\mathcal{X}_{\bar{\mathbf{b}}})$:

$$\begin{aligned}
\bar{\mathcal{H}}_f &= \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4,\\
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Definition: A *discrete Painlevé equation* is a discrete dynamical system on the family $\mathcal{X}_b$ induced by a translation in the $\Pi(R^\perp)$ affine symmetry sub-lattice of the corresponding surface.

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Questions: How, given a translation direction, obtain the corresponding *discrete Painlevé equation*? How to identify whether the two directions give the equivalent equations? Which equations are the simplest?

The Reverse Process: from Translations to Equations

Suppose we are now given the generalized Halphen surface of type $A_3^{(1)}$. It can be parameterized as above and we can define the corresponding roots α_i and D_i. How, given the translation

$$\text{q-}P_{\text{VI}} : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_0 + \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \alpha_4, \alpha_5 \rangle,$$

write down the corresponding discrete Painlevé equation?

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For **Step 1**, we use the following Lemma:

Reduction Lemma (V. Kac, **Infinite dimensional Lie algebras**, Lemma 3.11)

If $w(\alpha_i) < 0$, then

$$l(w \circ w_i) < l(w),$$

where $l(w)$ is length of $w \in W$, and α_i is a simple root.

Decomposing Translations into Elementary Reflections and Automorphisms

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So we get

$$\text{q-}P_{\mathrm{VI}} : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_0+\alpha_1+3\alpha_2+2\alpha_3+\alpha_4+\alpha_5, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_4-\alpha_5, \alpha_4, \alpha_5\rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2+\alpha_3, \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4+\alpha_5, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_5, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_4\rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2+\alpha_3, \alpha_4, \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_5, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_4\rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3, \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_5, \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4\rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0-\alpha_1-\alpha_2, \alpha_0+\alpha_1+2\alpha_2+\alpha_3, \alpha_5, \alpha_4\rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0-\alpha_1-\alpha_2, \alpha_0+\alpha_1+2\alpha_2+\alpha_3, \alpha_4, \alpha_5\rangle$$

Decomposing Translations into Elementary Reflections and Automorphisms

So we get

$$\text{q-}P_{\mathrm{VI}} : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_0 + \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \alpha_4, \alpha_5 \rangle$$

$$\begin{aligned}\text{q-}P_{\mathrm{VI}} \circ w_3 : \alpha \mapsto \langle &\alpha_0, \alpha_1, \alpha_2 + \alpha_3, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \\ &-\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 \rangle\end{aligned}$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2 + \alpha_3, \alpha_4, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 \rangle$$

$$\begin{aligned}\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 : \alpha \mapsto \langle &\alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_5, \\ &\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \rangle\end{aligned}$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0 - \alpha_1 - \alpha_2, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_5, \alpha_4 \rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0 - \alpha_1 - \alpha_2, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 : \alpha \mapsto \langle -\alpha_1 - \alpha_2, -\alpha_0 - \alpha_2, \alpha_0 + \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

Decomposing Translations into Elementary Reflections and Automorphisms

So we get

$$\text{q-}P_{\text{VI}} : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_0 + \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2 + \alpha_3, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2 + \alpha_3, \alpha_4, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_5, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0 - \alpha_1 - \alpha_2, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_5, \alpha_4 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0 - \alpha_1 - \alpha_2, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 : \alpha \mapsto \langle -\alpha_1 - \alpha_2, -\alpha_0 - \alpha_2, \alpha_0 + \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 : \alpha \mapsto \langle -\alpha_1 - \alpha_2, \alpha_0 + \alpha_2, \alpha_1, \alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

Decomposing Translations into Elementary Reflections and Automorphisms

So we get

$$\text{q-}P_{\mathrm{VI}} : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_0 + \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2 + \alpha_3, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,$$
$$-\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 \rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2 + \alpha_3, \alpha_4, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 \rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_5,$$
$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0 - \alpha_1 - \alpha_2, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_5, \alpha_4 \rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0 - \alpha_1 - \alpha_2, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 : \alpha \mapsto \langle -\alpha_1 - \alpha_2, -\alpha_0 - \alpha_2, \alpha_0 + \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 : \alpha \mapsto \langle -\alpha_1 - \alpha_2, \alpha_0 + \alpha_2, \alpha_1, \alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 \circ w_0 : \alpha \mapsto \langle \alpha_1 + \alpha_2, \alpha_0 + \alpha_2, -\alpha_2, \alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

Decomposing Translations into Elementary Reflections and Automorphisms

So we get

$$\text{q-}P_{\mathrm{VI}}: \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_0+\alpha_1+3\alpha_2+2\alpha_3+\alpha_4+\alpha_5, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_4-\alpha_5, \alpha_4, \alpha_5\rangle$$

$$\text{q-}P_{\mathrm{VI}}\circ w_3: \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2+\alpha_3, \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4+\alpha_5,$$
$$-\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_5, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_4\rangle$$

$$\text{q-}P_{\mathrm{VI}}\circ w_3\circ w_4: \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2+\alpha_3, \alpha_4, \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_5, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_4\rangle$$

$$\text{q-}P_{\mathrm{VI}}\circ w_3\circ w_4\circ w_5: \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3, \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_5,$$
$$\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4\rangle$$

$$\text{q-}P_{\mathrm{VI}}\circ w_3\circ w_4\circ w_5\circ w_3: \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0-\alpha_1-\alpha_2, \alpha_0+\alpha_1+2\alpha_2+\alpha_3, \alpha_5, \alpha_4\rangle$$

$$\text{q-}P_{\mathrm{VI}}\circ w_3\circ w_4\circ w_5\circ w_3\circ\sigma_1: \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0-\alpha_1-\alpha_2, \alpha_0+\alpha_1+2\alpha_2+\alpha_3, \alpha_4, \alpha_5\rangle$$

$$\text{q-}P_{\mathrm{VI}}\circ w_3\circ w_4\circ w_5\circ w_3\circ\sigma_1\circ w_2: \alpha \mapsto \langle -\alpha_1-\alpha_2, -\alpha_0-\alpha_2, \alpha_0+\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_4, \alpha_5\rangle$$

$$\text{q-}P_{\mathrm{VI}}\circ w_3\circ w_4\circ w_5\circ w_3\circ\sigma_1\circ w_2\circ w_1: \alpha \mapsto \langle -\alpha_1-\alpha_2, \alpha_0+\alpha_2, \alpha_1, \alpha_2+\alpha_3, \alpha_4, \alpha_5\rangle$$

$$\text{q-}P_{\mathrm{VI}}\circ w_3\circ w_4\circ w_5\circ w_3\circ\sigma_1\circ w_2\circ w_1\circ w_0: \alpha \mapsto \langle \alpha_1+\alpha_2, \alpha_0+\alpha_2, -\alpha_2, \alpha_2+\alpha_3, \alpha_4, \alpha_5\rangle$$

$$\text{q-}P_{\mathrm{VI}}\circ w_3\circ w_4\circ w_5\circ w_3\circ\sigma_1\circ w_2\circ w_1\circ w_0\circ w_2: \alpha \mapsto \langle \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5\rangle$$

Decomposing Translations into Elementary Reflections and Automorphisms

So we get

$$
\begin{aligned}
&\text{q-}P_{\mathrm{VI}} : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_0+\alpha_1+3\alpha_2+2\alpha_3+\alpha_4+\alpha_5, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_4-\alpha_5, \alpha_4, \alpha_5 \rangle\\
&\text{q-}P_{\mathrm{VI}} \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2+\alpha_3, \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4+\alpha_5,\\
&\qquad\qquad -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_5, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_4 \rangle\\
&\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2+\alpha_3, \alpha_4, \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_5, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3-\alpha_4 \rangle\\
&\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_0-\alpha_1-2\alpha_2-\alpha_3, \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_5,\\
&\qquad\qquad \alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4 \rangle\\
&\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0-\alpha_1-\alpha_2, \alpha_0+\alpha_1+2\alpha_2+\alpha_3, \alpha_5, \alpha_4 \rangle\\
&\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0-\alpha_1-\alpha_2, \alpha_0+\alpha_1+2\alpha_2+\alpha_3, \alpha_4, \alpha_5 \rangle\\
&\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 : \alpha \mapsto \langle -\alpha_1-\alpha_2, -\alpha_0-\alpha_2, \alpha_0+\alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_4, \alpha_5 \rangle\\
&\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 : \alpha \mapsto \langle -\alpha_1-\alpha_2, \alpha_0+\alpha_2, \alpha_1, \alpha_2+\alpha_3, \alpha_4, \alpha_5 \rangle\\
&\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 \circ w_0 : \alpha \mapsto \langle \alpha_1+\alpha_2, \alpha_0+\alpha_2, -\alpha_2, \alpha_2+\alpha_3, \alpha_4, \alpha_5 \rangle\\
&\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 \circ w_0 \circ w_2 : \alpha \mapsto \langle \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle\\
&\text{q-}P_{\mathrm{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 \circ w_0 \circ w_2 \circ \sigma_2 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle
\end{aligned}
$$

Decomposing Translations into Elementary Reflections and Automorphisms

So we get

$$\text{q-}P_{\text{VI}} : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_0 + \alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2 + \alpha_3, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2 + \alpha_3, \alpha_4, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_5, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_0 - \alpha_1 - 2\alpha_2 - \alpha_3, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_5, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0 - \alpha_1 - \alpha_2, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_5, \alpha_4 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 : \alpha \mapsto \langle \alpha_0, \alpha_1, -\alpha_0 - \alpha_1 - \alpha_2, \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 : \alpha \mapsto \langle -\alpha_1 - \alpha_2, -\alpha_0 - \alpha_2, \alpha_0 + \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 : \alpha \mapsto \langle -\alpha_1 - \alpha_2, \alpha_0 + \alpha_2, \alpha_1, \alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 \circ w_0 : \alpha \mapsto \langle \alpha_1 + \alpha_2, \alpha_0 + \alpha_2, -\alpha_2, \alpha_2 + \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 \circ w_0 \circ w_2 : \alpha \mapsto \langle \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle$$

$$\text{q-}P_{\text{VI}} \circ w_3 \circ w_4 \circ w_5 \circ w_3 \circ \sigma_1 \circ w_2 \circ w_1 \circ w_0 \circ w_2 \circ \sigma_2 : \alpha \mapsto \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle$$

Decomposition of q-P_{VI}:

$$\text{q-}P_{\text{VI}} = \sigma_2 \circ w_2 \circ w_0 \circ w_1 \circ w_2 \circ \sigma_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3$$

q-P_{VI} from Elementary Birational Transformations

Decomposition of q-P_{VI}:

$$\text{q-}P_{\mathrm{VI}} = \sigma_2 \circ w_2 \circ w_0 \circ w_1 \circ w_2 \circ \sigma_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3, \qquad q = (b_3 b_4 b_5 b_6)/(b_1 b_2 b_7 b_8)$$

q-P_{VI} from Elementary Birational Transformations

Decomposition of q-P_{VI}:

$$\text{q-}P_{\mathrm{VI}} = \sigma_2 \circ w_2 \circ w_0 \circ w_1 \circ w_2 \circ \sigma_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3, \qquad q = (b_3 b_4 b_5 b_6)/(b_1 b_2 b_7 b_8)$$

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix} ; f, g \Bigg) \xmapsto{w_3} \left(\begin{matrix} \frac{b_1 b_7}{b_5} & \frac{b_2 b_7}{b_5} & b_3 & b_4 \\ b_7 & b_6 & b_5 & b_8 \end{matrix} ; f ; g \frac{(f - b_7)}{(f - b_5)} \right)$$

q-P_{VI} from Elementary Birational Transformations

Decomposition of q-P_{VI}:

$$\text{q-}P_{\mathrm{VI}} = \sigma_2 \circ w_2 \circ w_0 \circ w_1 \circ w_2 \circ \sigma_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3, \qquad q = (b_3 b_4 b_5 b_6)/(b_1 b_2 b_7 b_8)$$

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \xmapsto{w_3} \left(\begin{matrix} \frac{b_1 b_7}{b_5} & \frac{b_2 b_7}{b_5} & b_3 & b_4 \\ b_7 & b_6 & b_5 & b_8 \end{matrix}; f; g\frac{(f-b_7)}{(f-b_5)}\right)$$

$$\xmapsto{w_4} \left(\begin{matrix} \frac{b_1 b_7}{b_5} & \frac{b_2 b_7}{b_5} & b_3 & b_4 \\ b_6 & b_7 & b_5 & b_8 \end{matrix}; f; g\frac{(f-b_7)}{(f-b_5)}\right)$$

q-P_{VI} from Elementary Birational Transformations

Decomposition of q-P_{VI}:

$$\text{q-}P_{\mathrm{VI}} = \sigma_2 \circ w_2 \circ w_0 \circ w_1 \circ w_2 \circ \sigma_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3, \qquad q = (b_3 b_4 b_5 b_6)/(b_1 b_2 b_7 b_8)$$

$$\begin{aligned}
\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \Bigg) &\xmapsto{w_3} \left(\begin{matrix} \frac{b_1 b_7}{b_5} & \frac{b_2 b_7}{b_5} & b_3 & b_4 \\ b_7 & b_6 & b_5 & b_8 \end{matrix}; f; g\,\frac{(f-b_7)}{(f-b_5)}\right) \\
&\xmapsto{w_4} \left(\begin{matrix} \frac{b_1 b_7}{b_5} & \frac{b_2 b_7}{b_5} & b_3 & b_4 \\ b_6 & b_7 & b_5 & b_8 \end{matrix}; f; g\,\frac{(f-b_7)}{(f-b_5)}\right) \\
&\xmapsto{w_5} \left(\begin{matrix} \frac{b_1 b_7}{b_5} & \frac{b_2 b_7}{b_5} & b_3 & b_4 \\ b_6 & b_7 & b_8 & b_5 \end{matrix}; f; g\,\frac{(f-b_7)}{(f-b_5)}\right)
\end{aligned}$$

q-P_{VI} from Elementary Birational Transformations

Decomposition of q-P_{VI}:

$$\text{q-}P_{\mathrm{VI}} = \sigma_2 \circ w_2 \circ w_0 \circ w_1 \circ w_2 \circ \sigma_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3, \qquad q = (b_3 b_4 b_5 b_6)/(b_1 b_2 b_7 b_8)$$

$$\begin{aligned}
\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix}; f, g\right) &\xmapsto{w_3} \left(\begin{matrix} \frac{b_1 b_7}{b_5} & \frac{b_2 b_7}{b_5} & b_3 & b_4 \\ b_7 & b_6 & b_5 & b_8 \end{matrix}; f; g\,\frac{(f - b_7)}{(f - b_5)}\right) \\
&\xmapsto{w_4} \left(\begin{matrix} \frac{b_1 b_7}{b_5} & \frac{b_2 b_7}{b_5} & b_3 & b_4 \\ b_6 & b_7 & b_5 & b_8 \end{matrix}; f; g\,\frac{(f - b_7)}{(f - b_5)}\right) \\
&\xmapsto{w_5} \left(\begin{matrix} \frac{b_1 b_7}{b_5} & \frac{b_2 b_7}{b_5} & b_3 & b_4 \\ b_6 & b_7 & b_8 & b_5 \end{matrix}; f; g\,\frac{(f - b_7)}{(f - b_5)}\right) \\
&\xmapsto{w_3} \left(\begin{matrix} \frac{b_3 b_4}{q b_2} & \frac{b_3 b_4}{q b_1} & b_3 & b_4 \\ b_8 & b_7 & b_6 & b_5 \end{matrix}; f; g\,\frac{(f - b_7)}{(f - b_5)}\,\frac{(f - b_8)}{(f - b_6)}\right)
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$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix};f,g\Bigg) \xmapsto{w_3} \begin{pmatrix} \frac{b_1b_7}{b_5} & \frac{b_2b_7}{b_5} & b_3 & b_4 \\ b_7 & b_6 & b_5 & b_8 \end{pmatrix};f;g\frac{(f-b_7)}{(f-b_5)}\Bigg)$$

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$$\xmapsto{w_3} \begin{pmatrix} \frac{b_3b_4}{qb_2} & \frac{b_3b_4}{qb_1} & b_3 & b_4 \\ b_8 & b_7 & b_6 & b_5 \end{pmatrix};f;g\frac{(f-b_7)}{(f-b_5)}\frac{(f-b_8)}{(f-b_6)}\Bigg)$$

$$\xmapsto{\sigma_1} \begin{pmatrix} qb_2 & qb_1 & b_4 & b_3 \\ b_6 & b_5 & b_8 & b_7 \end{pmatrix};f;\bar{g}=\frac{b_3b_4}{g}\frac{(f-b_5)}{(f-b_7)}\frac{(f-b_6)}{(f-b_8)}\Bigg)$$

$$\mapsto \cdots$$

q-P_{VI} from Elementary Birational Transformations (cont.)

Decomposition of q-P_{VI}:

$$\text{q-}P_{\mathrm{VI}} = \sigma_2 \circ w_2 \circ w_0 \circ w_1 \circ w_2 \circ \sigma_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3, \qquad q = (b_3 b_4 b_5 b_6)/(b_1 b_2 b_7 b_8)$$

$\dots$

q-P_{VI} from Elementary Birational Transformations (cont.)

Decomposition of q-P_{VI}:

$$\text{q-}P_{\mathrm{VI}} = \sigma_2 \circ w_2 \circ w_0 \circ w_1 \circ w_2 \circ \sigma_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3, \qquad q = (b_3 b_4 b_5 b_6)/(b_1 b_2 b_7 b_8)$$

$$\cdots \xmapsto{w_2} \left(\begin{matrix} b_4 & qb_1 & qb_2 & b_3 \\ \frac{b_4 b_6}{qb_2} & \frac{b_4 b_5}{qb_2} & b_8 & b_7 \end{matrix} ; f\,\frac{(\bar{g} - b_4)}{(\bar{g} - qb_2)} ; \bar{g} \right)$$

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$$\xmapsto{w_0} \left(\begin{matrix} b_4 & qb_1 & b_3 & qb_2 \\ \frac{b_4 b_6}{qb_2} & \frac{b_4 b_5}{qb_2} & b_8 & b_7 \end{matrix} ; f \frac{(\bar{g} - b_4)}{(\bar{g} - qb_2)} ; \bar{g} \right)$$

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Decomposition of q-P_{VI}:

$$\text{q-}P_{\mathrm{VI}} = \sigma_2 \circ w_2 \circ w_0 \circ w_1 \circ w_2 \circ \sigma_1 \circ w_3 \circ w_5 \circ w_4 \circ w_3, \qquad q = (b_3 b_4 b_5 b_6)/(b_1 b_2 b_7 b_8)$$

$$\cdots \xmapsto{w_2} \begin{pmatrix} b_4 & qb_1 & qb_2 & b_3 \\ \frac{b_4 b_6}{qb_2} & \frac{b_4 b_5}{qb_2} & b_8 & b_7 \end{pmatrix}; f\frac{(\bar{g}-b_4)}{(\bar{g}-qb_2)}; \bar{g}\Bigg)$$

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$$\xmapsto{w_1} \begin{pmatrix} qb_1 & b_4 & b_3 & qb_2 \\ \frac{b_4 b_6}{qb_2} & \frac{b_4 b_5}{qb_2} & b_8 & b_7 \end{pmatrix}; f\frac{(\bar{g}-b_4)}{(\bar{g}-qb_2)}; \bar{g}\Bigg)$$

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$$\begin{aligned}
\cdots \overset{w_2}{\longmapsto} &\left(\begin{matrix} b_4 & qb_1 & qb_2 & b_3 \\ \frac{b_4 b_6}{qb_2} & \frac{b_4 b_5}{qb_2} & b_8 & b_7 \end{matrix} ; f \frac{(\bar{g} - b_4)}{(\bar{g} - qb_2)} ; \bar{g} \right) \\
\overset{w_0}{\longmapsto} &\left(\begin{matrix} b_4 & qb_1 & b_3 & qb_2 \\ \frac{b_4 b_6}{qb_2} & \frac{b_4 b_5}{qb_2} & b_8 & b_7 \end{matrix} ; f \frac{(\bar{g} - b_4)}{(\bar{g} - qb_2)} ; \bar{g} \right) \\
\overset{w_1}{\longmapsto} &\left(\begin{matrix} qb_1 & b_4 & b_3 & qb_2 \\ \frac{b_4 b_6}{qb_2} & \frac{b_4 b_5}{qb_2} & b_8 & b_7 \end{matrix} ; f \frac{(\bar{g} - b_4)}{(\bar{g} - qb_2)} ; \bar{g} \right) \\
\overset{w_2}{\longmapsto} &\left(\begin{matrix} b_3 & b_4 & qb_1 & qb_2 \\ \frac{b_7 b_8}{qb_5} & \frac{b_7 b_8}{qb_6} & b_8 & b_7 \end{matrix} ; f \frac{(\bar{g} - b_4)}{(\bar{g} - qb_2)} \frac{(\bar{g} - b_3)}{(\bar{g} - qb_1)} ; \bar{g} \right)
\end{aligned}$$

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\xmapsto{w_1} & \begin{pmatrix} qb_1 & b_4 & b_3 & qb_2 \\ \frac{b_4 b_6}{qb_2} & \frac{b_4 b_5}{qb_2} & b_8 & b_7 \end{pmatrix}; f\,\frac{(\bar{g} - b_4)}{(\bar{g} - qb_2)}; \bar{g} \Bigg) \\
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\end{aligned}$$

as it should be!

Example of Applications: Discrete Orthogonal Polynomials

- **Observation 1:** The theory of Painlevé equations is *mathematically beautiful*. It is related to the theory of special functions and orthogonal polynomials, ordinary and partial differential equations, algebraic geometry, group and representation theory, the theory of cluster algebras, and this list goes on.

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- **Issue:** It can be very difficult to recognize a discrete Painlevé equation. Even when there are reasons to expect that some discrete recurrence will fall in the discrete Painlevé framework, it will most likely not be written in a standard "catalogue form". This is the issue of **coordinates** – the *application coordinates* that are natural to the applied problem can be very different from the good *Painlevé coordinates*. The necessary change of variables can be very difficult, if not impossible, to guess.

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- Here is one such example.

Discrete Orthogonal Polynomials with the Hypergeometric Weight

Consider the collection $\{p_n(x) = \gamma_n x^n + \cdots\}$ of polynomials that are orthonormal on the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ with respect to the *hypergeometric weight* w_k:

$$\sum_{k=0}^{\infty} p_n(k) p_m(k) w_k = \delta_{m,n}, \qquad w_k = \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} c^k, \quad \alpha, \beta, \gamma > 0,\ 0 < c < 1,$$

where $(\cdot)_k$ is the usual Pochhammer symbol and $\delta_{m,n}$ is the Kronecker delta.

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This collection of polynomials is known as the discrete orthogonal polynomials with hypergeometric weights since the moments of this weight function are given in terms of the Gauss hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; c)$ and its derivatives and it has been studied in [**FVA18**].

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These polynomials satisfy the usual three term recurrence relation

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

where $a_0 = 0$. The coefficients a_n and b_n are called the *recurrence coefficients*.

The corresponding *monic* orthogonal polynomials $P_n = p_n/\gamma_n$ satisfy a similar three term recurrence relation

$$x P_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x).$$

The Recurrence Coefficients

Following [**FVA18**], let us introduce two new variables x_n and y_n parameterizing the recurrence coefficients a_n^2 and b_n as follows:

$$a_n^2 \frac{1-c}{c} = y_n + \sum_{k=0}^{n-1} x_k + \frac{n(n+\alpha+\beta-\gamma-1)}{1-c}, \qquad b_n = x_n + \frac{n+(n+\alpha+\beta)c-\gamma}{1-c}.$$

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Note that $x_n = x(n,c;\alpha,\beta,\gamma)$, $y_n = y(n,c;\alpha,\beta,\gamma)$. As functions of the continuous parameter c, x_n, y_n satisfy the differential Toda system

$$\begin{cases} c\dfrac{d}{dc} a_n^2 = a_n^2(b_n - b_{n-1}), & n \geq 1, \\ c\dfrac{d}{dc} b_n = a_{n+1}^2 - a_n^2, & n \geq 0. \end{cases}$$

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This system can be reduced to the sigma-form of the sixth Painlevé equation. Namely, a simple linear change of variable transforms $S_n = \sum_{k=0}^{n-1} x_k$ into the solutions of the *σ-form of the sixth Painlevé equation*. Moreover, the differential equation for x_n can be directly reduced to the sixth Painlevé equation, [**HFC19**].

The Recurrence Coefficients

Following [**FVA18**], let us introduce two new variables x_n and y_n parameterizing the recurrence coefficients a_n^2 and b_n as follows:

$$a_n^2 \frac{1-c}{c} = y_n + \sum_{k=0}^{n-1} x_k + \frac{n(n+\alpha+\beta-\gamma-1)}{1-c}, \qquad b_n = x_n + \frac{n+(n+\alpha+\beta)c-\gamma}{1-c}.$$

Note that $x_n = x(n, c; \alpha, \beta, \gamma)$, $y_n = y(n, c; \alpha, \beta, \gamma)$. As functions of the continuous parameter c, x_n, y_n satisfy the differential Toda system

$$\begin{cases} c\dfrac{d}{dc} a_n^2 = a_n^2(b_n - b_{n-1}), & n \geq 1, \\ c\dfrac{d}{dc} b_n = a_{n+1}^2 - a_n^2, & n \geq 0. \end{cases}$$

This system can be reduced to the sigma-form of the sixth Painlevé equation. Namely, a simple linear change of variable transforms $S_n = \sum_{k=0}^{n-1} x_k$ into the solutions of the σ*-form of the sixth Painlevé equation*. Moreover, the differential equation for x_n can be directly reduced to the sixth Painlevé equation, [**HFC19**].

As functions of discrete parameter n, x_n, y_n satisfy a certain *recurrence relation*; this is the relation we are interested in. In view of the above, it is not surprising that this recurrence relation is related to the standard discrete Painlevé-V equation that describes certain Bäcklund transformations of P_{VI}. In fact, via a direct computation, is is possible to shown that the discrete system is a composition of the Bäcklund transformations of the sixth Painlevé equation, [**HFC19**].

The Discrete System

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$$\begin{aligned}(y_n - \alpha\beta + (\alpha + \beta + n)x_n - x_n^2)&(y_{n+1} - \alpha\beta + (\alpha + \beta + n + 1)x_n - x_n^2)\\ &= \frac{1}{c}(x_n - 1)(x_n - \alpha)(x_n - \beta)(x_n - \gamma),\end{aligned}$$

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The initial conditions for this recurrence are given by

$$x_0 = \frac{\alpha\beta c}{\gamma} \frac{{}_2F_1(\alpha + 1, \beta + 1; \gamma + 1; c)}{{}_2F_1(\alpha, \beta; \gamma; c)} + \frac{(\alpha + \beta)c - \gamma}{c - 1}, \quad y_0 = 0.$$

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We show that this system is equivalent to the standard d-P_{V} equation and provide the explicit change of variables transforming one system into the other.

Theorem (Main Result)

The above discrete system is equivalent to to the standard discrete Painlevé equation

$$\overline{f}f = \frac{tg(g-a_4)}{(g+a_2)(g+a_1+a_2)}, \qquad g + \underline{g} = a_0 + a_3 + a_4 + \frac{a_3}{f-1} + \frac{ta_0}{f-t},$$

with $\overline{a}_0 = a_0 - 1$, $\overline{a}_1 = a_1$, $\overline{a}_2 = a_2 + 1$, $\overline{a}_3 = a_3 - 1$, $\overline{a}_4 = a_4$, *and* $a_0 + a_1 + 2a_2 + a_3 + a_4 = 1$. *This equivalence is achieved via the following change of variables:*

$$\begin{aligned} x(f,g) &= \gamma - g - \frac{(n+\beta)f}{f-1}, \\ y(f,g) &= (g+\alpha+\beta+n-\gamma)(g+2\beta+2n-\gamma) - n\alpha - \frac{gt(g+\beta-\gamma)}{f} \\ &\quad + \frac{(n+\beta)((c-1)(2g+\alpha+3\beta+3n-2\gamma)+(\alpha+\beta+\gamma-n)+n)}{c(f-1)} + \frac{(c-1)(n+\beta)^2}{c(f-1)^2}. \end{aligned}$$

The inverse change of variables is given by

$$\begin{aligned} f(x,y) &= \frac{t(x-\beta)(x-\gamma)}{((x-\alpha)(x-\beta)-nx-y)}, \\ g(x,y) &= -\frac{(x-\gamma)(((x-\alpha)(x-\beta)-nx-y)-t(x-\beta)(x-\gamma+\beta+n))}{((x-\alpha)(x-\beta)-nx-y)-t(x-\beta)(x-\gamma)}. \end{aligned}$$

Parameters α, β, γ *and* c *of the weight are related to the standard Painlevé parameters (root variables) by*

$$\alpha = a_1 + 1, \quad \beta = a_0 + a_1 + a_2, \quad \gamma = 1 - a_2 - a_3, \quad n = a_2 + a_4 - 1, \qquad ct = 1.$$

The Sakai Classification Scheme for Discrete Painlevé Equations

The definitive classification scheme for discrete Painlevé equations is due to H. Sakai. It is quite intricate and is given in terms of certain algebro-geometric data, such as the configuration type of singular points of the equation. These points lie on some configuration of curves that, after the blowup procedure resolving the singularities, becomes a collection of -2 curves that are irreducible components of the unique anti-canonical divisor. The intersection configuration of these components are described by a certain affine Dynkin diagram; the type of this diagram is known as the *surface type* of the equation.

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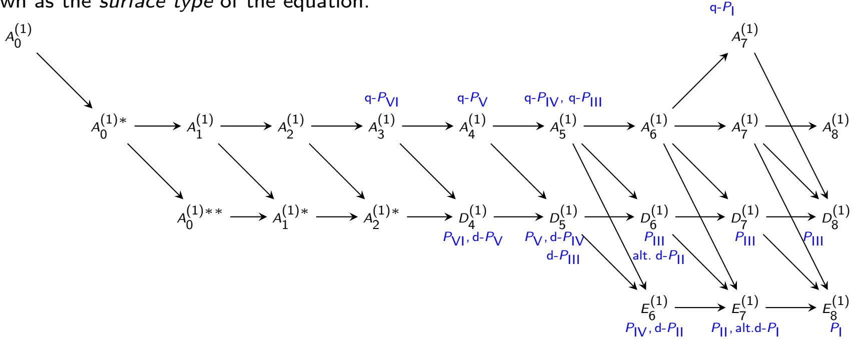

Surface-type classification scheme for Painlevé equations

The Sakai Classification Scheme for Discrete Painlevé Equations

A more traditional approach to Painlevé equations is through studying their symmetries, which gives us the *dual* symmetry-type classification scheme. It is also given in terms of affine Dynkin diagrams, although this time each diagram encodes the affine Weyl symmetry group structure of the equation (and its surface).

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$$\begin{tikzcd}
\left(E_8^{(1)}\right)^e \arrow[d] & & & & & & & \overset{q\text{-}P_{\mathrm{I}}}{\left(A_1^{(1)}\right)^q_{|\alpha|^2=8}} \arrow[rdd] & \\
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Symmetry-type classification schemes for Painlevé equations

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Symmetry-type classification schemes for Painlevé equations

Each discrete Painlevé equation then corresponds to a *translation element in this extended affine Weyl group*. There are infinitely-many non-equivalent translations, and hence there are infinitely-many non-equivalent discrete Painlevé equations of each type. Still, it is often possible to focus on certain simple equations (short translations).

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(b) If so, what is the surface type of this equation?
(c) Given the type of the equation, can we match it to any of the known examples? It is well-known that there are infinitely many non-equivalent discrete Painlevé equations of each type, but some simple forms of such equations are well-known, as in the canonical reference [**KNY17**], see also the original paper of Sakai [**Sak01**]. This can give access to special solutions and may provide links with other models.

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We now show how this procedure works in practice by considering an example related to the theory of discrete orthogonal polynomials. It concerns the computation of the recurrence coefficients of discrete orthogonal polynomials with the hypergeometric weight (joint work with G. Filipuk and A. Stokes).

The Identification Procedure

Here is an outline of a proposed general process of identifying a discere dynamical system as a discrete Painlevé equation and explicitly rewriting it in some standard form. This process consists of the following steps, where we assume that we indeed have some discrete Painlevé equation, otherwise the process will terminate at some step.

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(Step 1) **Identify the singularity structure of the problem.**

For that, if necessary, rewrite our recurrence equation as a system of two first-order recurrences, $(x_{n+1}, y_{n+1}) = \psi^{(n)}(x_n, y_n)$. The mapping $\psi^{(n)} : \mathbb{C}^2 \to \mathbb{C}^2$ should be a birational mapping that can depends on various parameters, including the iteration step n that we consider to be generic. Then compactify the configuration space from $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ to $\mathbb{P}^1 \times \mathbb{P}^1$. Find the base points of the mapping and resolve them using the blowup procesure. Continue doing that until all base points for both $\psi^{(n)}$ and $(\psi^n)^{-1}$ are resolved (for discrete Painlevé equations this process should terminate in finitely many steps). Thus, we get an isomorphism of resulting rational algebraic surfaces, $\psi^{(n)} : \mathcal{X}_n \xrightarrow{\simeq} \mathcal{X}_{n+1}$. In making this computation, it is important to keep in mind that positions of base points usually evolve with the mapping, so one needs to be careful distinguishing between the points in the domain and the points in the range. We also remark that sometimes the singularity structure can be seen before even studying the dynamics; e.g., singularities can occur as a result of a *parameterization of some moduli space* appearing in the problem, as in [**DK19**].

Recall: The Discrete System

We have the first-order system of non-linear non-autonomous difference equations

$$\begin{aligned}
&(y_n-\alpha\beta+(\alpha+\beta+n)x_n-x_n^2)(y_{n+1}-\alpha\beta+(\alpha+\beta+n+1)x_n-x_n^2)\\
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where $\alpha, \beta, \gamma > 0$ and $0 < c < 1$ are some parameters and

$$\mathfrak{Y}_n=\frac{y_n^2+y_n(n(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma)-\alpha\beta n(n+\alpha+\beta-\gamma-1)}{y_n(2n+\alpha+\beta-\gamma-1)+n((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma)}.$$

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where $\alpha, \beta, \gamma > 0$ and $0 < c < 1$ are some parameters and

$$
\mathfrak{Y}_n=\frac{y_n^2+y_n(n(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma)-\alpha\beta n(n+\alpha+\beta-\gamma-1)}{y_n(2n+\alpha+\beta-\gamma-1)+n((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma)}.
$$

This recurrence also naturally breaks into two mappings, the *forward* mapping $\psi_1^{(n)} : (x_n, y_n) \mapsto (x_n, y_{n+1})$ and the *backward* mapping $\psi_2^{(n)} : (x_n, y_n) \mapsto (x_{n-1}, y_n)$.

This is typical for many discrete Painlevé equations, in particular, for those that are obtained as deautonomizations of QRT mappings, [**CDT17**].

The Identification Procedure. Step 1: The Singularity Structure

The first step in the identification procedure is identifying the singularity structure of the problem.

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$$(\overline{x}, \overline{y}) = \left(x, \frac{(x-1)(x-\alpha)(x-\beta)(x-\gamma)}{c(y-(x-\alpha)(x-\beta)+nx)} + (x-\alpha)(x-\beta) - (n+1)x \right).$$

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We immediately see the following base points (in the affine coordinates (x, y)):

$$p_1(1, (1-\alpha)(1-\beta)-n), \quad p_2(\alpha, -n\alpha), \quad p_3(\beta, -n\beta), \quad p_4(\gamma, (\gamma-\alpha)(\gamma-\beta)-n\gamma).$$

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Rewriting the mapping for $\overline{y}$ in the (X, Y)-chart, $X = 1/x$, $Y = 1/y$ we get

$$\overline{y} = \frac{\begin{pmatrix} Y(1-X)(1-\alpha X)(1-\beta X)(1-\gamma X) \\ \qquad + c\Big(X^2 - Y(1-\alpha X)(1-\beta X) + nX\Big)\Big((1-\alpha X)(1-\beta X) - (n+1)X\Big) \end{pmatrix}}{cX^2\Big(X^2 - Y(1-\alpha X)(1-\beta X) + nX\Big)},$$

and so we get a new base point $p_5(x = \infty, y = \infty)$. These points are the only base points on $\mathbb{P}^1 \times \mathbb{P}^1$ for the forward dynamic. Thus, if this mapping is indeed in the discrete Painlevé family, there should be (at least) three more points on exceptional divisors.

The Identification Procedure. Step 1: The Singularity Structure

Indeed, we get the following cascade of "infinitely close" base points starting from the point $p_5(x=\infty, y=\infty)$:

$$p_5(X=0,Y=0) \leftarrow p_6\left(u_5 = X = 0, v_5 = \frac{Y}{X} = 0\right) \leftarrow p_7\left(u_6 = u_5 = 0, v_6 = \frac{v_5}{u_5} = \frac{c}{c-1}\right)$$

$$\leftarrow p_8\left(u_7 = u_6 = 0, v_7 = \frac{(c-1)v_6 - c}{(c-1)u_6} = \frac{c\Big(c(\alpha+\beta+n)+n-\gamma\Big)}{(c-1)^2}\right).$$

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Consider now the backward mapping. We put $x := x_n$, $\underline{x} = x_{n-1}$, $y := y_n$, $\underline{y} = y_{n-1}$. The backward mapping $\psi_2 : (x,y) \mapsto (\underline{x}, \underline{y})$ then becomes

$$(\underline{x}, \underline{y}) = \left(\frac{(y_n+n\alpha)(y_n+n\beta)(y_n+n\gamma-(\gamma-\alpha)(\gamma-\beta))(y_n-n-(1-\alpha)(1-\beta))}{(x_n+\mathfrak{Y}_n)\left(y_n(2n+\alpha+\beta-\gamma-1)+n\left((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma\right)\right)^2} - \mathfrak{Y}_n, y\right),$$

where $\mathfrak{Y}_n$ is given by

$$\mathfrak{Y}_n = \frac{y_n^2 + y_n(n(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma) - \alpha\beta n(n+\alpha+\beta-\gamma-1)}{y_n(2n+\alpha+\beta-\gamma-1)+n((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1)-\alpha\beta+\gamma)}.$$

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$$(\underline{x}, \underline{y}) = \left(\frac{(y_n + n\alpha)(y_n + n\beta)(y_n + n\gamma - (\gamma-\alpha)(\gamma-\beta))(y_n - n - (1-\alpha)(1-\beta))}{(x_n + \mathfrak{Y}_n)\left(y_n(2n+\alpha+\beta-\gamma-1) + n\left((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1) - \alpha\beta + \gamma\right)\right)^2} - \mathfrak{Y}_n, y\right),$$

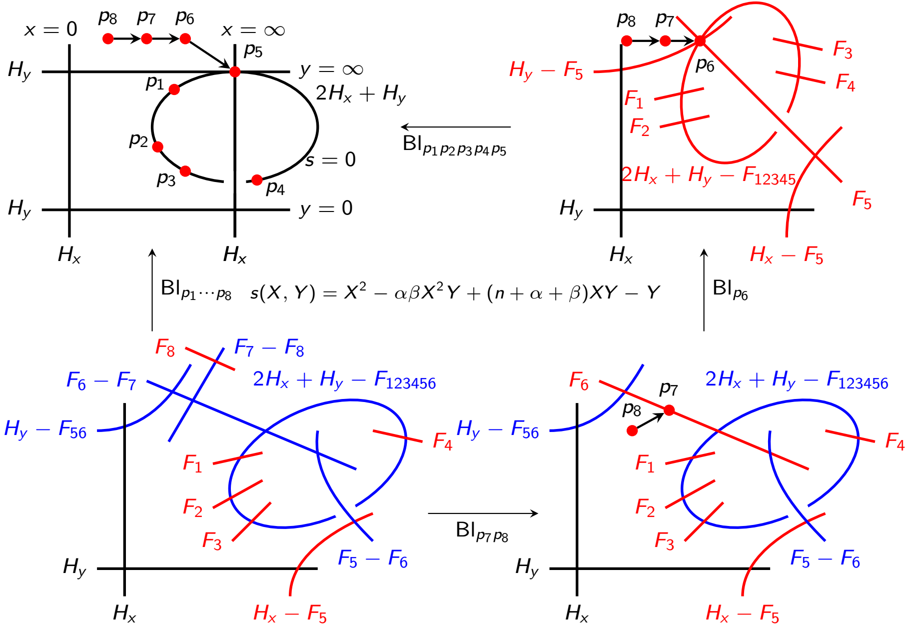
where $\mathfrak{Y}_n$ is given by

$$\mathfrak{Y}_n = \frac{y_n^2 + y_n(n(n+\alpha+\beta-\gamma-1) - \alpha\beta + \gamma) - \alpha\beta n(n+\alpha+\beta-\gamma-1)}{y_n(2n+\alpha+\beta-\gamma-1) + n((n+\alpha+\beta)(n+\alpha+\beta-\gamma-1) - \alpha\beta + \gamma)}.$$

The same standard computation shows that the only base points of the backwards dynamic are the same points $p_1, \ldots, p_4$

$$p_1(1, (1-\alpha)(1-\beta) - n), \quad p_2(\alpha, -n\alpha), \quad p_3(\beta, -n\beta), \quad p_4(\gamma, (\gamma-\alpha)(\gamma-\beta) - n\gamma),$$

but the singularity cascade at p_5 is not present. We then get the following picture.



$$s(X, Y) = X^2 - \alpha\beta X^2 Y + (n + \alpha + \beta)XY - Y$$

The Identification Procedure

Here is an outline of a proposed general process of identifying a discere dynamical system as a discrete Painlevé equation and explicitly rewriting it in some standard form. This process consists of the following steps, where we assume that we indeed have some discrete Painlevé equation, otherwise the process will terminate at some step.

(Step 1) **Identify the singularity structure of the problem.**

For that, if necessary, rewrite our recurrence equation as a system of two first-order recurrences, $(x_{n+1}, y_{n+1}) = \psi^{(n)}(x_n, y_n)$. The mapping $\psi^{(n)} : \mathbb{C}^2 \to \mathbb{C}^2$ should be a birational mapping that can depends on various parameters, including the iteration step n that we consider to be generic. Then compactify the configuration space from $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ to $\mathbb{P}^1 \times \mathbb{P}^1$. Find the base points of the mapping and resolve them using the blowup procesure. Continue doing that until all base points for both $\psi^{(n)}$ and $(\psi^n)^{-1}$ are resolved (for discrete Painlevé equations this process should terminate in finitely many steps). Thus, we get an isomorphism of resulting rational algebraic surfaces, $\psi^{(n)} : \mathcal{X}_n \xrightarrow{\simeq} \mathcal{X}_{n+1}$. In making this computation, it is important to keep in mind that positions of base points usually evolve with the mapping, so one needs to be careful distinguishing between the points in the domain and the points in the range. We also remark that sometimes the singularity structure can be seen before even studying the dynamics; e.g., singularities can occur as a result of a *parameterization of some moduli space* appearing in the problem, as in [**DK19**].

(Step 2) **Linearize the mapping on** $\operatorname{Pic}(\mathcal{X})$**.**

This can be done explicitly in relatively simple cases. Sometimes, however, the evolution mapping can be too complicated even for a Computer Algebra System. In this case, it may be possible to deduce the action of the mapping on $\operatorname{Pic}(\mathcal{X})$ from the knowledge of parameter evolution via the *Period Map*, see [**DK19**].

The Identification Procedure. Step 2: The Action on $\mathrm{Pic}(\mathcal{X})$

In parallel with resolving the base points of the mapping, we can compute the induced linear action on the Picard Lattice.

Lemma (The Forward Action)

The action of the forward dynamic $(\psi_1)_* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\overline{\mathcal{X}})$ *is given by*

$$\begin{aligned}
&\mathcal{H}_x \mapsto \overline{\mathcal{H}}_x, && \mathcal{F}_1 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_1,\ \mathcal{F}_3 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_3,\ \mathcal{F}_5 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_8,\ \mathcal{F}_7 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_6,\\
&\mathcal{H}_y \mapsto 4\overline{\mathcal{H}}_x + \overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{12345678}, && \mathcal{F}_2 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_2,\ \mathcal{F}_4 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_4,\ \mathcal{F}_6 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_7,\ \mathcal{F}_8 \mapsto \overline{\mathcal{H}}_x - \overline{\mathcal{F}}_5,
\end{aligned}$$

and the evolution of base points $\overline{p}_i = \psi_1(p_i)$ *is given by*

$$\overline{p}_1(1, (1-\alpha)(1-\beta) - (n+1)),\ \overline{p}_2(\alpha, -(n+1)\alpha),\ \overline{p}_3(\beta, -(n+1)\beta),\ \overline{p}_4(\gamma, (\gamma-\alpha)(\gamma-\beta) - (n+1)\gamma),$$

for finite points, and for the degeneration cascade we get

$$\begin{aligned}
\overline{p}_5(\overline{X}=0, \overline{Y}=0) &\leftarrow p_6\left(\overline{u}_5 = \overline{X} = 0, \overline{v}_5 = \frac{\overline{Y}}{\overline{X}} = 0\right) \leftarrow \overline{p}_7\left(\overline{u}_6 = \overline{u}_5 = 0, \overline{v}_6 = \frac{\overline{v}_5}{\overline{u}_5} = \frac{c}{c-1}\right)\\
&\leftarrow \overline{p}_8\left(\overline{u}_7 = \overline{u}_6 = 0, \overline{v}_7 = \frac{(c-1)\overline{v}_6 - c}{(c-1)\overline{u}_6} = \frac{c\Big(c(\alpha+\beta+n+1) + n - \gamma - 1\Big)}{(c-1)^2}\right).
\end{aligned}$$

From the evolution of base points we see that $\psi_1(n) = n + 1$.

The Identification Procedure. Step 2: The Action on $\operatorname{Pic}(\mathcal{X})$

Lemma (The Backward Action)

The action of the backwards dynamic $(\psi_2)_* : \operatorname{Pic}(\mathcal{X}) \to \operatorname{Pic}(\underline{\mathcal{X}})$ *is given by*

$$\begin{aligned}
&\mathcal{H}_x \mapsto \underline{\mathcal{H}}_x + 2\underline{\mathcal{H}}_y - \underline{\mathcal{F}}_{1234}, \quad && \mathcal{F}_1 \mapsto \underline{\mathcal{H}}_y - \underline{\mathcal{F}}_1, \quad && \mathcal{F}_3 \mapsto \underline{\mathcal{H}}_y - \underline{\mathcal{F}}_3, \quad && \mathcal{F}_5 \mapsto \underline{\mathcal{F}}_5, \quad && \mathcal{F}_7 \mapsto \underline{\mathcal{F}}_7,\\
&\mathcal{H}_y \mapsto \underline{\mathcal{H}}_y, && \mathcal{F}_2 \mapsto \underline{\mathcal{H}}_y - \underline{\mathcal{F}}_2, && \mathcal{F}_4 \mapsto \underline{\mathcal{H}}_y - \underline{\mathcal{F}}_4, && \mathcal{F}_6 \mapsto \underline{\mathcal{F}}_6, && \mathcal{F}_8 \mapsto \underline{\mathcal{F}}_8.
\end{aligned}$$

From this we can also easily compute the evolution of base points. We get

$$\underline{p}_1(1, (1-\alpha)(1-\beta) - n),\ \underline{p}_2(\alpha, -n\alpha),\ \underline{p}_3(\beta, -n\beta),\ \underline{p}_4(\gamma, (\gamma-\alpha)(\gamma-\beta) - n\gamma),$$

as well as the degeneration cascade

$$\begin{aligned}
\underline{p}_5(\underline{X} = 0, \underline{=}0) \leftarrow p_6\left(\underline{u}_5 = \underline{X} = 0, \underline{v}_5 = \frac{\underline{Y}}{\underline{X}} = 0\right) \leftarrow \underline{p}_7\left(\underline{u}_6 = \underline{u}_5 = 0, \underline{v}_6 = \frac{\underline{v}_5}{\underline{u}_5} = \frac{c}{c-1}\right)\\
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\end{aligned}$$

From the evolution of base points we see that $\psi_2(n) = n$.

The Identification Procedure. Step 2: The Action on $\operatorname{Pic}(\mathcal{X})$

Lemma (The Composed Action)

The action of the composed mapping $\psi_*^{(n)} = \psi_* = (\psi_2)_*^{-1} \circ (\psi_1)_* : \operatorname{Pic}(\mathcal{X}_n) \to \operatorname{Pic}(\mathcal{X}_{n+1})$ *is given by*

$$\begin{aligned}
\mathcal{H}_x &\mapsto \overline{\mathcal{H}}_x + 2\overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{1234}, & \mathcal{H}_y &\mapsto 4\overline{\mathcal{H}}_x + 5\overline{\mathcal{H}}_y - 3\overline{\mathcal{F}}_{1234} - \overline{\mathcal{F}}_{5678},\\
\mathcal{F}_1 &\mapsto \overline{\mathcal{H}}_x + \overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{234}, & \mathcal{F}_5 &\mapsto \overline{\mathcal{H}}_x + 2\overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{12348}\\
\mathcal{F}_2 &\mapsto \overline{\mathcal{H}}_x + \overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{134}, & \mathcal{F}_6 &\mapsto \overline{\mathcal{H}}_x + 2\overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{12347},\\
\mathcal{F}_3 &\mapsto \overline{\mathcal{H}}_x + \overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{124}, & \mathcal{F}_7 &\mapsto \overline{\mathcal{H}}_x + 2\overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{12346},\\
\mathcal{F}_4 &\mapsto \overline{\mathcal{H}}_x + \overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{123}, & \mathcal{F}_8 &\mapsto \overline{\mathcal{H}}_x + 2\overline{\mathcal{H}}_y - \overline{\mathcal{F}}_{12345}.
\end{aligned}$$

The evolution of the base points (here $\overline{p}_i = \psi^{(n)}(p_i)$*) is*

$$\overline{p}_1(1, (1-\alpha)(1-\beta) - (n+1)),\ \overline{p}_2(\alpha, -(n+1)\alpha),\ \overline{p}_3(\beta, -(n+1)\beta),\ \overline{p}_4(\gamma, (\gamma-\alpha)(\gamma-\beta) - (n+1)\gamma),$$

for finite points, and

$$\begin{aligned}
\overline{p}_5(\overline{X} = 0, \overline{Y} = 0) &\leftarrow p_6\left(\overline{u}_5 = \overline{X} = 0, \overline{v}_5 = \frac{\overline{Y}}{\overline{X}} = 0\right) \leftarrow \overline{p}_7\left(\overline{u}_6 = \overline{u}_5 = 0, \overline{v}_6 = \frac{\overline{v}_5}{\overline{u}_5} = \frac{c}{c-1}\right)\\
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\end{aligned}$$

for the degeneration cascade.

From the evolution of base points we see that $\psi^{(n)}(n) = n+1$.

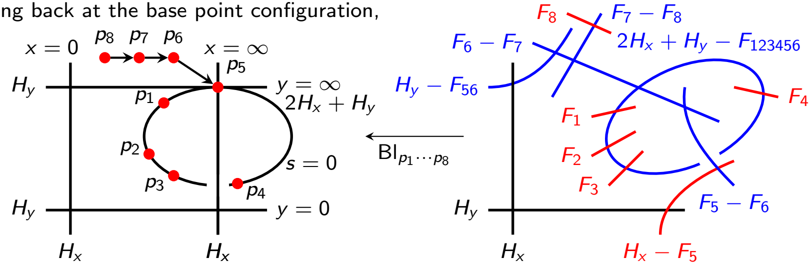
The Identification Procedure (continued)

(Step 3) **Determine the surface type, according to Sakai's classification scheme.**

For a discrete Painlevé equation, although the positions of base points may evolve, the *configuration* will stay fixed, and so the surfaces $\{\mathcal{X}_n\}$ will all have the same type. There should be *eight* such base points; those points will lie on some (generically unique) biquadratic curve on $\mathbb{P}^1 \times \mathbb{P}^1$ (i.e., a curve whose defining polynomial, when written in a coordinate chart, has a bi-degree $(2,2)$) and the *point configuration* is defined to be the configuration of the irreducible components of this curve. Each such component should have self-intersection index -2 and is associated with a node in an *affine Dynkin diagram*, nodes are connected when the corresponding components intersect. The type of this Dynkin diagram is called the *surface type* of the equation. This description assumes that the surfaces $\mathcal{X}_n$ are *minimal*, but can happen that after the initial blowup procedure is complete, some -1-curves would have to be blown down. This will also result in some irreducible components having higher negative self-intersection index. The blowing down procedure is quite delicate, so here we assume that the surfaces $\mathcal{X}_n$ are indeed minimal, but see [**DST13**] and [**DK19**] for examples requiring a blowing down.

The Identification Procedure. Step 3: Determining the Surface Type

Looking back at the base point configuration,



we see that base points lie on a *reducible* $(2, 2)$ curve $\Gamma = V\left(Ys(X, Y)\right)$, where $s(X, Y) = X^2 - \alpha\beta X^2 Y + (n + \alpha + \beta)XY - Y$.

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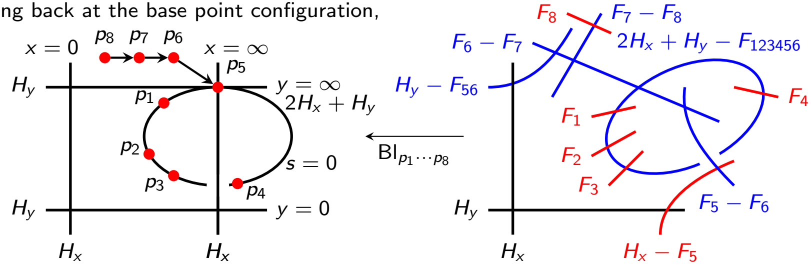

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Γ is the *pole divisor* of a symplectic form $\omega = k\,\frac{dX\wedge dY}{s(X,Y)\,Y} = k\,\frac{dX\wedge ds}{s(s-X^2)}$, and the anti-canonical divisor class $-\mathcal{K}_{\mathcal{X}} = -[\omega_{\mathcal{X}}]$ decomposes into irreducible components as follows:

$$-[\omega]_{\mathcal{X}} = (2H_x + H_y - F_1 - F_2 - F_3 - F_4 - F_5 - F_6) + (H_y - F_5 - F_6) + (F_5 - F_6) + 2(F_6 - F_7) + (F_7 - F_8).$$

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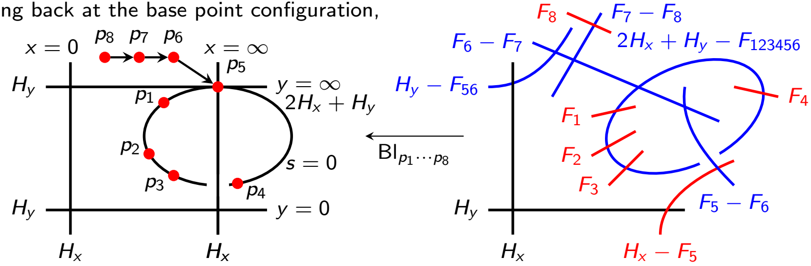

we see that base points lie on a *reducible* $(2,2)$ curve $\Gamma=V\left(Ys(X,Y)\right)$, where $s(X,Y)=X^2-\alpha\beta X^2Y+(n+\alpha+\beta)XY-Y$.

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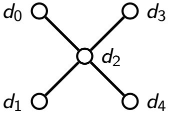
$$-[\omega]_{\mathcal{X}}=(2H_x+H_y-F_1-F_2-F_3-F_4-F_5-F_6)+(H_y-F_5-F_6)+(F_5-F_6)+2(F_6-F_7)+(F_7-F_8).$$

The intersection structure of irreducible components is given by the $D_4^{(1)}$ affine Dynkin diagram



$d_0=F_5-F_6,$
$d_1=2H_x+H_y-F_{123456},$
$d_2=F_6-F_7,$
$d_3=F_7-F_8,$
$d_4=H_y-F_{56}.$

Detour: The Standard d-P_{V} Mapping. The Point Configuration

Let us briefly review discrete Painlevé equations of type d-$P\left(D_4^{(1)}/D_4^{(1)}\right)$, and in particular, the usual d-P$_V$ equation, following the standard reference [**KNY17**] for choices of root bases.

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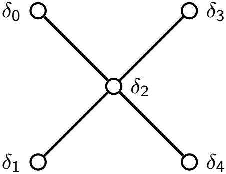
We start with the root basis of the surface sub-lattice that is given by the classes $\delta_i = [d_i]$ of the irreducible components of the anti-canonical divisor

$$\delta = -\mathcal{K}_{\mathcal{X}} = 2\mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8 = \delta_0 + \delta_1 + 2\delta_2 + \delta_3 + \delta_4.$$

The intersection configuration of those roots is given by the Dynkin diagram of type $D_4^{(1)}$.



$$\delta_0 = \mathcal{E}_3 - \mathcal{E}_4, \qquad \delta_3 = \mathcal{E}_7 - \mathcal{E}_8,$$
$$\delta_1 = \mathcal{H}_f - \mathcal{E}_1 - \mathcal{E}_2, \qquad \delta_4 = \mathcal{H}_f - \mathcal{E}_5 - \mathcal{E}_6.$$
$$\delta_2 = \mathcal{H}_g - \mathcal{E}_3 - \mathcal{E}_7,$$

$$\delta = \delta_0 + \delta_1 + 2\delta_2 + \delta_3 + \delta_4.$$

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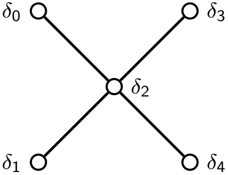


$$\delta_0 = \mathcal{E}_3 - \mathcal{E}_4, \qquad \delta_3 = \mathcal{E}_7 - \mathcal{E}_8,$$
$$\delta_1 = \mathcal{H}_f - \mathcal{E}_1 - \mathcal{E}_2, \qquad \delta_4 = \mathcal{H}_f - \mathcal{E}_5 - \mathcal{E}_6.$$
$$\delta_2 = \mathcal{H}_g - \mathcal{E}_3 - \mathcal{E}_7,$$

$$\delta = \delta_0 + \delta_1 + 2\delta_2 + \delta_3 + \delta_4.$$

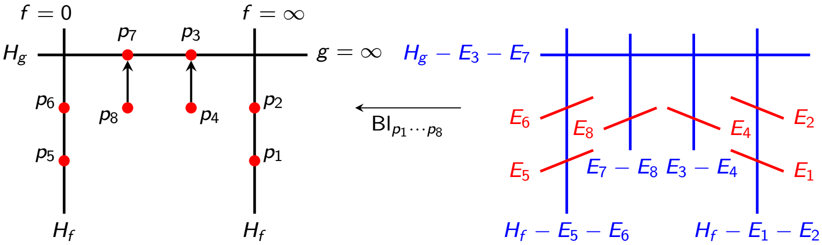
Using the action of the $\mathbf{PGL}_2(\mathbb{C}) \times \mathbf{PGL}_2(\mathbb{C})$ gauge group (i.e., the action of a Möbius group on each of the factors of $\mathbb{P}^1 \times \mathbb{P}^1$), we can, without loss of generality, put d_i, with $\delta_i = [d_i]$ to be

$$d_1 = V(F) = \{f = \infty\}, \qquad d_2 = V(G) = \{g = \infty\}, \qquad d_4 = V(f) = \{f = 0\},$$

which then reduces the gauge group action to that of a three-parameter subgroup, $(f, g) \mapsto (\lambda f, \mu g + \nu)$.

The Standard d-P_{V} Mapping: The Point Configuration

The corresponding point configuration and the Sakai surface are as follows:



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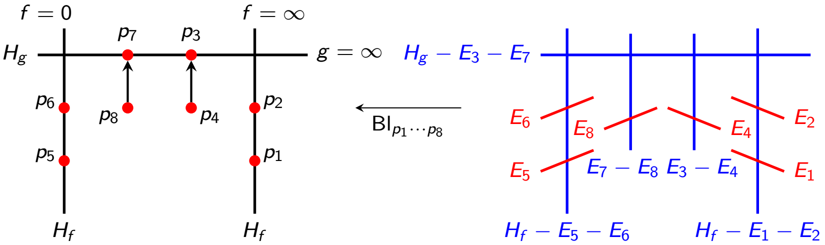

This point configuration can be parameterized by eight parameters $b_1, \ldots, b_8$ as follows:

$$\begin{aligned} &p_1(\infty, b_1), \quad p_2(\infty, b_2), \quad p_3(b_3, \infty) \leftarrow p_4(b_3, \infty; g(f - b_3) = b_4), \\ &p_5(0, b_5), \quad\ \ p_6(0, b_6), \quad\ \ p_7(b_7, \infty) \leftarrow p_8(b_7, \infty; g(f - b_7) = b_8). \end{aligned}$$

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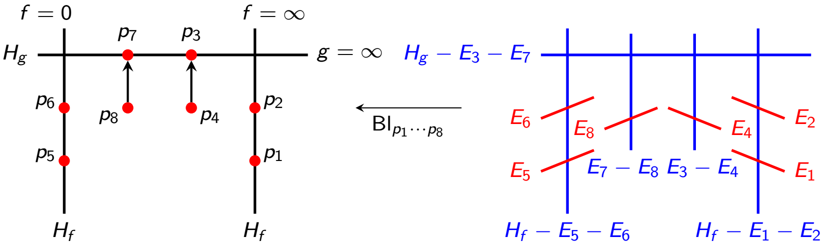

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The three-parameter gauge group above acts on these configurations via

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \begin{matrix} f \\ g \end{matrix} \sim \begin{pmatrix} \mu b_1+\nu & \mu b_2+\nu & \lambda b_3 & \lambda\mu b_4 \\ \mu b_5+\nu & \mu b_6+\nu & \lambda b_7 & \lambda\mu b_8 \end{pmatrix}; \begin{matrix} \lambda f \\ \mu g+\nu \end{matrix}, \quad \lambda,\mu\neq 0,$$

and so the true number of parameters is five. The correct gauge-invariant parameterization is given by the *root variables*.

The Standard d-P_{V} Mapping: The Period Map

To define the root variables we begin by choosing a root basis in the *symmetry sub-lattice* $Q = \Pi(R^{\perp}) \triangleleft \operatorname{Pic}(\mathcal{X})$ and defining the symplectic form ω whose polar divisor $-K_{\mathcal{X}}$ is the configuration of -2-curves shown on on the previous figure.

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A symplectic form $\omega \in -\mathcal{K}_{\mathcal{X}}$ such that $[\omega] = \delta_0 + \delta_1 + 2\delta_2 + \delta_3 + \delta_4$ can be given in local coordinate charts as

$$\omega = k\frac{df \wedge dg}{f} = -k\frac{dF \wedge dg}{F} = -k\frac{df \wedge dG}{fG^2} = k\frac{dF \wedge dG}{FG^2} = -k\frac{dU_3 \wedge dV_3}{(b_3 + U_3V_3)V_3} = -k\frac{dU_7 \wedge dV_7}{(b_7 + U_7V_7)V_7},$$

where $F = 1/f$, $G = 1/g$ are the coordinates centered at infinity, the blowup coordinates (U_i, V_i) at the points p_i, $i = 3, 7$, are given by $f = b_i + U_iV_i$ and $g = V_i$, and k is some non-zero proportionality constant that we normalize later.

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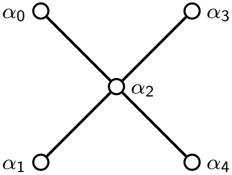
We take the following symmetry root basis, following [**KNY17**]:



$$\alpha_0 = \mathcal{H}_f - \mathcal{E}_3 - \mathcal{E}_4, \qquad \alpha_3 = \mathcal{H}_f - \mathcal{E}_7 - \mathcal{E}_8,$$
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$$\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4.$$

The Standard d-P_{V} Mapping: The Period Map

Lemma

(i) The residues of the symplectic form ω along the irreducible components of the polar divisor is given by

$$\operatorname{res}_{d_0}\omega = k\frac{dU_3}{b_3},\ \operatorname{res}_{d_1}\omega = -kdg,\ \operatorname{res}_{d_2}\omega = 0,\ \operatorname{res}_{d_3}\omega = k\frac{dU_7}{b_7},\ \operatorname{res}_{d_4}\omega = kdg.$$

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(ii) The root variables a_i *are given by*

$$a_0 = -k\frac{b_4}{b_3},\ a_1 = k(b_2 - b_1),\ a_2 = k(b_1 - b_5),\ a_3 = -k\frac{b_8}{b_7},\ a_4 = k(b_5 - b_6).$$

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It is convenient to take $k = -1$. We can then use the gauge action to normalize $b_5 = 0$, $b_7 = 1$, and $\chi(\delta) = a_0 + a_1 + 2a_2 + a_3 + a_4 = 1$. In view of the relation of this example to P_{VI}, it is also convenient to denote b_3 by t.

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$$b_1 = -a_2,\ b_2 = -a_1 - a_2,\ b_3 = t,\ b_4 = ta_0,\ b_5 = 0,\ b_6 = a_4,\ b_7 = 1,\ b_8 = a_3.$$

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The proof of this Lemma is a standard computation.

The Standard d-P_{V} Mapping: The Symmetry Group

The symmetry group of $D_4^{(1)}$ surface family is the extended affine Weyl symmetry group $\widetilde{W}\left(D_4^{(1)}\right) = \mathrm{Aut}\left(D_4^{(1)}\right) \ltimes W\left(D_4^{(1)}\right)$, which is a *semi-direct product* of the usual affine Weyl group $W\left(D_4^{(1)}\right)$ and the group of Dynkin diagram automorphisms $\mathrm{Aut}\left(D_4^{(1)}\right)$.

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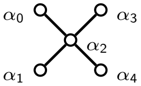

$$W\left(D_4^{(1)}\right) = W\left(\quad\right) = \left\langle w_0, \ldots, w_4 \;\middle|\; \begin{array}{ll} w_i^2 = e, \quad w_i \circ w_j = w_j \circ w_i & \text{when } \underset{\alpha_i}{\circ} \quad \underset{\alpha_j}{\circ} \\ w_i \circ w_j \circ w_i = w_j \circ w_i \circ w_j & \text{when } \underset{\alpha_i}{\circ}\!\!-\!\!-\!\!\underset{\alpha_j}{\circ} \end{array} \right\rangle .$$

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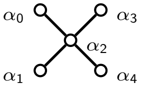

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The natural action of this group on $\mathrm{Pic}(\mathcal{X})$ is given by reflections in the roots α_i,

$$w_i(\mathcal{C}) = w_{\alpha_i}(\mathcal{C}) = \mathcal{C} - 2\frac{\mathcal{C} \bullet \alpha_i}{\alpha_i \bullet \alpha_i}\alpha_i = \mathcal{C} + (\mathcal{C} \bullet \alpha_i)\,\alpha_i, \qquad \mathcal{C} \in \mathrm{Pic}(\mathcal{X}),$$

which can be extended to an action on point configurations by elementary birational maps (which lifts to isomorphisms $w_i : \mathcal{X}_{\mathbf{b}} \to \mathcal{X}_{\overline{\mathbf{b}}}$ on the family of Sakai's surfaces), this is known as a birational representation of $W\left(D_4^{(1)}\right)$.

Theorem

Reflections w_i on $\operatorname{Pic}(\mathcal{X})$ are induced by the elementary birational mappings given below and also denoted by w_i, on the family $\mathcal{X}_{\mathbf{b}}$. To ensure the group structure, we require that each map preserves our normalization

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & t & b_4 \\ 0 & b_6 & 1 & b_8 \end{pmatrix} = \begin{pmatrix} -a_2 & -a_1 - a_2 & t & ta_0 \\ 0 & a_4 & 1 & a_3 \end{pmatrix}.$$

For the initial configuration

$$\begin{pmatrix} b_1 & b_2 & t & b_4 \\ 0 & b_6 & 1 & b_8 \end{pmatrix} ; \begin{matrix} f \\ g \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_3 & a_4 & t \end{matrix} ; \begin{pmatrix} f \\ g \end{pmatrix},$$

the action of w_i is given by the following expressions.

$$\begin{aligned}
w_0 &: \left(\begin{matrix} b_1 - \frac{b_4}{t} & b_2 - \frac{b_4}{t} & t & -b_4 \\ 0 & b_6 & 1 & b_8 \end{matrix} ; \begin{matrix} f \\ g - \frac{b_4 f}{t(f-t)} \end{matrix}\right) = \left(\begin{matrix} -a_0 & a_1 & a_0 + a_2 \\ a_3 & a_4 & t \end{matrix} ; \begin{matrix} f \\ g - \frac{a_0 f}{f-t} \end{matrix}\right), \\
w_1 &: \left(\begin{matrix} b_2 & b_1 & t & b_4 \\ 0 & b_6 & 1 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) = \left(\begin{matrix} a_0 & -a_1 & a_1 + a_2 \\ a_3 & a_4 & t \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right), \\
w_2 &: \left(\begin{matrix} -b_1 & b_2 - b_1 & t & b_4 - tb_1 \\ 0 & b_6 - b_1 & 1 & b_8 - b_1 \end{matrix} ; \begin{matrix} f - \frac{b_1 f}{g} \\ g - b_1 \end{matrix}\right) = \left(\begin{matrix} a_0 + a_2 & a_1 + a_2 & -a_2 \\ a_2 + a_3 & a_2 + a_4 & t \end{matrix} ; \begin{matrix} f + \frac{a_2 f}{g} \\ g + a_2 \end{matrix}\right), \\
w_3 &: \left(\begin{matrix} b_1 - b_8 & b_2 - b_8 & t & b_4 \\ 0 & b_6 & 1 & -b_8 \end{matrix} ; \begin{matrix} f \\ g - \frac{b_8 f}{f-1} \end{matrix}\right) = \left(\begin{matrix} a_0 & a_1 & a_2 + a_3 \\ -a_3 & a_4 & t \end{matrix} ; \begin{matrix} f \\ g - \frac{a_3 f}{f-1} \end{matrix}\right), \\
w_4 &: \left(\begin{matrix} b_1 - b_6 & b_2 - b_6 & t & b_4 \\ 0 & -b_6 & 1 & b_8 \end{matrix} ; \begin{matrix} f \\ g - b_6 \end{matrix}\right) = \left(\begin{matrix} a_0 & a_1 & a_2 + a_4 \\ a_3 & -a_4 & t \end{matrix} ; \begin{matrix} f \\ g - a_4 \end{matrix}\right).
\end{aligned}$$

Let us now describe the group of Dynkin diagram automorphisms. It is clear that $\mathrm{Aut}\left(D_4^{(1)}\right) \simeq \mathcal{S}_4$, so we only describe three transpositions that generate the whole group.

Theorem

Consider the following generators $\sigma_1, \ldots, \sigma_3$ of $\mathrm{Aut}\left(D_4^{(1)}\right)$ that act on the symmetry and the surface root bases as follows (here we use the standard cycle notations for permutations):

$$\sigma_1 = (\alpha_3\alpha_4) = (\delta_3\delta_4), \qquad \sigma_2 = (\alpha_0\alpha_3) = (\delta_0\delta_3), \qquad \sigma_3 = (\alpha_1\alpha_4) = (\delta_1\delta_4).$$

Then σ_i act on the Picard lattice as

$$\sigma_1 = (\mathcal{E}_6\mathcal{E}_8)w_\rho, \qquad \sigma_2 = (\mathcal{E}_3\mathcal{E}_7)(\mathcal{E}_4\mathcal{E}_8), \qquad \sigma_3 = (\mathcal{E}_1\mathcal{E}_5)(\mathcal{E}_2\mathcal{E}_6),$$

where w_ρ is a reflection in the root $\rho = \mathcal{H}_f - \mathcal{E}_5 - \mathcal{E}_7$ (note also that a transposition $(\mathcal{E}_i\mathcal{E}_j) = w_{\mathcal{E}_i - \mathcal{E}_j}$). The induced elementary birational mappings are then given by the following expressions.

$$\sigma_1 : \begin{pmatrix} b_1 & b_2 & 1-t & \frac{\frac{(1-t)b_4}{t}}{b_6} & ; & \frac{1-f}{\frac{(f-1)g}{f}} \\ 0 & b_8 & 1 & & & \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & a_2 & ; & 1-f \\ a_4 & a_3 & 1-t & ; & \frac{(f-1)g}{f} \end{pmatrix},$$

$$\sigma_2 : \begin{pmatrix} b_1 & b_2 & \frac{1}{t} & \frac{b_8}{t} & ; & \frac{f}{t} \\ 0 & b_6 & 1 & \frac{b_4}{t} & ; & g \end{pmatrix} = \begin{pmatrix} a_3 & a_1 & a_2 & ; & \frac{f}{t} \\ a_0 & a_4 & \frac{1}{t} & ; & g \end{pmatrix},$$

$$\sigma_3 : \begin{pmatrix} b_1 & b_1 - b_6 & \frac{1}{t} & \frac{b_4}{t^2} & ; & \frac{1}{f} \\ 0 & b_1 - b_2 & 1 & b_8 & ; & b_1 - g \end{pmatrix} = \begin{pmatrix} a_0 & a_4 & a_2 & ; & \frac{1}{f} \\ a_3 & a_1 & \frac{1}{t} & ; & -g - a_2 \end{pmatrix}.$$

The Standard d-P_{V} Mapping

Recall that there are infinitely many different discrete Painlevé equations of the same type, since they correspond to the non-conjugate translations in the affine symmetry sub-lattice Q. Some of these equations are special, since they either appear in applications, or have a particularly nice form, or have degenerations to other known equations. In the d-$P\left(D_4^{(1)}/D_4^{(1)}\right)$ family one such equation is known as a discrete Painlevé-V equation, since it has a continuous limit to the differential Painlevé-V equation.

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In [**KNY17**] this equation is given in the form

$$\overline{f}f = \frac{tg(g-a_4)}{(g+a_2)(g+a_1+a_2)}, \qquad g + \underline{g} = a_0 + a_3 + a_4 + \frac{a_3}{f-1} + \frac{ta_0}{f-t},$$

with $\overline{a}_0 = a_0 - 1$, $\overline{a}_1 = a_1$, $\overline{a}_2 = a_2 + 1$, $\overline{a}_3 = a_3 - 1$, $\overline{a}_4 = a_4$, and $a_0 + a_1 + 2a_2 + a_3 + a_4 = 1$.

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From this root variable evolution we immediately see that the corresponding translation on the root lattice is

$$\varphi_* : \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \mapsto \varphi_*(\alpha) = \alpha + \langle 1, 0, -1, 1, 0 \rangle \delta.$$

Using the standard techniques we get the following decomposition of ψ in terms of the generators of $\widetilde{W}\left(D_4^{(1)}\right)$:

$$\varphi = \sigma_3\sigma_2 w_3 w_0 w_2 w_4 w_1 w_2.$$

The Standard d-P_{V} Mapping

This mapping can be further decomposed, in the natural way, as $\varphi = \varphi_2^{-1} \circ \varphi_1$, where φ_1 is a forward mapping $\varphi_1 : (f, g) \mapsto (\overline{f}, -g)$ and φ_2 is a backward mapping $\varphi_2 : (f, g) \mapsto (f, -\underline{g})$; the additional negative sign is necessary for the mapping to be a Cremona isometry.

The Standard d-P_{V} Mapping

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The individual mappings $\varphi_{1,2}$ do not correspond to translations on the symmetry sub-lattice, but they can also be written in terms of generators, in two natural, but slightly different ways; $\varphi = \varphi_2^{-1} \circ \varphi_1 = \tilde{\varphi}_2^{-1} \circ \tilde{\varphi}_1$:

$\varphi_1 = \sigma_3\sigma_2 w_1 w_2 w_4 w_1 w_2 : (f, g) \mapsto (\overline{f}, -g)$; $\quad \overline{a}_0 = 1 - a_0,\ \overline{a}_1 = a_1,\ \overline{a}_2 = -a_1 - a_2,\ \overline{a}_3 = 1 - a_3,\ \overline{a}_4 = -a_4$;

$\varphi_2 = w_0 w_3 w_4 : (f, g) \mapsto (f, -\underline{g})$; $\quad \underline{a}_0 = -a_0,\ \underline{a}_1 = a_1,\ \underline{a}_2 = 1 - a_1 - a_2,\ \underline{a}_3 = -a_3,\ \underline{a}_4 = -a_4$,

or

$\tilde{\varphi}_1 = \sigma_3\sigma_2 w_1 w_2 w_4 w_1 w_2 w_1 : (f, g) \mapsto (\overline{f}, -g)$; $\quad \overline{a}_0 = 1 - a_0,\ \overline{a}_1 = -a_1,\ \overline{a}_2 = -a_2,\ \overline{a}_3 = 1 - a_3,\ \overline{a}_4 = -a_4$;

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The additional negative sign disappears if we consider complete forward or backward steps in the dynamics,

$$(\underline{f}, \underline{g}) \xleftarrow{\varphi_1^{-1}} (f, -\underline{g}) \xleftarrow{\varphi_2} (f, g) \xrightarrow{\varphi_1} (\bar{f}, -g) \xrightarrow{\varphi_2^{-1}} (\bar{f}, \bar{g}).$$

The Standard d-P_{V} Mapping

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This is the equation that we want to match with our discrete dynamical system for the recurrence coefficients for discrete orthogonal polynomials with the hypergeometric weight. We do this next.

The Identification Procedure (continued)

(Step 3) **Determine the surface type, according to Sakai's classification scheme.**

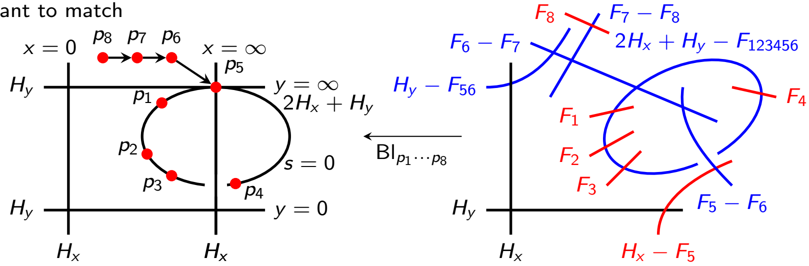
For a discrete Painlevé equation, although the positions of base points may evolve, the *configuration* will stay fixed, and so the surfaces $\{\mathcal{X}_n\}$ will all have the same type. There should be *eight* such base points; those points will lie on some (generically unique) biquadratic curve on $\mathbb{P}^1 \times \mathbb{P}^1$ (i.e., a curve whose defining polynomial, when written in a coordinate chart, has a bi-degree $(2, 2)$) and the *point configuration* is defined to be the configuration of the irreducible components of this curve. Each such component should have self-intersection index -2 and is associated with a node in an *affine Dynkin diagram*, nodes are connected when the corresponding components intersect. The type of this Dynkin diagram is called the *surface type* of the equation. This description assumes that the surfaces $\mathcal{X}_n$ are *minimal*, but can happen that after the initial blowup procedure is complete, some -1-curves would have to be blown down. This will also result in some irreducible components having higher negative self-intersection index. The blowing down procedure is quite delicate, so here we assume that the surfaces $\mathcal{X}_n$ are indeed minimal, but see [**DST13**] and [**DK19**] for examples requiring a blowing down.

(Step 4) **Find a preliminary change of basis of** $\mathrm{Pic}(\mathcal{X})$**.**

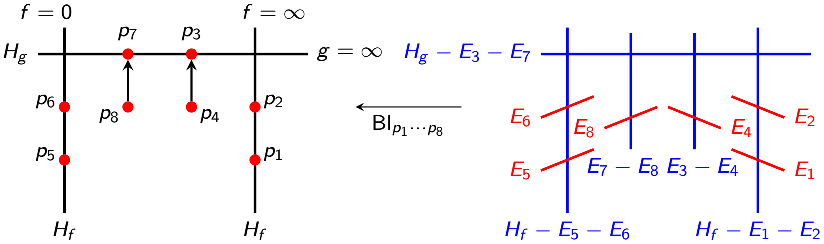
At this step, we only need to ensure that this change of basis identifies the *surface roots* (or nodes of the Dynkin diagrams of our surface) with the standard example. This essentially matches the geometry of the application problem with the geometry of the model example. However, matching the geometries does not guarantee matching of the dynamics. This will be adjusted at the next step.

The Identification Procedure. Step 4: Preliminary Geometry Identification

We want to match

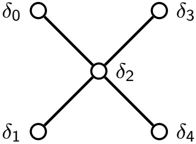


with



The Identification Procedure. Step 4: Preliminary Geometry Identification

We match the geometries by looking at the surface roots:



$$\begin{aligned}
\delta_0 &= \mathcal{E}_3 - \mathcal{E}_4, &&= \mathcal{F}_5 - \mathcal{F}_6,\\
\delta_1 &= \mathcal{H}_f - \mathcal{E}_{12} &&= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_{123456},\\
\delta_2 &= \mathcal{H}_g - \mathcal{E}_{37} &&= \mathcal{F}_6 - \mathcal{F}_7\\
\delta_3 &= \mathcal{E}_7 - \mathcal{E}_8 &&= \mathcal{F}_7 - \mathcal{F}_8,\\
\delta_4 &= \mathcal{H}_f - \mathcal{E}_{56} &&= \mathcal{H}_y - \mathcal{F}_{56}.
\end{aligned}$$

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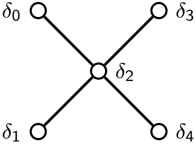

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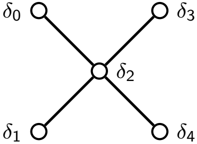

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\end{aligned}$$

Lemma

The following change of bases of $\operatorname{Pic}(\mathcal{X})$ *identifies the root bases between the standard* $D_4^{(1)}$ *surface and the surface that we obtained for the hypergeometric weight recurrence:*

$$\begin{aligned}
\mathcal{H}_x &= \mathcal{H}_g, & \mathcal{H}_f &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_5 - \mathcal{F}_6,\\
\mathcal{H}_y &= \mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6, & \mathcal{H}_g &= \mathcal{H}_x,\\
\mathcal{F}_1 &= \mathcal{E}_1, & \mathcal{E}_1 &= \mathcal{F}_1,\\
\mathcal{F}_2 &= \mathcal{E}_2, & \mathcal{E}_2 &= \mathcal{F}_2,\\
\mathcal{F}_3 &= \mathcal{H}_g - \mathcal{E}_6, & \mathcal{E}_3 &= \mathcal{H}_x - \mathcal{F}_6,\\
\mathcal{F}_4 &= \mathcal{H}_g - \mathcal{E}_5, & \mathcal{E}_4 &= \mathcal{H}_x - \mathcal{F}_5,\\
\mathcal{F}_5 &= \mathcal{H}_g - \mathcal{E}_4, & \mathcal{E}_5 &= \mathcal{H}_x - \mathcal{F}_4,\\
\mathcal{F}_6 &= \mathcal{H}_g - \mathcal{E}_3, & \mathcal{E}_6 &= \mathcal{H}_x - \mathcal{F}_3,\\
\mathcal{F}_7 &= \mathcal{E}_7, & \mathcal{E}_7 &= \mathcal{F}_7,\\
\mathcal{F}_8 &= \mathcal{E}_8, & \mathcal{E}_8 &= \mathcal{F}_8.
\end{aligned}$$

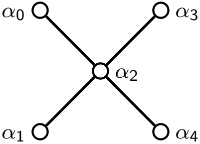
The Identification Procedure (continued)

(Step 5) **Find the translation vector and compare it with the standard dynamic.**

Using this preliminary change of basis we can define the *symmetry roots* for our surface that match the standard example. Using the action φ_* of the mapping on $\mathrm{Pic}(\mathcal{X})$ we can then see the induced action on the symmetry sub-lattice and, in particular, on the symmetry roots. For the discrete Painlevé equations, this action on each root should be a translation by some multiple of the anti-canonical divisor class. Even when this vector is not the same as the translation vector for the reference dynamic, it may be *conjugate* to it. To find out whether this is the case, we represent each translation as a word in the generators of the extended affine Weyl group and solve the conjugacy problem for words in groups. If they are conjugate, the conjugation element is the necessary adjustment to our preliminary change of basis.

The Identification Procedure. Step 5: Comparing the Translations

We are now in the position to compare the dynamics. Using this preliminary change of bases we get the following expressions for the symmetry roots for the applied problem:



$$\alpha_0 = \mathcal{H}_y - \mathcal{F}_{34}, \qquad \alpha_3 = 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_{345678},$$
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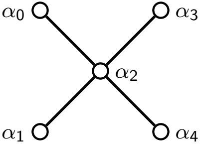

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From the action of ψ_* on $\mathrm{Pic}(\mathcal{X})$ we immediately see that the corresponding translation on the root lattice is

$$\psi_* : \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \mapsto \psi_*(\alpha) = \alpha + \langle 1, 0, 0, -1, 0 \rangle \delta,$$

which is *different* than the standard translation vector $\langle 1, 0, -1, 1, 0 \rangle$.

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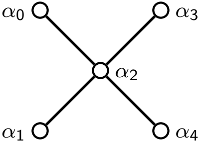

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which is *different* than the standard translation vector $\langle 1, 0, -1, 1, 0 \rangle$.

However, decomposing ψ in terms of generators of the extended affine Weyl symmetry group and comparing it with the expression for φ,

$$\psi = \sigma_3 \sigma_2 w_3 w_2 w_4 w_1 w_2 w_3, \qquad \varphi = \sigma_3 \sigma_2 w_3 w_0 w_2 w_4 w_1 w_2,$$

we immediately see that $\psi = w_3 \circ \varphi \circ w_3^{-1}$ (recall that $w_3 \sigma_3 \sigma_2 = \sigma_3 \sigma_2 w_0$ and that w_3 is an involution, $w_3^{-1} = w_3$). Thus, our dynamic is indeed equivalent to the standard d-P_{V} equation, but the preliminary change of bases in the previous Lemma needs to be adjusted by acting by w_3.

The Identification Procedure. Step 5: Comparing the Translations

We now adjust the change of basis by w_3 action to get the following Lemma.

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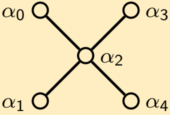
Lemma

After the change of bases of $\operatorname{Pic}(\mathcal{X})$ *given by*

$$\begin{aligned}
\mathcal{H}_x &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_7 - \mathcal{E}_8, & \mathcal{H}_f &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_5 - \mathcal{F}_6,\\
\mathcal{H}_y &= 3\mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - 2\mathcal{E}_7 - 2\mathcal{E}_8, & \mathcal{H}_g &= 3\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_7 - \mathcal{F}_8,\\
\mathcal{F}_1 &= \mathcal{E}_1, & \mathcal{E}_1 &= \mathcal{F}_1,\\
\mathcal{F}_2 &= \mathcal{E}_2, & \mathcal{E}_2 &= \mathcal{F}_2,\\
\mathcal{F}_3 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8, & \mathcal{E}_3 &= \mathcal{H}_x - \mathcal{F}_6,\\
\mathcal{F}_4 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_5 - \mathcal{E}_7 - \mathcal{E}_8, & \mathcal{E}_4 &= \mathcal{H}_x - \mathcal{F}_5,\\
\mathcal{F}_5 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8, & \mathcal{E}_5 &= \mathcal{H}_x - \mathcal{F}_4,\\
\mathcal{F}_6 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_3 - \mathcal{E}_7 - \mathcal{E}_8, & \mathcal{E}_6 &= \mathcal{H}_x - \mathcal{F}_3,\\
\mathcal{F}_7 &= \mathcal{H}_f - \mathcal{E}_8, & \mathcal{E}_7 &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_8,\\
\mathcal{F}_8 &= \mathcal{H}_f - \mathcal{E}_7, & \mathcal{E}_8 &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_7,
\end{aligned}$$

the recurrence relations for variables x_n *and* y_n *coincides with the standard d-*P_V *discrete Painlevé equation. The resulting identification of the symmetry root bases (the surface root bases do not change) is*



$$\begin{aligned}
\alpha_0 &= \mathcal{H}_y - \mathcal{F}_{34}, & \alpha_3 &= -2\mathcal{H}_x - \mathcal{H}_y + \mathcal{F}_{345678},\\
\alpha_1 &= \mathcal{F}_1 - \mathcal{F}_2, & \alpha_4 &= \mathcal{F}_3 - \mathcal{F}_4.\\
\alpha_2 &= 2\mathcal{H}_x + \mathcal{H}_y - \mathcal{F}_{135678}, & &
\end{aligned}$$

The Identification Procedure (continued)

(Step 5) **Find the translation vector and compare it with the standard dynamic.**

Using this preliminary change of basis we can define the *symmetry roots* for our surface that match the standard example. Using the action φ_* of the mapping on $\mathrm{Pic}(\mathcal{X})$ we can then see the induced action on the symmetry sub-lattice and, in particular, on the symmetry roots. For the discrete Painlevé equations, this action on each root should be a translation by some multiple of the anti-canonical divisor class. Even when this vector is not the same as the translation vector for the reference dynamic, it may be *conjugate* to it. To find out whether this is the case, we represent each translation as a word in the generators of the extended affine Weyl group and solve the conjugacy problem for words in groups. If they are conjugate, the conjugation element is the necessary adjustment to our preliminary change of basis.

(Step 6) **Find the change of variables reducing the applied problem to the standard example**.

Adjusting the change of bases in $\mathrm{Pic}(\mathcal{X})$, if necessary, we now have the identification on the level of the Picard lattice. Next, we need to fing the actual change of variables that induces that linear change of basis. For that, identify the curves that form the basis in the corresponding coordinate pencils. Those curves then are our projective coordinates, up to a Möbius transformation. To fix the Möbius transformations, use the mapping of coordinate divisors. An importnat part of this computation is the identification of various parameters between the two problems. This, in fact, can be done ahead of time by using the *Period Map*, which gives the parameterization in terms of canonical (for the given choice of root bases) *root variables*. Expressing these root variables in terms of parameters of the problem gives the necessary identification of parameters.

The Identification Procedure. Step 6: The Change of Coordinates

Next we need to realize this change of basis on $\operatorname{Pic}(\mathcal{X})$ by the explicit change of coordinates. For that, it is convenient to first match the parameters between the applied problem and the reference example. This is done with the help of the *Period Map*.

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Lemma

(i) The residues of the symplectic form $\omega = k\frac{dX\wedge dY}{s(X,Y)\,Y} = k\frac{dX\wedge ds}{s(s-X^2)}$ *along the irreducible components of the polar divisor are given by*

$$\mathrm{res}_{d_0}\,\omega = -k\frac{dv_5}{v_5^2}, \qquad \mathrm{res}_{d_2}\,\omega = -k\frac{(n+\alpha+\beta)dv_6}{(v_6-1)^2}, \qquad \mathrm{res}_{d_4}\,\omega = -k\frac{X}{X^2}.$$

$$\mathrm{res}_{d_1}\,\omega = k\frac{dX}{X^2}, \qquad \mathrm{res}_{d_3}\,\omega = -k\frac{(c-1)^2dv_7}{c},$$

(ii) The root variables are given by

$$a_0 = k(\gamma-n-\alpha), \quad a_1 = k(\alpha-1), \quad a_2 = k(1+n+\beta-\gamma), \quad a_3 = -k(n+\beta), \quad a_4 = k(\gamma-\beta).$$

The normalization condition $a_0+a_1+2a_2+a_3+a_4 = 1$ *then implies that* $k=1$, *and we get the following relations between our parameters and the root variables:*

$$\alpha = a_1+1, \quad \beta = a_0+a_1+a_2, \quad \gamma = 1-a_2-a_3, \quad n = a_2+a_4-1.$$

Note that the root variable evolution, which is the same as in the standard case, is consistent with what we expect: $\overline{\alpha}=\alpha$, $\overline{\beta}=\beta$, $\overline{\gamma}=\gamma$, *and* $\overline{n}=n+1$. *Also observe that we can not yet see the relationship between parameters* t *and* c *in this identification.*

The Identification Procedure. Step 6: The Change of Coordinates

Theorem (Main Result)

The discrete dynamical system on recurrence coefficients of discrete orthogonal polynomials with the hypergeometric weight is equivalent to to the standard difference Painlevé-V equation

$$\overline{f}f = \frac{tg(g-a_4)}{(g+a_2)(g+a_1+a_2)}, \qquad g + \underline{g} = a_0 + a_3 + a_4 + \frac{a_3}{f-1} + \frac{ta_0}{f-t},$$

with $\overline{a}_0 = a_0 - 1$, $\overline{a}_1 = a_1$, $\overline{a}_2 = a_2 + 1$, $\overline{a}_3 = a_3 - 1$, $\overline{a}_4 = a_4$, *and* $a_0 + a_1 + 2a_2 + a_3 + a_4 = 1$. *This equivalence is achieved via the following change of variables:*

$$x(f,g) = \gamma - g - \frac{(n+\beta)f}{f-1},$$

$$y(f,g) = (g+\alpha+\beta+n-\gamma)(g+2\beta+2n-\gamma) - n\alpha - \frac{gt(g+\beta-\gamma)}{f} + \frac{(n+\beta)((c-1)(2g+\alpha+3\beta+3n-2\gamma)+(\alpha+\beta+\gamma-n)+n)}{c(f-1)} + \frac{(c-1)(n+\beta)^2}{c(f-1)^2}.$$

The inverse change of variables is given by

$$f(x,y) = \frac{t(x-\beta)(x-\gamma)}{((x-\alpha)(x-\beta)-nx-y)},$$

$$g(x,y) = -\frac{(x-\gamma)(((x-\alpha)(x-\beta)-nx-y)-t(x-\beta)(x-\gamma+\beta+n))}{((x-\alpha)(x-\beta)-nx-y)-t(x-\beta)(x-\gamma)}.$$

Parameters α, β, γ *and* c *of the weight are related to the standard Painlevé parameters (root variables) by* $\alpha = a_1 + 1$, $\beta = a_0 + a_1 + a_2$, $\gamma = 1 - a_2 - a_3$, $n = a_2 + a_4 - 1$, *and* $ct = 1$.

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