2.e Exercises for Lecture 2

Here are all the exercises from the lecture notes, reorganized and renumbered to make an exercise sheet. There are also a few new questions, to keep things interesting. Solve only what you like.

Exercise 2.e.1. (**Lineland**) We consider the one-dimensional version of the Domino problem: a *Wang bar* is a unit segment with colored vertices, and a finite set of Wang bars is a barset. Given a barset B, a *one-dimentional tiling* of a set of points $S \subseteq \mathbb{Z}$ by B is a function $S \to B$. A tiling of the whole $\mathbb{Z}$ is called a *tiling of the line.* Of course Wang bars also have color constraints: a tiling is *valid* if for each pair of neighbouring bars, the adjacent vertices have the same color. For example, these two bars match: .
Here is an example of a valid tiling of the line:

A barset is *solvable* if it has at least one valid tiling of the line.

A tiling of the line T is *periodic* if there exists a nonzero integer p, called the *period*, such that $T(x+p) = T(x)$ for all x in $\mathbb{Z}$.

(a) Prove the following statement (called the one-dimensional Compactness theorem). Let A denote a barset; if there is a valid tiling of the set $\{-n, \dots, +n\}$ by A for every integer n, then there is a valid tiling of $\mathbb{Z}$ by A.

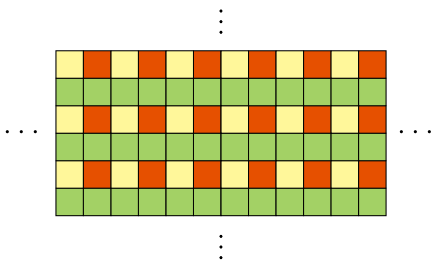
(b) Show that Wang's conjecture holds for one-dimensional tilings: if a barset is solvable, then it has at least one periodic tiling of the line.

(c) Give an algorithm solving the one-dimensional Domino Problem.

Back to Flatland... Only Exercise 2.e.1 talks about Wang bars. All other exercises talk about two-dimensional tiles.

Exercise 2.e.2 ()**.** Let A denote a solvable tileset. Prove that if A has a valid tiling T with one period $\vec{v}$, then it has a tiling (maybe different from T) with two periods $\vec{v_1}$ and $\vec{v_2}$ which are not colinear (proportional).

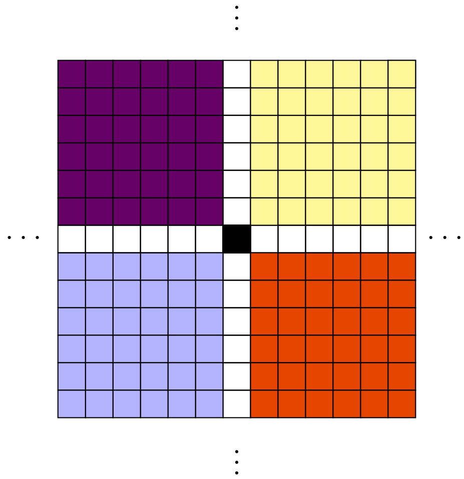
Exercise 2.e.3. Design a set of 3 Wang tiles A such that all tilings of the plane by A, up to translation, look like this:



where squares of different colors denote Wang tiles of different types.

Exercise 2.e.4.

(a) Design a set of 9 Wang tiles A such that one of its tilings of the plane looks like this:



where squares of different colors denote Wang tiles of different types. (One color can correspond to several tiles, but one tile corresponds to only one color on the picture).
(If necessary, start by designing a tileset with any number of tiles inside, and then try to reduce it.)

(b) Explain why the tileset from your solution of (a) also can assemble in tilings where all the tiles are “orange” (respectively “blue”, “violet”, “yellow”).

Exercise 2.e.5. Find an algorithm that, given a **directed** graph G, generates a set of Wang tiles A such that G has a cycle if and only if A is solvable. The algorithm should work with a constant memory. **The algorithm itself should not answer whether the graph has a cycle or not,** it only should translate to a tileset.

Exercise 2.e.6. Why is the condition “has at least one cell with $\mathcal{H}$ on its south color” needed in Lemma 2.4.11 of the lecture notes?

Exercise 2.e.7 ( ). Let us modify the definition of a Turing machine and of a configuration (§1.2.4) so that the memory is *biinfinite*, i.e., a function $\mathbb{Z} \to \mathbb{I}$ instead of $\mathbb{N} \to \mathbb{I}$. Prove that a function is computable with an $\mathbb{N}$-Turing machine if and only if it is computable with a $\mathbb{Z}$-Turing machine.

Exercise 2.e.8 ( ). Write a Python program that converts a Turing machine into a set of tiles, as described in Section 2.4.

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