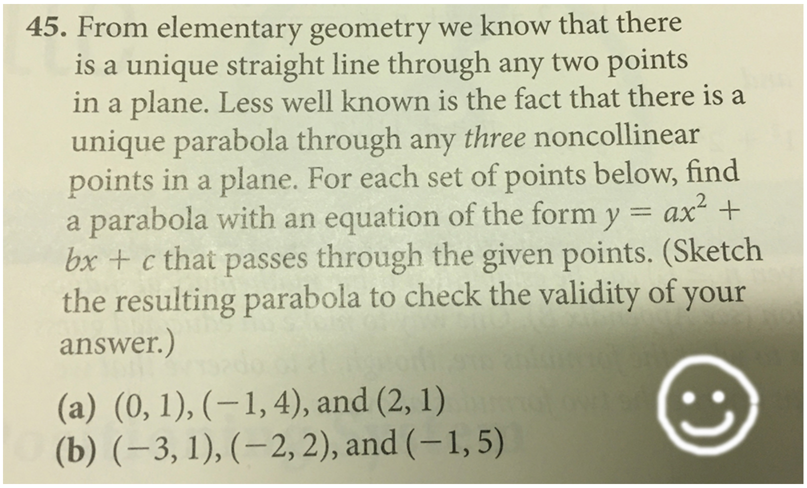
EXERCISES

DUBNA 2018: LINES ON CUBIC SURFACES

Exercise 1. The following problem is from *Linear Algebra, A Modern Introduction* by David Poole (2014).



45. From elementary geometry we know that there is a unique straight line through any two points in a plane. Less well known is the fact that there is a unique parabola through any *three* noncollinear points in a plane. For each set of points below, find a parabola with an equation of the form $y = ax^2 + bx + c$ that passes through the given points. (Sketch the resulting parabola to check the validity of your answer.)

(a) $(0, 1), (-1, 4)$, and $(2, 1)$
(b) $(-3, 1), (-2, 2)$, and $(-1, 5)$

The sentence "Less well known is the fact that there is a unique parabola through any three noncollinear points in a plane" is mathematically wrong. In this problem, Poole assumes that parabola is the curve in $\mathbb{R}^2$ that is given by the equation

$$y = ax^2 + bx + c$$

for some real numbers a, b and c. This assumption is a bit weird, since parabolas were used long before René Descartes introduced Cartesian coordinates. Moreover, this definition of parabola discriminates x-coordinate, which is not appropriate ☺. The goal of this exercise is to solve this problem using *good* definition of parabola: parabola is a subset in $\mathbb{R}^2$ such that there exists a composition of rotations and translations that maps it to the curve given by

$$y = px^2,$$

where p is a positive real number. Do the following.

(a) Find all parabolas in $\mathbb{R}^2$ that pass through the points $(0, 1)$, $(-1, 4)$, $(2, 1)$, $(19, 20)$.
(b) Find all parabolas in $\mathbb{R}^2$ that pass through the points $(0, 1)$, $(-1, 4)$, $(2, 1)$, $(9, 10)$.
(c) Describe all parabolas in $\mathbb{R}^2$ that pass through the points $(0, 1)$, $(-1, 4)$, $(2, 1)$.
(d) Let P be a point in $\mathbb{R}^2$ that is different from $(0, 1)$, $(-1, 4)$, $(2, 1)$. Explain when there exists a parabola that contains $(0, 1)$, $(-1, 4)$, $(2, 1)$ and P.

Solution. Let C be a conic in $\mathbb{R}^2$. Then it is given by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

for some real numbers a, b, c, d, e, f such that $(a, b, c) \neq (0, 0, 0)$. Rewrite this equation as

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

Denote this 3×3 matrix by M. Then C is a parabola $\iff$ $b^2 - 4ac = 0$ and $\det(M) \neq 0$.

(a) Suppose that C contains the points $(0,1)$, $(-1,4)$, $(2,1)$ and $(19,20)$. Substituting their coordinates in the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$, we obtain the system of equations

$$\begin{cases} c + e + f = 0, \\ a - 4b + 16c - d + 4e + f = 0, \\ 4a + 2b + c + 2d + e + f = 0, \\ 361a + 380b + 400c + 19d + 20e + f = 0. \end{cases}$$

Moreover, if the conic C is parabola, then $b^2 - 4ac = 0$. Thus, we get the system of equations

$$\begin{cases} c + e + f = 0, \\ a - 4b + 16c - d + 4e + f = 0, \\ 4a + 2b + c + 2d + e + f = 0, \\ 361a + 380b + 400c + 19d + 20e + f = 0, \\ b^2 - 4ac = 0. \end{cases}$$

If $b = 0$, then either $a = 0$ or $c = 0$ (or both). In both these cases, this system has only trivial solution: $(a,b,c,d,e,f) = (0,0,0,0,0,0)$. Thus, we may assume that $b = 1$. This gives us exactly two solutions: $(a,b,c,d,e,f) = \frac{1}{4}(4,-4,1,-4,-13,12)$ and $(a,b,c,d,e,f) = \frac{1}{4}(1,-4,4,2,-25,21)$. These solutions give us two conics in $\mathbb{R}^2$ that contains $(0,1)$, $(-1,4)$, $(2,1)$ and $(19,20)$. The first conic is given by

$$4x^2 - 4xy - 4x + y^2 - 13y + 12 = 0,$$

and the second conic is given by

$$x^2 - 4xy + 2x + 4y^2 - 25y + 21 = 0.$$

Both of them are parabolas. Indeed, the first one is given by

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & -\frac{13}{2} \\ -2 & -\frac{13}{2} & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

The determinant of this 3×3 matrix is -225, so that the first conic is parabola. Similarly, the second conic is given by

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -\frac{25}{2} \\ 1 & -\frac{25}{2} & 21 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0,$$

and the determinant of this 3×3 matrix is $-\frac{441}{4}$, so that it is also parabola.

(b) Now we suppose that C contains the points $(0,1)$, $(-1,4)$, $(2,1)$, $(9,10)$. Arguing as in the solution to part (a), we obtain the system of equations

$$\begin{cases} c + e + f = 0, \\ a - 4b + 16c - d + 4e + f = 0, \\ 4a + 2b + c + 2d + e + f = 0, \\ 81a + 90b + 100c + 9d + 10e + f = 0, \\ b^2 - 4ac = 0. \end{cases}$$

If $b = 0$, then this system has only trivial solution: $(a,b,c,d,e,f) = (0,0,0,0,0,0)$, so that we may assume that $b = 1$. Then either $(a,b,c,d,e,f) = -\frac{1}{6}(9,-6,1,-12,-20,19)$ or $(a,b,c,d,e,f) = -\frac{1}{6}(1,-6,9,4,-52,43)$. These solutions give us two conics in $\mathbb{R}^2$ that contains $(0,1)$, $(-1,4)$, $(2,1)$ and $(9,10)$. The first conic is given by

$$9x^2 - 6xy + y^2 - 12x - 20y + 19 = 0,$$

and the second conic is given by

$$x^2 - 6xy + 9y^2 + 4x - 52y + 43 = 0.$$

Both of them are parabolas, because

$$\det \begin{pmatrix} 9 & -3 & -6 \\ -3 & 1 & -10 \\ -6 & -10 & 19 \end{pmatrix} = -1296 \neq 0 \neq -400 = \det \begin{pmatrix} 1 & -3 & 2 \\ -3 & 9 & -26 \\ 2 & -26 & 43 \end{pmatrix}.$$

(c) There are infinitely many parabolas in $\mathbb{R}^2$ that pass through $(0,1)$, $(-1,4)$, $(2,1)$. To describe all of them, let t be a real number, and let C_t be a conic in $\mathbb{R}^2$ that is given by

$$(\blacklozenge) \qquad -x^2 + 2txy - t^2y^2 + (2-2t)x + (5t^2+2t+1)y - 4t^2 - 2t - 1 = 0.$$

Then C_t contains the points $(0,1)$, $(-1,4)$, $(2,1)$ for every $t \in \mathbb{R}$. Moreover, computing the determinant of the corresponding 3×3 matrix, we see that

$$\det \begin{pmatrix} -1 & t & 1-t \\ t & -t^2 & \frac{5t^2+2t+1}{2} \\ 1-t & \frac{5t^2+2t+1}{2} & -4t^2-2t-1 \end{pmatrix} = \frac{(t+1)^2(3t+1)^2}{4}.$$

Thus, if $t \neq -\frac{1}{3}$ and $t \neq -1$, then the conic C_t is a parabola. For instance, if we want to obtain the parabolas in the part (a), we substitute $x = 19$ and $y = 20$ into $(\blacklozenge)$. This gives

$$-304t^2 + 760t - 304 = 0,$$

so that either $t = \frac{1}{2}$ or $t = 2$. If $t = \frac{1}{2}$, we obtain the parabola $4x^2 - 4xy - 4x + y^2 - 13y + 12 = 0$. Similarly, if $t = 2$, we obtain the parabola $x^2 - 4xy + 2x + 4y^2 - 25y + 21 = 0$.

The *bad* values of the parameter t correspond to the case when C_t is a union of two parallel lines. Namely, if $t = -1$, then C_t is given by

$$(\blacktriangle) \qquad (x+y-3)(x+y-1) = 0,$$

so that C is a union of parallel lines. Likewise, if $t = -\frac{1}{3}$, then C_t is a union of parallel lines given by

$$(\blacktriangledown) \qquad (3x+y-7)(3x+y-1) = 0.$$

Note that the conic $y(y-4) = 0$ is also a union of two parallel lines that contains $(0,1)$, $(-1,4)$, $(2,1)$. This conic corresponds to $t = \infty$. Aside from these three *degenerate* cases, all other conics given by $(\blacklozenge)$ are parabolas.

In fact, every parabola that passes through $(0,1)$, $(-1,4)$, $(2,1)$ is given by $(\blacklozenge)$ for an appropriate $t \in \mathbb{R}$. Namely, let C be a conic in $\mathbb{R}^2$ that is given by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

where a, b, c, d, e, f are real numbers such that $(a,b,c) \neq (0,0,0)$. Then we can rewrite this equation in the matrix form:

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

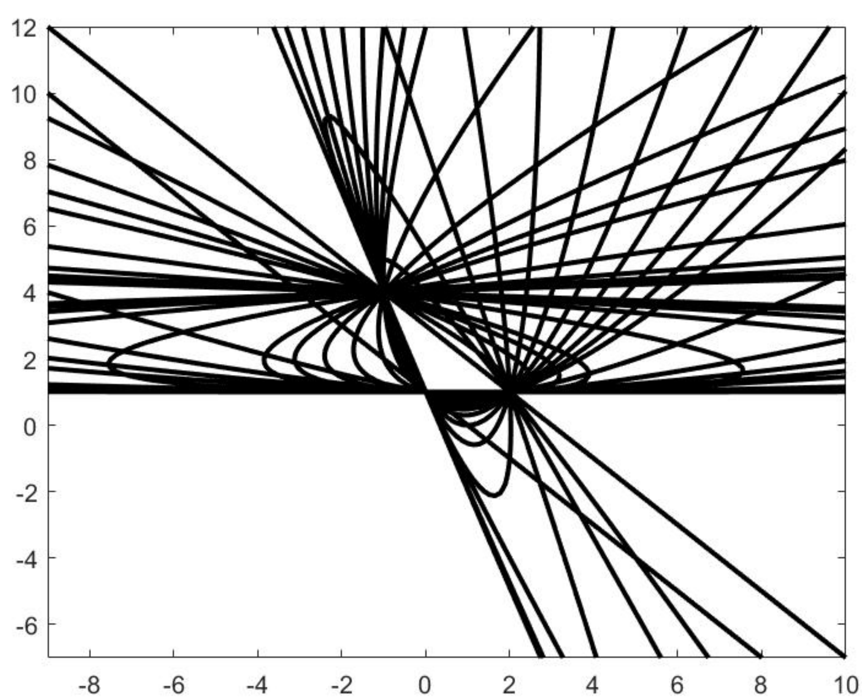
Let M be the 3×3 matrix in the equation. Then C is parabola if and only if $\det(M) \neq 0$ and $b^2 = 4ac$. The condition $b^2 = 4ac$ simply means that the projectivization of the conic C intersects the *infinite line* by one point. We already know from the thirds worksheet that this may happen only if the infinite line is tangent to the projectivization of the conic C. Moreover, for every point $[t_1 : t_0 : 0] \in \mathbb{P}^2$, there exists a unique conic in $\mathbb{P}^2$ that contains the points $[0 : 1 : 1]$, $[-1 : 4 : 1]$, $[2 : 1 : 1]$, and intersects the infinite line only at the point $[t_1 : t_0 : 0]$. Furthermore, the equation (♦) defines such conic for $[t_1 : t_0 : 0] = [t : 1 : 0]$, and $(y-1)(y-4) = 0$ defines such conic for $[t_1 : t_0 : 0] = [1 : 0 : 0]$. Therefore, every parabola in $\mathbb{R}^2$ that passes through the points $(0,1)$, $(-1,4)$, $(2,1)$ is given by (♦) for an appropriate real number t.

(d) If P is contained in one of the lines $y = 1$, $x+y-3 = 0$ and $3x+y-1 = 0$, then there is no parabola that contains the points $(0,1)$, $(-1,4)$, $(2,1)$ and P. Indeed, these three lines are the lines that pass through two points among $(0,1)$, $(-1,4)$, $(2,1)$. Namely, the line $y = 1$ contains the points $(0,1)$ and $(2,1)$, the line $x+y-3 = 0$ contains the points $(-1,4)$ and $(2,1)$, and the line $3x+y-1 = 0$ contains the points $(0,1)$ and $(-1,4)$. Thus, if P is contained in any of the lines $y = 1$, $x+y-3 = 0$ or $3x+y-1 = 0$, then there exists no parabola passing through $(0,1)$, $(-1,4)$, $(2,1)$ and P, because line and parabola intersect by at most 2 points. Suppose that

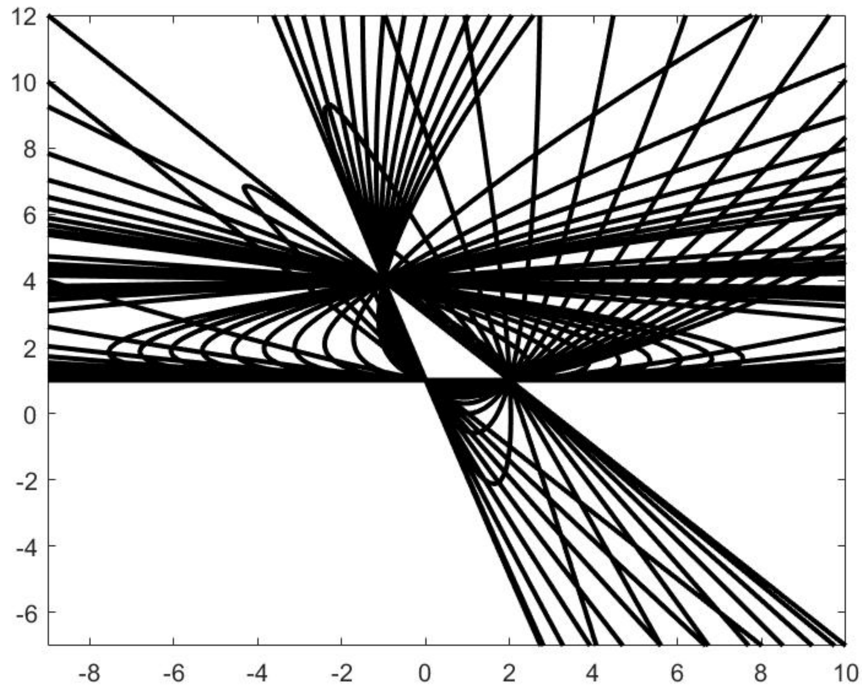
the point P is not contained in the lines $y = 1$, $x+y-3 = 0$ and $3x+y-1 = 0$.

Does it exist a parabola that contains $(0,1)$, $(-1,4)$, $(2,1)$ and P? Not always. For example, if $P = (-1,-1)$, then there is no parabola that contains $(0,1)$, $(-1,4)$, $(2,1)$ and P. But $(-1,-1)$ is not contained in any of the lines $y = 1$, $x+y-3 = 0$ and $3x+y-1 = 0$. On the other hand, if $P = (19,20)$, then there are exactly two parabolas that pass through $(0,1)$, $(-1,4)$, $(2,1)$ and P. Similarly, if $P = (9,10)$, then there are two parabolas that contain the points $(0,1)$, $(-1,4)$, $(2,1)$ and P.

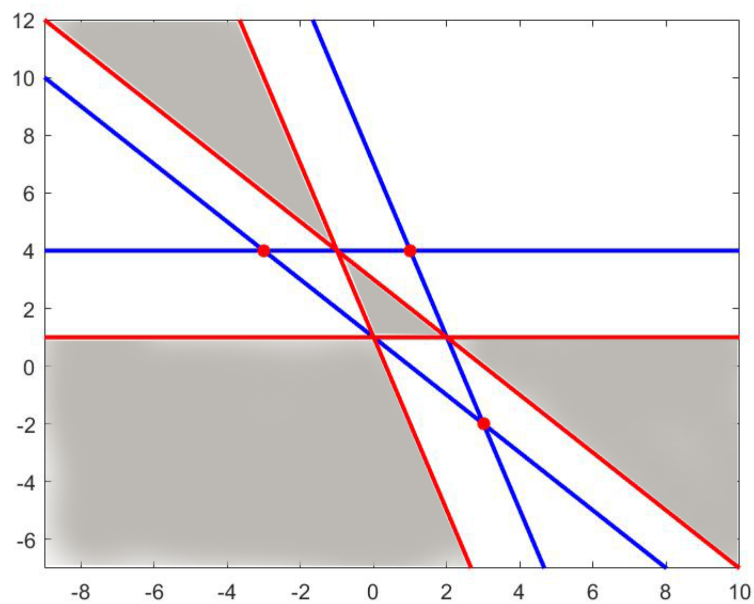
Let us describe explicitly for which $P \in \mathbb{R}^2$ there exists a parabola that contains $(0,1)$, $(-1,4)$, $(2,1)$ and P, and for which $P \in \mathbb{R}^2$ such parabola does not exist. The answer is quite interesting. One can guess it by plotting many parabolas given by (♦). For instance, the following pictures displays 30 parabolas that pass through the points $(0,1)$, $(-1,4)$, $(2,1)$.



The following pictures displays 60 parabolas that pass through $(0,1)$, $(-1,4)$, $(2,1)$.



Looking at these pictures, we can guess the answer. The lines $y = 1$, $x + y - 3 = 0$ and $3x + y - 1 = 0$ split the plane $\mathbb{R}^2$ into seven ares. If P is contained in four of them, then there exists no parabola that contains $(0, 1)$, $(-1, 4)$, $(2, 1)$ and P. To formulate the answer more precisely, observe first that one of the following cases holds:
(a) there exists two parabolas that contain $(0, 1)$, $(-1, 4)$, $(2, 1)$ and P;
(b) there exists exactly one parabola that contains $(0, 1)$, $(-1, 4)$, $(2, 1)$ and P;
(c) there are no parabolas that contain $(0, 1)$, $(-1, 4)$, $(2, 1)$ and P.
Moreover, the following picture describes when these cases hold:



Here the red lines are the lines $y=1$, $x+y-3=0$ and $3x+y-1=0$. The red points are the points $(1,4)$, $(-3,4)$ and $(3,-2)$. The blue lines are the lines $y=4$, $x+y-1=0$ and $3x+y-7=0$. Then we have case (a) if P is in the white area. Similarly, we have case (b) if P is contained in one of the blue lines, it is not contained in the red lines, and P is not one of the red points $(1,4)$, $(-3,4)$ and $(3,-2)$. Finally, we have case (c) if P is in the gray area, or P is contained in one of the red lines, or P is one of the red points $(1,4)$, $(-3,4)$ and $(3,-2)$. Let us prove this.

Write $P=(s,t)$, where s and t are some real numbers. Suppose that P is not one of the points $(0,1)$, $(-1,4)$, $(2,1)$, and suppose that P is not contained in one of the lines $y=1$, $x+y-3=0$ and $3x+y-1=0$. Suppose that C contains $(0,1)$, $(-1,4)$, $(2,1)$ and P. Then

$$(\clubsuit)\qquad \begin{cases} c+e+f=0, \\ a-4b+16c-d+4e+f=0, \\ 4a+2b+c+2d+e+f=0, \\ as^2+bst+t^2+ds+et+f=0 \end{cases}$$

If $b^2=4ac$, then either $\det(M)\neq 0$ and C is a parabola, or $\det(M)=0$ and C is a union of two parallel lines.

If $a=0$ and $b^2=4ac$, then $c\neq 0$, so that we may assume that $c=5$, which implies that $b=0$, $c=5$, $d=0$, $e=-25$ and $f=20$ by $(\clubsuit)$. In this case, the conic C is given by

$$(y-1)(y-4)=0.$$

This is not parabola. This is a union of two parallel lines $y=1$ and $y=4$. Hence, we may assume that $a\neq 0$. Multiplying the equation of C by $\frac{1}{a}$, we may assume that $a=1$. If $b^2=4ac$, then $(\clubsuit)$ gives

$$\begin{cases} c+e+f=0, \\ 1-4b+16c-d+4e+f=0, \\ 4+2b+c+2d+e+f=0, \\ as^2+bst+ct^2+ds+et+f=0, \\ b^2=4c. \end{cases}$$

This gives

$$\begin{cases} a=1, \\ c=\dfrac{b^2}{4}, \\ d=-2-b, \\ e=-\dfrac{5}{4}b^2-1+b, \\ f=b^2-b+1, \\ as^2+bst+ct^2+ds+et+f=0. \end{cases}$$

Thus, if $b^2=4ac$, then substituting these expressions for a, c, d, e, f into the equation $as^2+bst+ct^2+ds+et+f=0$, we obtain

$$(\bigstar)\qquad \frac{1}{4}(t-1)(t-4)b^2+(s+1)(t-1)b+s^2-2s-t+1=0.$$

Recall that s and t are some fixed real numbers such that $s\neq 1$, because P is not contained in the line $y=1$ by assumption. If $t=4$, then $(\bigstar)$ gives

$$3(s+1)b+(s+1)(s-3)=0,$$

so that $b=\frac{3-s}{3}$, because $s\neq -1$, since $(-1,4)$ is contained in the line $x+y-3=0$. Thus, if $t=4$ and $b^2=4ac$, then

$$\begin{cases} a=1, \\ b=\dfrac{3-s}{3}, \\ c=\dfrac{(s-3)^2}{36}, \\ d=\dfrac{s-9}{3}, \\ e=-\dfrac{5s^2-18s+45}{36}, \\ f=\dfrac{s^2-3s+9}{9}, \end{cases}$$

so that the conic C is given by

$$36x^2+(36-12s)xy+(s^2-6s+9)y^2+(12s-108)x-(5s^2-18s+45)y+4s^2-12s+36=0.$$

In the matrix form this equation can be rewritten as

$$\begin{pmatrix} x & y & 1 \end{pmatrix}\begin{pmatrix} 36 & 18-6s & 6s-54 \\ 18-6s & s^2-6s+9 & -\frac{5s^2-18s+45}{2} \\ 6s-54 & -\frac{5s^2-18s+45}{2} & 4s^2-12s+36 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}=0.$$

The determinant of this 3×3 matrix is

$$-81(s+3)^2(s-1)^2.$$

Thus, if $t=4$ and $b^2=4ac$, then C is a parabola if and only if P is not one of the points $(-3,4)$ and $(1,4)$. Similarly, if $P=(-3,4)$ and $b^2=4ac$, then C is a union of two parallel lines. These are the lines $y=1$ and $y=4$. Likewise, if $P=(1,4)$ and and $b^2=4ac$, then C is a union of two parallel lines $y=1$ and $y=4$. Thus, we proved the following: if P is contained in the line $y=4$, $P\neq(-3,4)$ and $P\neq(1,4)$, then there exists unique parabola that passes through $(0,1)$, $(-1,4)$, $(2,1)$ and P. Moreover, if $P=(-3,4)$ or $P=(1,4)$, then there exists no parabola that passes through $(0,1)$, $(-1,4)$, $(2,1)$ and P. Hence, to complete the proof, we may assume that P is not contained in the line $y=4$. Then $t\neq 4$.

Since $t\neq 4$, the equation ($\bigstar$) is a quadratic equation in b. It has real solution if and only if its discriminant is positive. Denote this discriminant by $\Delta(s,t)$. Then

$$\Delta(s,t)=(s+t-3)(3s+t-1)(t-1).$$

For example, we have $\Delta(-1,-1)<0$, so that there is no parabola that contains $(0,1)$, $(-1,4)$, $(2,1)$ and $(-1,-1)$. We already mentioned this earlier. Similarly, we have $\Delta(19,20)>0$ and $\Delta(9,10)>0$, which we already know.

By assumption, the point P is not contained in any of the lines $y=1$, $x+y-3=0$ and $3x+y-1=0$. Thus, we have $\Delta(s,t)\neq 0$. Moreover, each time the point $P=(s,t)$ crosses one of this lines, the sign of $\Delta(s,t)$ changes, so that $\Delta(s,t)<0$ if and only if the point P is contained in the grey area in our picture. Thus, to complete the proof, we may assume that $\Delta(s,t)>0$. In this case, the equation ($\bigstar$) has exactly two solutions. Thus, if $b^2=4ac$, then there are exactly two possibilities for the conic C. In each case, either $\det(M)\neq 0$ and C is a parabola, or $\det(M)=0$ and C is a union of two parallel lines. Since $t\neq 4$, if $\det(M)=0$, then one of the following two cases holds:

- the point P is contained in the line $x+y-1=0$, the conic C is given by ($\blacktriangle$), and C is union of two parallel lines $x+y-1=0$ and $x+y-3=0$;

- the point P is contained in the line $x+y-1=0$, the conic C is given by (▼), and C is union of two parallel lines $3x+y-7=0$ and $3x+y-1=0$

Thus, if P is in the white area in our picture, then there exists exactly two parabolas that contain $(0,1)$, $(-1,4)$, $(2,1)$ and P.

If $P=(3,-2)$, then two solution of the equation (★) gives us the conics (▲) and (▼). Thus, there exists no parabola that contains $(0,1)$, $(-1,4)$, $(2,1)$ and $(3,-2)$. One the other hand, if P is contained in the line $x+y-1=0$ and $P\neq(3,-2)$, then one solution of the equation (★) gives us the conic(▲), and another solution of the equation (★) gives us the unique parabola that contains the points $(0,1)$, $(-1,4)$, $(2,1)$ and P. Likewise, if P is contained in the line $3x+y-7=0$ and $P\neq(3,-2)$, then one solution of the equation (★) gives us the conic (▼), and another solution of the equation (★) gives us the unique parabola that contains the points $(0,1)$, $(-1,4)$, $(2,1)$ and P.

□

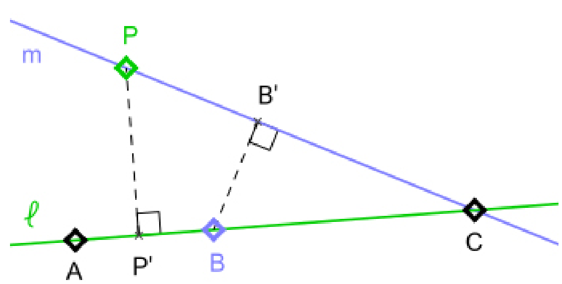
Exercise 2. Let Σ be a subset in $\mathbb{P}^2_{\mathbb{C}}$ such that Σ is not contained in one line in $\mathbb{P}^2_{\mathbb{C}}$.

(a) Suppose that $|\Sigma|\leqslant 6$. Prove that there exists a line $L\subset\mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ.

(b) Suppose that $|\Sigma|=7$. Prove that there exists a line $L\subset\mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ.

(c) Suppose that $|\Sigma|=8$. Prove that there exists a line $L\subset\mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ.

Solution. The Sylvester–Gallai theorem in geometry states that, given a finite number of points in $\mathbb{R}^2$, either all the points lie on a single line; or there is a line which contains exactly two of the points. It is named after James Sylvester, who posed it as a problem in 1893, and Tibor Gallai, who proved it in 1944. Later, a simpler proof of this result was found by Leroy Kelly. This proof is easy to describe. Namely, let S be a finite subset in $\mathbb{R}^2$ such that S is not contained in one line. Choose a point $P\in S$ and a line ℓ such that ℓ contains at least two points in S, it does not contain the point P, and the distance between P and ℓ is the smallest possible. Then then ℓ cannot contain three points of the set S, so that it contains exactly two points of S. Indeed, let P' be the perpendicular projection of P to the line l, i.e. the point in ℓ such that the vector $\overrightarrow{PP'}$ is orthogonal to the line ℓ. Then the distance between P and P' equals to the distance between P and ℓ. Suppose that ℓ contains at least three points of the set S, Then at least two of them are on the same side of P'. Denote them B and C such that B is the closest among them to the point P'. Then the distance between B and the line passing through P and C is smaller that $|\overrightarrow{PP'}|$, which contradicts to the choice of the point P and the line ℓ. This proof is illustrated by the following picture:



It is more natural to consider this problem for lines in projective planes. Unfortunately, the assertion of Gallai–Silvester theorem does not hold for points in $\mathbb{P}^2_{\mathbb{C}}$. This follows, for example, from Exercise 5. The goal of this exercise is to show that this assertion still holds for at most 8 points in $\mathbb{P}^2_{\mathbb{C}}$. Note that this result is sharp (see Exercise 5).

Before we proceed, let us make small observation. Let $[a_{11} : a_{12} : a_{13}]$, $[a_{21} : a_{22} : a_{23}]$, and $[a_{31} : a_{32} : a_{33}]$ be points in $\mathbb{P}^2_{\mathbb{C}}$. Then these three points are contained in one line in $\mathbb{P}^2_{\mathbb{C}}$ if and only if the determinant of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is zero. Likewise, the determinant of this matrix is zero if and only if the lines $a_{11}x + a_{12}y + a_{13}z = 0$, $a_{21}x + a_{22}y + a_{23}z = 0$ and $a_{31}x + a_{32}y + a_{33}z = 0$ all pass through one point in $\mathbb{P}^2_{\mathbb{C}}$.

(a) We have a finite subset Σ in $\mathbb{P}^2_{\mathbb{C}}$ such that $|\Sigma| \leqslant 6$ and Σ is not contained in one line in $\mathbb{P}^2_{\mathbb{C}}$. We have to show that there exists a line $L \subset \mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ. To do this, denote by n the largest number of points in Σ that are contained in a single line in $\mathbb{P}^2_{\mathbb{C}}$. By assumption, $n < |\Sigma| \leqslant 6$. If $n = 2$, then we are done. Thus, we assume that $n \geqslant 3$. Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ that contains n points in Σ. We proceed in three steps.

(i) Suppose that $|\Sigma| = n + 1$. Then Σ contains exactly one point P that is not contained in L. Let Q be any point in $\Sigma \cap L$, and let L' be a line in $\mathbb{P}^2_{\mathbb{C}}$. that passes through P and Q. Then $L \cap L' = Q$, because two different lines in $\mathbb{P}^2_{\mathbb{C}}$ intersect by one point. Thus, L' contains exactly two points of the set Σ, which are the points P and Q.

(ii) Suppose that $|\Sigma| = n + 2$. Then Σ contains exactly two points that are not contained in L. Denote them by P_1 and P_2. Let Q be any point in $\Sigma \cap L$, and let L' be a line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P_1 and Q. Then $L \cap L' = Q$. Thus, either L' contains exactly two points of the set Σ, which are the points P_1 and Q, or L' contains exactly three points of the set Σ, which are the points P_1, P_2 and Q. In the former case, we are done: L' is the line we are looking for. Thus, we may assume that L' contains the points P_1, P_2 and Q. Let $\widehat{Q}$ be a point in $\Sigma \cap L$ that is different from Q (it exists because $n \geqslant 3$), and let $\widehat{L}'$ be a line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P_1 and $\widehat{Q}$. Then $L \cap \widehat{L}' = \widehat{Q}$ and $L' \cap \widehat{L}' = P_1$. This shows that $\widehat{L}'$ does not contain P_2, and $\widehat{L}'$ does not contain any point in $\Sigma \cap L$ that is different from $\widehat{Q}$. Thus, $\widehat{L}'$ is the line we are looking for.

(iii) Suppose that $|\Sigma| \geqslant n + 3$. Since $n \geqslant 3$ and $n + 3 \leqslant |\Sigma| \leqslant 6$, we see that $|\Sigma| = 6$ and $n = 3$, so that Σ contains exactly three points that are not in L. Denote them by P_1, P_2, and P_3. Similarly, denote the points in $\Sigma \cap L$ by Q_1, Q_2, and Q_3. Then denote by L_{ij} the line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P_i and Q_j. Then

$$L_{ij} \cap L = Q_i.$$

Take the line L_{11}. Then it does not contain Q_2 and Q_3, because $L_{11} \cap L = Q_1$. Thus, if L_{11} does not contain P_2 and P_3, then we are done: L_{11} is the line we need. Without loss of generality, we may assume that $P_2 \in L_{11}$. Then P_3 is not contained in L_{11}, because $n = 3$. Now let us do the same trick with the line L_{12}. Since $L_{11} \cap L_{12} = P_1$, the point P_2 is not contained in L_{12}. Hence, if P_3 is not contained in L_{12}, then L_{12} is the line we are looking for. Thus, we may assume that L_{12} contains P_3. Now (finally) we take the line L_{13}. It is different from L_{11} and L_{12}, because $L_{11} \cap L = Q_1$, $L_{12} \cap L = Q_2$, $L_{13} \cap L = Q_3$. On the other hand, the points P_1 and P_2 are not contained in the line L_{13}, because $L_{11} \cap L_{12} \cap L_{13} = P_1$. Hence, the line L_{13} is the line we are looking for.

(b) Now Σ is a finite subset in $\mathbb{P}^2_{\mathbb{C}}$ such that $|\Sigma| = 7$ and Σ is not contained in one line in $\mathbb{P}^2_{\mathbb{C}}$. We have to show that there exists a line $L \subset \mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ. As above, we denote by n the largest number of points in Σ that are contained in a single line in $\mathbb{P}^2_{\mathbb{C}}$. By assumption, $n < |\Sigma| \leqslant 7$. If $n = 2$, then we are done. Thus, we assume that $n \geqslant 3$. Let L be a line in $\mathbb{P}^2_{\mathbb{C}}$ that contains n points in Σ. We proceed in four steps.

(i) Suppose that $n = 6$. Then Σ contains exactly one point P that is not contained in L, so that every line that passes through P and any point in $\Sigma \cap L$ contains exactly two points in Σ.

(ii) Suppose that $n = 5$. Then Σ contains exactly two points that are not contained in L. Denote them by P_1 and P_2. Let Q be any point in $\Sigma \cap L$, and let L' be a line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P_1 and Q. Then $L \cap L' = Q$. Thus, either L' contains exactly two points of the set Σ, which are the points P_1 and Q, or L' contains exactly three points of the set Σ, which are the points P_1, P_2 and Q. In the former case, we are done: L' is the line we are looking for. Thus, we may assume that L' contains the points P_1, P_2 and Q. Let $\widehat{Q}$ be a point in $\Sigma \cap L$ that is different from Q (it exists because $n = 5$), and let $\widehat{L}'$ be a line in $\mathbb{FP}^2$ that passes through P_1 and $\widehat{Q}$. Then

$$L \cap \widehat{L}' = \widehat{Q}$$

and $L' \cap \widehat{L}' = P_1$. This shows that $\widehat{L}'$ does not contain P_2, and $\widehat{L}'$ does not contain any point in $\Sigma \cap L$ that is different from $\widehat{Q}$. Thus, $\widehat{L}'$ is the line we are looking for.

(iii) Suppose that $n = 4$. Then Σ contains exactly three points that are not in L. Denote them by P_1, P_2, and P_3. Similarly, denote the points in $\Sigma \cap L$ by Q_1, Q_2, and Q_3. Denote by L_{ij} the line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P_i and Q_j. Then L_{11} does not contain Q_2 and Q_3, because

$$L_{11} \cap L = Q_1.$$

Thus, if L_{11} does not contain P_2 and P_3, then we are done: the line L_{11} is the line we need. Thus, without loss of generality, we may assume that $P_2 \in L_{11}$. Then P_3 is not contained in L_{11}, because $n = 3$. Since $L_{11} \cap L_{12} = P_1$, the point P_2 is not contained in L_{12}. Hence, if P_3 is not contained in L_{12}, then L_{12} is the line we are looking for. Thus, we may assume that L_{12} contains P_3. Then the points P_2 and P_3 are not contained in the line L_{13}, because

$$L_{11} \cap L_{12} \cap L_{13} = P_1.$$

Hence, the line L_{13} is the line we are looking for.

(iv) Suppose that $n = 3$. Then L contains exactly three points in Σ. Denote them by Q_1, Q_2, Q_3. Since $|\Sigma| = 7$, the subset Σ contains exactly four points that are not in L. Denote them by P_1, P_2, P_3, and P_4. Let L_{ij} be the line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P_i and Q_j. If $L_{11} \cap \Sigma = \{P_1, Q_1\}$, then we are done ($L_{11}$ is the line we are looking for). So, we may assume that $P_2 \in L_{11}$, so that $L_{11} = L_{21}$ and

$$L_{11} \cap \Sigma = \big\{P_1, Q_1, P_2\big\},$$

because $n = 3$. Similarly, if $L_{12} \cap \Sigma = \{P_1, Q_2\}$, then we are done ($L_{12}$ is the line we are looking for). Thus, we ay assume that L_{12} contains one more point in Σ. This point is not Q_1, Q_3 or P_2, because $L_{12} \cap L = Q_2$ and $L_{12} \cap L_{11} = P_1$. Hence, either L_{12} contains P_3 or L_{12} contains P_4. Without loss of generality, we may assume that L_{12} contains P_3, so that

$$L_{12} \cap \Sigma = \big\{P_1, Q_2, P_3\big\}.$$

Applying the same arguments to the line $L13$, we see that we may assume that

$$L_{13} \cap \Sigma = \{P_1, Q_3, P_4\}.$$

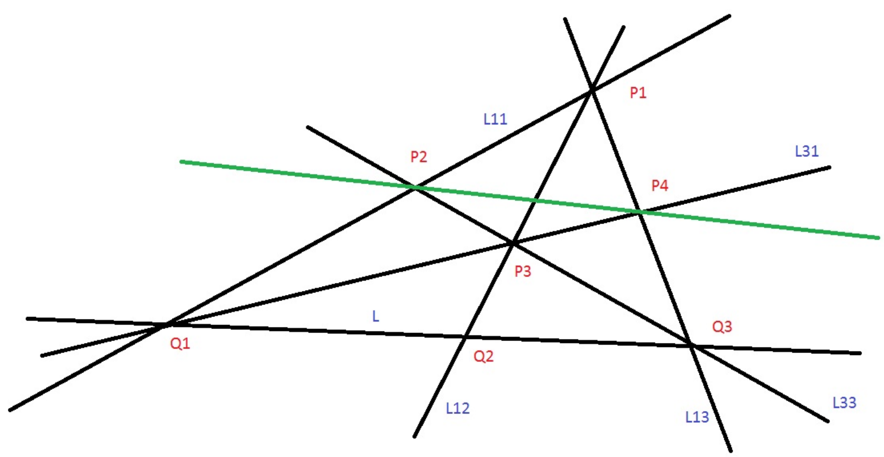
Now let us look at the points Q_1, P_1, Q_3, and P_3. No three of them are contained in one line in $\mathbb{P}^2_{\mathbb{C}}$, because the lines L, L_{11} and L_{13} contain exactly two points among Q_1, P_1, Q_3, and P_3. Thus, there exists projective transformation $\mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ that maps the points Q_1, P_1, Q_3, P_3 to the points $[0:0:1]$, $[0:1:0]$, $[0:0:1]$, $[1:1:1]$, respectively. This was proved in lecture 2. Thus, we may assume that $Q_1 = [0:0:1]$, $P_1 = [0:1:0]$, $Q_3 = [1:0:0]$, $P_3 = [1:1:1]$. Then L is given by $y = 0$, the line L_{11} is given by $x = 0$, the line L_{13} is given by $z = 0$, and the line L_{12} is given by $x = z$. Similarly, the line L_{33} is given by $y = z$. In particular, this implies that $Q_3 = [1:0:1]$, because $Q_3 = L \cap L_{13}$. Moreover, if $L_{33} \cap \Sigma = \{P_3, Q_3\}$, then we are done. Thus, we may assume that L_{33} contains another point in Σ. Since

$$L_{33} \cap L = L_{33} \cap L_{13} = Q_3,$$

the line L_{33} does not contain the points Q_1, Q_2, P_1, and P_4. Thus, the line L_{33} contains the point P_2, which implies that P_2 is given by $x = y - z = 0$, because $P_2 \in L_{11}$. Thus, $P_3 = [0:1:1]$. Similarly, the line L_{31} is given by $x = y$. If $L_{31} \cap \Sigma = \{P_3, Q_1\}$, then we are done. Thus, we may assume that L_{31} contains another point in Σ. Since $L_{31} \cap L = L_{31} \cap L_{13} = Q_3$, the line L_{33} does not contain the points Q_2, Q_3, P_1, and P_2. Thus, the line L_{31} contains the point P_4, which implies that P_4 is given by

$$z = x - y = 0,$$

because $P_4 \in L_{13}$. Thus, $P_4 = [1:1:0]$. Thus, our subset Σ is explicitly described. Let ℓ be the line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P_2 and P_4. Then ℓ is given by the equation $x - y + z = 0$. This line does not contain P_1, Q_1, Q_3, and P_3. So far, we never used any property of the field $\mathbb{C}$ that is specific to $\mathbb{C}$. We are going to use one of them now: $2 \neq 0$ in $\mathbb{C}$, so that the proof works for any field of characteristic $\neq 2$. This implies that ℓ does not contain Q_2, because $2 \neq 0$ in $\mathbb{C}$. Thus, ℓ is the line we are looking for! The proof can be illustrated by this picture:



The last step of the proof crucially depends on the fact that $2 \neq 0$. In fact, this is the only point that we used explicit properties of complex numbers. Thus, the whole proof is valid for all projective planes including the projective plane $\mathbb{P}^2_{\mathbb{F}_p}$ with $p \neq 2$. However, the proof in the case $|\Sigma| = 7$ and $n = 3$ does not work over $\mathbb{F}_2$, because

$2=0$ in $\mathbb{F}_2$. Moreover, in this case, the required assertion is wrong. Indeed, the finite projective plane $\mathbb{P}^2_{\mathbb{F}_2}$ consists of 7 points. These points are

$$[0:0:1], [0:1:0], [1:0:0], [0:1:1], [1:0:1], [1:1:0], [1:1:1].$$

On the other hand, there are exactly 7 lines in $\mathbb{P}^2_{\mathbb{F}_2}$. They are given by equations

$$x=0, y=0, z=0, x+y=0, x+z=0, y+z=0, x+y+z=0,$$

respectively. Substituting seven points of $\mathbb{P}^2_{\mathbb{F}_2}$ into these equations, we immediately see that every line L in $\mathbb{P}^2_{\mathbb{F}_2}$ contains exactly three points.

The projective plane $\mathbb{P}^2_{\mathbb{F}_2}$ is called *Fano plane*. It can be illustrated by the following tatoo:

It display all 7 lines and all 7 points in $\mathbb{P}^2_{\mathbb{F}_2}$.

(c) Now Σ is a finite subset in $\mathbb{P}^2_{\mathbb{C}}$ such that $|\Sigma|=8$ and Σ is not contained in one line in $\mathbb{P}^2_{\mathbb{C}}$. We have to show that there exists a line $L\subset\mathbb{P}^2_{\mathbb{C}}$ that contains exactly two points of the set Σ. As above, we denote by n the largest number of points in Σ that are contained in a single line in $\mathbb{P}^2_{\mathbb{C}}$. Then $n\leqslant 7$. We may assume that $n\geqslant 3$.

Let L be the line in $\mathbb{P}^2_{\mathbb{C}}$ that contains n points in Σ.

(i) Suppose that $n=6$ or $n=7$. Let Q be any point in $\Sigma\cap L$. If $n=7$, then Σ contains exactly one point P that is not contained in the line L. In this case, the line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P and Q is the line we are looking for. Thus, we may assume that $n=6$. Then Σ contains exactly two points that are not contained in the line L. Denote them by P_1 and P_2. Let Q be any point in $\Sigma\cap L$, and let L' be a line in $\mathbb{P}^2$ that passes through P_1 and Q. Then

$$L\cap L'=Q.$$

Thus, either L' contains exactly two points of the set Σ, which are the points P_1 and Q, or L' contains exactly three points of the set Σ, which are the points P_1,

P_2 and Q. In the former case, we are done: the line L' is the line we are looking for. Thus, we may assume that L' contains the points P_1, P_2 and Q. Let $\widehat{Q}$ be a point in $\Sigma \cap L$ that is different from Q (it exists because $n \geqslant 3$), and let $\widehat{L}'$ be a line in $\mathbb{P}^2$ that passes through P_1 and $\widehat{Q}$. Then

$$L \cap \widehat{L}' = \widehat{Q}$$

and $L' \cap \widehat{L}' = P_1$. This shows that $\widehat{L}'$ does not contain P_2, and $\widehat{L}'$ does not contain any point in $\Sigma \cap L$ that is different from $\widehat{Q}$. Thus, the line $\widehat{L}'$ is the line we are looking for.

(ii) Suppose that $n = 5$. Then Σ contains exactly three points that are not in the line L. Denote them by P_1, P_2, and P_3. Since $n \geqslant 3$, the set $\Sigma \cap L$ contains at least three points. Let Q_1, Q_2 and Q_3 be any three of them. Denote by L_{ij} the line in $\mathbb{P}^2$ that passes through P_i and Q_j. Then

$$L_{ij} \cap L = Q_i.$$

Take the line L_{11}. Then it does not contain any point in $\Sigma \cap L$, because

$$L_{11} \cap L = Q_1.$$

Thus, if L_{11} does not contain P_2 and P_3, then we are done: the line L_{11} is the line we need. Without loss of generality, we may assume that $P_2 \in L_{11}$. Then P_3 is not contained in L_{11}, because $n = 3$. Now let us do the same trick with the line L_{12}. Since

$$L_{11} \cap L_{12} = P_1,$$

the point P_2 is not contained in L_{12}. Hence, if P_3 is not contained in L_{12}, then L_{12} is the line we are looking for. Thus, we may assume that L_{12} contains P_3. Now (finally) we take the line L_{13}. It is different from L_{11} and L_{12}, because $L_{11} \cap L = Q_1$, $L_{12} \cap L = Q_2$ and $L_{13} \cap L = Q_3$. On the other hand, the points P_1 and P_2 are not contained in the line L_{13}, because

$$L_{11} \cap L_{12} \cap L_{13} = P_1.$$

Hence, the line L_{13} is the line we are looking for.

(iii) Suppose that $n = 4$. Then Σ contains exactly four points that are not in L. Denote them by P_1, P_2, P_3 and P_4. The set $\Sigma \cap L$ contains exactly 4 points. Denote them by Q_1, Q_2, Q_3 and Q_4. Denote by L_{ij} the line in $\mathbb{P}^2$ that passes through P_i and Q_j. Then each line among L_{11}, L_{12}, L_{13} and L_{14} contains exactly one point in $\Sigma \cap L$. Moreover, any two of these four lines intersects only in the point P_1. Hence, at least one of them does not contain any points among P_2, P_3 and P_4, so that it contains exactly two points in Σ. This case is done.

(iv) Finally we suppose that $n = 3$. This case is similar to the one we just considered. Indeed, the set Σ contains exactly five points that are not in L. Denote them by P_1, P_2, P_3, P_4 and P_5. The set $\Sigma \cap L$ contains exactly 3 points. Denote them by Q_1, Q_2, Q_3 Denote by L_{1i} the line in $\mathbb{P}^2$ that passes through the points P_1 and Q_i. Then each line among L_{11}, L_{12} and L_{13} contains exactly one points in $\Sigma \cap L$. Moreover, each of them cannot contain more than one point among P_2, P_3, P_4 and P_5, because $n = 3$. Thus, without loss of generality, we may assume that $P_2 \in L_{11}$, $P_3 \in L_{12}$, $P_4 \in L_{13}$. Then

$$P_5 \notin L \cup L_{11} \cup L_{12} \cup L_{13}.$$

Then the line that passes through P_1 and P_5 does not contain other points of Σ.

$\square$

Exercise 3. Do the following:

(a) Find all lines in $\mathbb{P}^2_{\mathbb{C}}$ that contains exactly 2 points among

$$[0:0:1], [0:1:1], [1:1:-1], [1:3:1], [2:5:1], [1:1:1], [1:4:2].$$

(b) Find a smooth conic $C \subset \mathbb{P}^2_{\mathbb{C}}$ such that C contains the points

$$[0:0:1], [0:1:0], [1:0:0],$$

the line in $\mathbb{P}^2_{\mathbb{C}}$ that tangents the conic C at the point $[1:0:0]$ is given by $y - z = 0$, and the line in $\mathbb{P}^2_{\mathbb{C}}$ that tangents C at the point $[0:0:1]$ is given by $y + 2x = 0$.

(c) Find all smooth conics in $\mathbb{P}^2_{\mathbb{C}}$ that passes through

$$[1:0:2], [3:1:2], [1:2:1], [1:1:1],$$

and tangent to the line $x + 2y + z = 0$.

Solution. (a) Put $P_1 = [0:0:1]$, $P_2 = [0:1:1]$, $P_3 = [1:1:-1]$, $P_4 = [1:3:1]$, $P_5 = [2:5:1]$, $P_6 = [1:1:1]$ and $P_7 = [1:4:2]$. For every two points P_i and P_j with $i < j$, there is a unique line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through them. Denote this line by L_{ij}. A priori this gives us 21 lines L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{17}, L_{23}, L_{24}, L_{25}, L_{26}, L_{27}, L_{34}, L_{35}, L_{36}, L_{37}, L_{45}, L_{46}, L_{47}, L_{56}, L_{57} and L_{67}. However, many of them coincide.

Let us find the equations of the lines L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{17}, L_{23}, L_{24}, L_{25}, L_{26}, L_{27}, L_{34}, L_{35}, L_{36}, L_{37}, L_{45}, L_{46}, L_{47}, L_{56}, L_{57} and L_{67}. The line L_{12} is given by $x = 0$, the line L_{13} is given by $x - y = 0$, the line L_{14} is given by $3x - y = 0$, the line L_{15} is given by $5x - 2y = 0$, the line L_{16} is given by $x - y = 0$, the line L_{17} is given by $4x - y = 0$. Thus, we have $L_{13} = L_{16}$.

The line L_{23} is given by

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & -1 \\ x & y & z \end{vmatrix} = y - z - 2x = 0,$$

which can be rewritten as $2x - y + z = 0$. The line L_{24} is given by

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ x & y & z \end{vmatrix} = y - z - 2x = 0,$$

so that $L_{24} = L_{23}$. The line L_{25} is given by

$$\begin{vmatrix} 0 & 1 & 1 \\ 2 & 5 & 1 \\ x & y & z \end{vmatrix} = 2y - 2z - 4x = 0,$$

which implies that $L_{25} = L_{24} = L_{23}$. The line L_{26} is given by $y - z = 0$. The line L_{27} is given by

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 2 \\ x & y & z \end{vmatrix} = y - z - 2x = 0,$$

so that $L_{27} = L_{25} = L_{24} = L_{23}$. Thus, we see that the point P_2, P_3, P_4, P_5 and P_7 are all contained in one line L_{23}. This gives

$$L_{23} = L_{24} = L_{25} = L_{27} = L_{34} = L_{35} = L_{37} = L_{45} = L_{47} = L_{57}.$$

It remains to find L_{36}, L_{46}, L_{56} and L_{67}. We already know that $L_{13} = L_{16}$, so that $L_{36} = L_{13} = L_{16}$. The line L_{46} is given by

$$\begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = 2x - 2z = 0,$$

so that L_{46} is given by $x - z = 0$. The line L_{56} is given by

$$\begin{vmatrix} 2 & 5 & 1 \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = 4x - y - 3z = 0,$$

so that L_{56} is given by $4x - y - 3z = 0$. Finally L_{67} is given by

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ x & y & z \end{vmatrix} = 2x - y + 3z = 0,$$

so that L_{67} is given by $2x - y - 3z = 0$.

Let us sum up what we found. The line L_{12} is given by $x = 0$, the line $L_{13} = L_{16} = L_{36}$ is given by $x - y = 0$, the line L_{14} is given by $3x - y = 0$, the line L_{15} is given by $5x - 2y = 0$, the line L_{17} is given by $4x - y = 0$, the line $L_{23} = L_{24} = L_{25} = L_{27} = L_{34} = L_{35} = L_{37} = L_{45} = L_{47} = L_{57}$ is given by $2x - y + z = 0$, the line L_{26} is given by $y - z = 0$, the line L_{46} is given by $x - z = 0$, the line L_{56} is given by $4x - y - 3z = 0$, and the line L_{67} is given by $2x - y - 3z = 0$. Thus, we have a picture like this

We found 10 lines that contains at lease two points among P_1, P_2, P_3, P_4, P_5, P_6, P_7. Among them only the lines L_{12}, L_{14}, L_{15}, L_{17}, L_{26}, L_{46}, L_{56} and L_{67} contains exactly 2 points among P_1, P_2, P_3, P_4, P_5, P_6, P_7.

(b) Let C be a smooth smooth conic in $\mathbb{P}^2_{\mathbb{C}}$ that passes through the points $[0:0:1]$, $[0:1:0]$, $[1:0:0]$. Then C is given by

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \delta xy + \epsilon xz + \zeta yz = 0$$

for some $[\alpha : \beta : \gamma : \delta : \epsilon : \zeta] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\begin{cases} \gamma = 0, \\ \beta = 0, \\ \gamma = 0, \end{cases}$$

so that C is given by $\delta xy + \epsilon xz + \zeta yz = 0$.

Put $f(x, y, z) = \delta xy + \epsilon xz + \zeta yz$. For every point $[a : b : c] \in C$, the line in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$\frac{\partial f(a,b,c)}{\partial x} x + \frac{\partial f(a,b,c)}{\partial y} y + \frac{\partial f(a,b,c)}{\partial z} z = 0$$

tangents the conic C at the point $[a : b : c]$. On the other hand, we have

$$\begin{cases} \dfrac{\partial f(x,y,z)}{\partial x} = \delta y + \epsilon z, \\ \dfrac{\partial f(x,y,z)}{\partial y} = \delta x + \zeta z, \\ \dfrac{\partial f(x,y,z)}{\partial z} = \epsilon x + \zeta y. \end{cases}$$

We get

$$\begin{cases} \dfrac{\partial f(1,0,0)}{\partial x} = 0, \\ \dfrac{\partial f(1,0,0)}{\partial y} = \delta, \\ \dfrac{\partial f(1,0,0)}{\partial z} = \epsilon. \end{cases}$$

so that the tangent line to the conic C at the point $[1:0:0]$ is given by $\delta y+\epsilon z=0$. Similarly, we see that the tangent line to the conic C at the point $[0:0:1]$ is given by $\epsilon x+\zeta y=0$.

Note that we can find the tangent lines to C at the points $[1:0:0]$ and $[0:0:1]$ simply by taking the Taylor expansion of the affine equation of the curve C in the appropriate charts of $\mathbb{P}^2_{\mathbb{C}}$. For instance, let U be the open subset in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $x\neq 0$. Then we can identify $U=\mathbb{C}^2$ with coordinates $\overline{y}=\frac{y}{x}$ and $\overline{z}=\frac{z}{x}$. Then $C\cap U$ is given by

$$\delta\overline{y}+\epsilon\overline{z}+\zeta\overline{yz}=0,$$

so that the tangent line in U to C at the point $(0,0)$ is just the line $\delta\overline{y}+\epsilon\overline{z}=0$. Thus, the tangent line in $\mathbb{P}^2_{\mathbb{C}}$ to C at the point $[1:0:0]$ is given by $\delta y+\epsilon z=0$..

If the line in $\mathbb{P}^2_{\mathbb{C}}$ that tangents the conic C at the point $[1:0:0]$ is given by $y-z=0$, then the lines $\delta y+\epsilon z=0$ and $y-z$ coincide, so that $\delta=-\epsilon$. Similarly if the line in $\mathbb{P}^2_{\mathbb{C}}$ that tangents C at the point $[0:0:1]$ is given by $y+2x=0$, then the lines $\epsilon x+\zeta y=0$ and $y+2x=0$ coincide, so that $\epsilon=2\zeta$. Thus, we have

$$\begin{cases}\delta=-\epsilon,\\ \epsilon=2\zeta.\end{cases}$$

We can put $\zeta=1$, so that $\epsilon=2$ and $\delta=-2$. Thus, the conic C, so that $f(x,y,z)=-2xy+2xz+yz$, and C is given by $f(x,y,z)=0$.

We must check that C is smooth. If $[a:b:c]$ is a singular point of the conic C, then

$$\frac{\partial f(a,b,c)}{\partial x}=\frac{\partial f(a,b,c)}{\partial y}=\frac{\partial f(a,b,c)}{\partial z}=0.$$

On the other hand, we have

$$\begin{cases}\dfrac{\partial f(x,y,z)}{\partial x}=-2y+2z,\\ \dfrac{\partial f(x,y,z)}{\partial y}=-2x+z,\\ \dfrac{\partial f(x,y,z)}{\partial z}=2x+y.\end{cases}$$

Thus, if $[a:b:c]$ is a singular point of the conic C, then

$$\begin{cases}-2b+2c=0,\\ -2a+c=0,\\ 2a+b=0.\end{cases}$$

This system of linear equations has unique solution $a=b=c=0$, which does not correspond to any point in $\mathbb{P}^2_{\mathbb{C}}$. Thus, the conic C is smooth.

(c) Let C be a smooth smooth conic in $\mathbb{P}^2_{\mathbb{C}}$. Then C is given by

$$ax^2+by^2+cz^2+dxy+exz+fyz=0$$

for some $[a:b:c:d:e:f]\in\mathbb{P}^2_{\mathbb{C}}$. Suppose that C contains the points $[1:0:2]$, $[3:1:2]$, $[1:2:1]$ and $[1:1:1]$. Then

$$\begin{cases}a+4c+2e=0,\\ 9a+b+4c+3d+6e+2f=0,\\ a+4b+c+2d+e+2f=0,\\ a+b+c+d+e+f.\end{cases}$$

Solving this system of equations, we get

$$[a:b:c:d:e:f]=[-8s-4t:-3s-t:2s:49s+19t:2t:-40s-16t]$$

for any $[s:t] \in \mathbb{P}^1_{\mathbb{C}}$.

Thus, if $e=0$, then C is given by

$$-8x^2+49xy-3y^2-40yz+2z^2=0.$$

In this case, the intersection of C and the line $x+2y+z=0$ consists of two different points $[5:-6:7]$ and $[-17:-1:19]$, which implies that C does not tangent the line $x+2y+z=0$. Thus, we may assume that $e \neq 0$. Then, scaling by the defining equation of C by $\frac{1}{e}$, we may assume that $e=1$.

We see that C is given by

$$(-8s-4)x^2+(-3s-1)y^2+2sz^2+(49s+19)xy+2xz+(-40s-16)yz=0.$$

Then its intersection with the line $x+2y+z=0$ is given by

$$\begin{cases} (-8s-4)x^2+(-3s-1)y^2+2sz^2+(49s+19)xy+2xz+(-40s-16)yz=0, \\ x+2y+z=0. \end{cases}$$

This gives $-133sy^2-121syz-6sz^2-55y^2-55yz-6z^2=0$, so that

$$[y:z]=[-121s-55\pm\sqrt{11449s^2+8798s+1705}:266s+110].$$

Thus, the line $x+2y+z=0$ is tangent to C if and only if $11449s^2+8798s+1705=0$. This gives

$$s=-\frac{4399}{11449}\pm\frac{168\sqrt{6}}{11449}i$$

Thus, we see that either C is given by

$$\left(5302+672i\sqrt{6}\right)x^2-\left(990+4116i\sqrt{6}\right)yx-11449zx-\left(874-252i\sqrt{6}\right)y^2+ \\ +\left(3612+3360i\sqrt{6}\right)zy+\left(4399+168i\sqrt{6}\right)z^2=0$$

or by a complex conjugated equation. Taking partial derivatives, we see that in both cases, the conic C is smooth.

□

Exercise 4. Observe that no three points among the four points $[1:2:3]$, $[1:0:-1]$, $[2:5:1]$ and $[-1:7:1]$ in $\mathbb{P}^2_{\mathbb{C}}$ are collinear.

(a) Find the projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi([1:2:3])=[1:0:0]$, $\phi([1:0:-1])=[0:1:0]$, $\phi([2:5:1])=[0:0:1]$ and $\phi([-1:7:1])=[1:1:1]$.

(b) Let $\mathcal{C}$ be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$-xy+2y^2-3xz+7yz+3z^2=0.$$

Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C})$ is given by $xy=0$.

(c) Let $\mathcal{C}$ be the conic in $\mathbb{P}^2$ that is given by

$$x^2+xy-2y^2+3xz+3yz+z^2=0.$$

Then $\mathcal{C}$ contains the point $[-2:1:3]$. Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi([-2:1:3])=[0:0:1]$ and $\phi(\mathcal{C})$ is given by $xz+y^2=0$.

Solution. No three points among $[1:2:3]$, $[1:0:-1]$ $[2:5:1]$ and $[-1:7:1]$ are collinear, because

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 2 & 5 & 1 \end{pmatrix} = 14 \neq 0, \det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ -1 & 7 & 1 \end{pmatrix} = 28 \neq 0,$$

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ -1 & 7 & 1 \end{pmatrix} = 49 \neq 0, \det\begin{pmatrix} 1 & 0 & -1 \\ 2 & 5 & 1 \\ -1 & 7 & 1 \end{pmatrix} = -21 \neq 0.$$

(a) By definition, the transformation ϕ is given by

$$\big[x:y:z\big] \mapsto \big[a_{11}x + a_{12}y + a_{13}z : a_{21}x + a_{22}y + a_{23}z : a_{31}x + a_{32}y + a_{33}z\big]$$

for some complex numbers a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32} and a_{33}. Let us find these numbers by *brute force*. By assumption, we have

$$\begin{cases} \phi\big([1:2:3]\big) = \big[a_{11} + 2a_{12}y + 3a_{13} : a_{21} + 2a_{22} + 3a_{23} : a_{31} + 2a_{32} + 3a_{33}\big] = [1:0:0], \\ \phi\big([1:0:-1]\big) = \big[a_{11} - a_{13} : a_{21} - a_{23} : a_{31} - a_{33}\big] = [0:1:0], \\ \phi\big([2:5:1]\big) = \big[2a_{11} + 5a_{12} + a_{13} : 2a_{21} + 5a_{22} + a_{23} : 2a_{31} + 5a_{32} + a_{33}\big] = [0:0:1], \\ \phi\big([-1:7:1]\big) = \big[-a_{11} + 7a_{12} + a_{13} : -a_{21} + 7a_{22} + a_{23} : -a_{31} + 7a_{32} + a_{33}\big] = [1:1:1]. \end{cases}$$

This gives us system of equations

$$\begin{cases} a_{11} + 2a_{12} + 3a_{13} = a, \\ a_{21} + 2a_{22} + 3a_{23} = 0, \\ a_{31} + 2a_{32} + 3a_{33} = 0, \\ a_{11} - a_{13} = 0, \\ a_{21} - a_{23} = b, \\ a_{31} - a_{33} = 0, \\ 2a_{11} + 5a_{12} + a_{13} = 0, \\ 2a_{21} + 5a_{22} + a_{23} = 0, \\ 2a_{31} + 5a_{32} + a_{33} = c, \\ -a_{11} + 7a_{12} + a_{13} = d, \\ -a_{21} + 7a_{22} + a_{23} = d, \\ -a_{31} + 7a_{32} + a_{33} = d, \end{cases}$$

where a, b, c and d are some complex numbers. Thus, we have 12 linear equations and 13 variables: a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}, a, b, c and d. Using the rank–nullity theorem, we see that solutions form at least one-dimensional vector space. However, we do not want solutions with $d = 0$, because there exists no such point in $\mathbb{P}^2_{\mathbb{C}}$ as $[0:0:0]$. Thus, we may add one extra equation $d = 1$. Solving the resulting system, we get $a_{11} = -\frac{5}{21}$, $a_{12} = \frac{1}{7}$, $a_{13} = -\frac{5}{21}$, $a_{21} = -\frac{13}{49}$, $a_{22} = \frac{5}{49}$, $a_{23} = \frac{1}{49}$, $a_{31} = -\frac{1}{14}$, $a_{32} = \frac{1}{7}$, $a_{33} = -\frac{1}{14}$, $a = -\frac{2}{3}$, $b = -\frac{2}{7}$, $c = \frac{1}{2}$ and $d = 1$. Thus, the required projective transformation ϕ is given by

$$\big[x:y:z\big] \mapsto \Big[-\frac{5x}{21} + \frac{y}{7} - \frac{z}{21} : -\frac{13x}{49} + \frac{5y}{49} + \frac{z}{49} : -\frac{x}{14} + \frac{y}{7} - \frac{z}{14}\Big].$$

Multiplying all entries by $49 \cdot 2 \cdot 3 = 294$ or recomputing the system of equation with $d = 294$, we can can rewrite the formula for ϕ as

$$\big[x:y:z\big] \mapsto \big[-70x + 42y - 70z : -78x + 30y + 6z : -21x + 42y - 21z\big].$$

Now let us find ϕ again using the idea described in lecture 2. Let α be the projective transformation that is induced by the linear transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 5 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and let β be the inverse of α. Then $\beta([1:2:3]) = [1:0:0]$, $\beta([1:0:-1]) = [0:1:0]$ and $\beta([2:5:1]) = [0:0:1]$. Observe that

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 2 & 5 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{5}{14} & \frac{-3}{14} & \frac{5}{14} \\ \frac{13}{14} & -\frac{5}{14} & -\frac{1}{14} \\ -\frac{2}{14} & \frac{4}{14} & -\frac{2}{14} \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 5 & -3 & 5 \\ 13 & -5 & -1 \\ -2 & 4 & -2 \end{pmatrix}.$$

This shows that β is given by

$$\big[x:y:z\big] \mapsto \big[5x - 3y + 5z : 13x - 5y - z : -2x + 4y - 2z\big].$$

Then $\beta([-1:7:1]) = [-21:-49:28]$. Let γ be the projective transformation that is given by

$$\big[x:y:z\big] \mapsto \big[-\frac{x}{21} : -\frac{y}{49} : \frac{z}{28}\big] = \big[28x : 12y : -21z\big].$$

Then the composition $\gamma \circ \beta$ is the projective transformation ϕ, which we already found by brute force. This can be verified by as follows:

$$\begin{pmatrix} 28 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 21 \end{pmatrix} \begin{pmatrix} 5 & 13 & -2 \\ -3 & -5 & 4 \\ 5 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 140 & -84 & 140 \\ 156 & -60 & -12 \\ -42 & 84 & -42 \end{pmatrix} = -2 \begin{pmatrix} -70 & 42 & -70 \\ -78 & 30 & 6 \\ -21 & 42 & -21 \end{pmatrix}.$$

(b) Observe that

$$-xy + 2y^2 - 3xz + 7yz + 3z^2 = -\big(y + 3z\big)\big(x - 2y - z\big),$$

so that $\mathcal{C}$ is a union of the lines $y + 3z = 0$ and $x - 2y - z = 0$. Let

$$\begin{cases} \mathbf{x} = y + 3z, \\ \mathbf{y} = x - 2y - z, \\ \mathbf{z} = z. \end{cases}$$

Then $x = 2\mathbf{x} + \mathbf{y} - 5\mathbf{z}$, $y = \mathbf{x} - 3\mathbf{z}$, $z = \mathbf{z}$, which gives

$$-xy + 2y^2 - 3xz + 7yz + 3z^2 = \mathbf{xy}.$$

Let ϕ be the map $[x:y:z] \mapsto [y + 3z : x - 2y - z : z]$. Then $\phi(\mathcal{C})$ is given by $xy = 0$.

(c) First, we want to map the point $[-2:1:3]$ to the point $[0:0:1]$. To do this, we should choose x_1, y_1, z_1 in terms of x, y, z such that $[-2:1:3]$ is given by $x_1 = 0$ and $y_1 = 0$. For instance, we can choose x_1, y_1 and z_1 using this formula:

$$\begin{cases} x_1 = x + 2y, \\ y_1 = z - 3y, \\ z_1 = z. \end{cases}$$

Then the old coordinates x, y and z are expressed by

$$\begin{cases} x = x_1 + \frac{2}{3}y_1 - \frac{2}{3}z_1, \\ y = \frac{z_1}{3} - \frac{y_1}{3}, \\ z = z_1. \end{cases}$$

Substituting this into $x^2 + xy - 2y^2 + 3xz + 3yz + z^2$, we see that $\mathcal{C}$ is given by the equation
$$x_1y_1 + 2x_1z_1 + y_1z_1 + x_1^2 = 0.$$
Observe that $x_1y_1 + 2x_1z_1 + y_1z_1 + x_1^2 = x_1y_1 + (2x_1 + y_1)z_1 + x_1^2$. Now we choose the coordinates x_2, y_2 and z_2 as follows:
$$\begin{cases} x_2 = 2x_1 + y_1, \\ y_2 = y_1, \\ z_2 = z_1. \end{cases}$$
Let us explain the geometrical meaning of this step. Observe that the line $2x+y+z = 0$ is tangent to $\mathcal{C}$ at the point $[-2 : 1 : 3]$. In new coordinates this line is given by $2x_1 + y_1 = 0$. So we introduced new coordinates x_2, y_2 and z_2 such that this tangent line is given by $x_2 = 0$. Expressing x_1, y_1, z_1 in terms of x_2, y_2 and z_2, we get
$$\begin{cases} x_1 = \dfrac{x_2}{2} - \dfrac{y_2}{2}, \\ y_1 = y_2, \\ z_1 = z_2. \end{cases}$$
Substituting this into the polynomial $x_1y_1 + 2x_1z_1 + y_1z_1 + x_1^2$, we see that $\mathcal{C}$ is given by
$$\frac{x_2^2}{4} + z_2x_2 - \frac{y_2^2}{4} = 0.$$
Now we introduce new coordinates x_3, y_3 and z_3 by the formula
$$\begin{cases} x_3 = x_2, \\ y_3 = y_2, \\ z_3 = z_2 + Ax_2 + By_2, \end{cases}$$
where A and B are some complex numbers to be chosen later. Then $x_2 = x_3$, $y_2 = y_3$, $z_2 = z_3 - Ax_3 - By_3$. Substituting this into $\frac{x_2^2}{4} + z_2x_2 - \frac{y_2^2}{4}$, we get the polynomial
$$x_3z_3 - Ax_3^2 + \frac{x_3^2}{4} - \frac{y_3^2}{4} - Bx_3y_3 = \Big(\frac{1}{4} - A\Big)x_3^2 - Bx_3y_3 + z_3x_3 - \frac{y_3^2}{4}.$$
We can simplify it a lot if we chose $A = \frac{1}{4}$ and $B = 0$. Thus, we let
$$\begin{cases} x_3 = x_2, \\ y_3 = y_2, \\ z_3 = z_2 + \dfrac{1}{4}x_2. \end{cases}$$
Then $x_2 = x_3$, $y_2 = y_3$ and $z_2 = z_3 - \frac{x_3}{4}$. Substituting this into $\frac{x_2^2}{4} + z_2x_2 - \frac{y_2^2}{4}$, we obtain the polynomial $x_3z_3 - \frac{y_3^2}{4}$. Thus, the conic $\mathcal{C}$ is given by
$$-4x_3z_3 + y_3^2 = 0.$$
This is almost what we want. To simplify the equation $-4x_3z_3 + y_3^2 = 0$ further, we let $x_4 = -4x_3$, $y_4 = x_3$, $z_4 = z_3$. Then
$$\begin{cases} x_3 = -\dfrac{x_4}{4}, \\ y_3 = y_4, \\ z_3 = z_4. \end{cases}$$
Substituting this into $x_3z_3 - \frac{y_3^2}{4}$, we obtain the polynomial $-\frac{1}{4}(x_4z_4 + y_3^2)$. This shows that the conic $\mathcal{C}$ is given by $x_4z_4 + y_3^2 = 0$ as required. Now we have to combine all

coordinate changes we did together. First we express x, y and z in terms of x_4, y_4, z_4. We have

$$\begin{cases} x = \dfrac{y_4}{6} - \dfrac{x_4}{6} - \dfrac{2}{3}z_4, \\ y = \dfrac{x_4}{48} - \dfrac{y_4}{3} + \dfrac{z_4}{3}, \\ z = \dfrac{x_4}{16} + z_4. \end{cases}$$

Substituting this into $x^2 + xy - 2y^2 + 3xz + 3yz + z^2$, we double check that

$$x^2 + xy - 2y^2 + 3xz + 3yz + z^2 = -\frac{1}{4}\Big(x_4z_4 + y_3^2\Big).$$

Now we express x_4, y_4 and z_4 in terms of x, y and z. We get

$$\begin{cases} x_4 = -8x - 4y - 4z, \\ y_4 = z - 3y, \\ z_4 = \dfrac{x}{2} + \dfrac{y}{4} + \dfrac{5}{4}z. \end{cases}$$

Using this, we define the projective transformation $\phi\colon \mathbb{P}^2 \mapsto \mathbb{P}^2$ by

$$\phi\Big([x : y : z]\Big) = \Big[-8x - 4y - 4z : z - 3y : \frac{x}{2} + \frac{y}{4} + \frac{5}{4}z\Big].$$

Then ϕ is the required projective transformation.

$\square$

Exercise 5. Let λ be a complex number. Put

$$f(x, y, z) = x^3 + y^3 + z^3 + \lambda xyz.$$

Let C be a subset in $\mathbb{P}^2_{\mathbb{C}}$ given by $f(x, y, z) = 0$. Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, so that $\omega^3 = 1$. Denote by Σ the subset in $\mathbb{P}^2_{\mathbb{C}}$ consisting of the following 9 points:

$$\begin{aligned} &[1 : -1 : 0], [1 : -\omega : 0], [1 : -\omega^2 : 0], \\ &[1 : 0 : -1], [1 : 0 : -\omega], [1 : 0 : -\omega^2], \\ &[0 : 1 : -1], [0 : 1 : -\omega], [0 : 1 : -\omega^2]. \end{aligned}$$

(a) Check that C contains Σ. Show that the set Σ is not contained in any line in $\mathbb{P}^2_{\mathbb{C}}$. Going through all pairs of points in Σ, one can see that every line $L \subset \mathbb{P}^2_{\mathbb{C}}$ that passes through two points in Σ contains another point in Σ. Check this in some cases.

(b) Suppose that $\lambda^3 \neq -27$. Show that there is no point $[a : b : c] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a, b, c)}{\partial x} = \frac{\partial f(a, b, c)}{\partial y} = \frac{\partial f(a, b, c)}{\partial z} = 0.$$

Use Bezout theorem to show that the homogeneous polynomial $f(x, y, z)$ is irreducible. Conclude that C is a smooth irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 3. Pick a point $P \in \Sigma$. Find the equation of the line $L_P \subset \mathbb{P}^2_{\mathbb{C}}$ that is tangent to the curve C at the point P. Show that $L_P \cap C = P$.

(c) Suppose that $\lambda^3 = -27$. Show that there are 3 points $[a : b : c] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a, b, c)}{\partial x} = \frac{\partial f(a, b, c)}{\partial y} = \frac{\partial f(a, b, c)}{\partial z} = 0.$$

Use Bezout theorem to deduce that the curve C is a union of 3 different lines in $\mathbb{P}^2_{\mathbb{C}}$. Conclude that $f(x, y, z)$ is a product of 3 different polynomials in $\mathbb{C}[x, y, z]$ of degree 1. Find these 3 polynomials explicitly.

Solution. (a) Plugging points from Σ into $x^3+y^3+z^3+\lambda xyz$ and using $\omega^3=1$, we get

$$\Sigma \subset C.$$

It is easy to see that Σ is not contained in one line in $\mathbb{P}^2_{\mathbb{C}}$. For example, the points

$$[0:1:-1], [0:1:-\omega], [1:0:-\omega^2]$$

are not contained in one line in $\mathbb{P}^2_{\mathbb{C}}$ by Exercise 2(a), because

$$\det\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -\omega \\ 1 & 0 & -\omega^2 \end{pmatrix} = 1-\omega \neq 0,$$

One can show that for every two points in Σ, we can find a third point in Σ such these three points are all contained in one line in $\mathbb{P}^2_{\mathbb{C}}$. It can be done explicitly or by using Exercise 2(a). For instance, if we take the points

$$[0:1:-1], [0:1:-\omega],$$

Similarly, if we pick the points $[1:-1:0]$ and $[1:0:-1]$ in Σ, then the equation of the line that passes through them is $x+y+z=0$. This line also contains the point

$$[0:1:-1] \in \Sigma.$$

To illustrate how to use Exercise 2(a), pick two points $[1:-\omega:0]$ and $[1:0:-1]$ in Σ. Then the line in $\mathbb{P}^2_{\mathbb{C}}$ that passes though them must contain the point $[0:1:-\omega]$, because

$$\det\begin{pmatrix} 1 & -\omega & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -\omega \end{pmatrix} = 0.$$

Note that C posses rather big group of symmetries. Namely, we can permute coordinates (x,y,z), which gives us 6 permutations. Moreover, for every a and b in $\{0,1,2\}$, we can consider a map $\mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that

$$x \mapsto \omega^a x, y \mapsto \omega^b y, z \mapsto \frac{z}{\omega^{a+b}}.$$

This gives us 9 symmetries. Composing them with with permutations of coordinates, we obtain a subgroup $G \subset \mathrm{PGL}_3(\mathbb{C})$ of order 36 such that C is invariant with respect to the action of this group on $\mathbb{P}^2_{\mathbb{C}}$. One can easily check that G acts transitively on the set Σ. This can help to reduce the computations. In fact, one can show that C is invariant with respect to a larger finite subgroup in $\mathrm{PGL}_3(\mathbb{C})$, which is classically known as the Hessian group. It consists of 216 elements. The Hessian group was introduced by Jordan back in 1877 who named it for Otto Hesse. Because of this the family of curves we study in this exercise is called the Hesse pencil. See a very nice paper "The Hesse pencil of plane cubic curves" by Michela Artebani and Igor Dolgachev at http://arxiv.org/abs/math/0611590.

(b) Let us proceed in three steps.

(i) Suppose that $\lambda^3 \neq -27$, and suppose that there is a point $[a:b:c] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.$$

Then

$$\begin{cases} 3a^2+\lambda bc = 0, \\ 3b^2+\lambda ac = 0, \\ 3c^2+\lambda ab = 0, \end{cases}$$

which implies that $\lambda \neq 0$, since $(a,b,c) \neq (0,0,0)$. Since the equation of the curve C is symmetric with respect to permutation of x, y, and z, we may assume that $a \neq 0$. Then we can put $a = 1$. Thus, we have

$$\begin{cases} 3 + \lambda bc = 0, \\ 3b^2 + \lambda c = 0, \\ 3c^2 + \lambda b = 0, \end{cases}$$

which implies that $c = -3\frac{b^2}{\lambda}$. Then $3 - \lambda b \frac{3b^2}{\lambda}$, which implies that $b^3 = 1$. But

$$b = -\frac{3c^2}{\lambda} = -\frac{3(3b^2/\lambda)^2}{\lambda} = -\frac{27b^4}{\lambda^3} = -\frac{27b}{\lambda^3},$$

which implies that $\lambda^3 = -27$. The latter contradicts to our assumption.

(ii) Suppose that the polynomial $f(x,y,z)$ is reducible. Let us seek for a contradiction. We have

$$f(x,y,z) = l(x,y,z)g(x,y,z),$$

for some homogeneous polynomial $l(x,y,z)$ of degree 1 and some (possibly reducible) homogenous polynomial $g(x,y,z)$ of degree 2. Then there exists a solution $[a:b:c] \in \mathbb{P}^2_{\mathbb{C}}$ to the system of equations

$$\begin{cases} l(x,y,z) = 0, \\ g(x,y,z) = 0. \end{cases}$$

Indeed, if the line $l(x,y,z) = 0$ is not contained in the (possibly degenerate) conic $g(x,y,z) = 0$, then this follow from Bezout theorem (actually from its very very simple subcase). Moreover, if the line $l(x,y,z) = 0$ is contained in the conic $g(x,y,z) = 0$, which simply means that $g(x,y,z)$ is divisible by $l(x,y,z)$, then every point in the line $l(x,y,z)$ does the job. Thus, we have

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial l(a,b,c)}{\partial x} g(a,b,c) + l(a,b,c)\frac{\partial g(a,b,c)}{\partial x} = 0.$$

Similarly, we see that

$$\frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.$$

This is impossible by part (2). The obtained contradiction shows that the polynomial $f(x,y,z)$ is irreducible. So, we can conclude that C is a smooth irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 3.

(iii) Now let us pick the point $P \in \Sigma$, find the tangent line L_P to the curve C at this point, and prove that $L_P \cap C = P$. Note that it does not matter which point $P \in \Sigma$ to pick in order to prove that $L_P \cap C = P$, because the curve C has a lot of symmetries. For simplicity, let us put $P = [1:-1:0] \in \Sigma$. For every point $[\alpha:\beta:\gamma] \in C$, the line

$$\frac{\partial f(\alpha,\beta,\gamma)}{\partial x}x + \frac{\partial f(\alpha,\beta,\gamma)}{\partial y}y + \frac{\partial f(\alpha,\beta,\gamma)}{\partial z}z = 0$$

is the line tangent to the curve C at the point $[\alpha:\beta:\gamma]$. Thus, the equation

$$3(x+y) - \lambda z = 0$$

defines the line L_P in $\mathbb{P}^2_{\mathbb{C}}$ that is tangent to the curve C at the point P. To find the intersection $C \cap L_P$, we have to solve the system of equations

$$\begin{cases} 3(x+y) - \lambda z = 0, \\ x^3 + y^3 + z^3 + \lambda xyz = 0. \end{cases}$$

If $\lambda = 0$, then this system of equations gives $x = y$ and $z = 0$, so that

$$L_P \cap C = [1 : -1 : 0] = P.$$

Thus, we may assume that $\lambda \neq 0$. Then $z = 3\frac{x+y}{\lambda}$ and

$$x^3 + y^3 + 27\frac{(x+y)^3}{\lambda^3} + 3xy(x+y) = 0,$$

which can be rewritten as

$$(x+y)\left(x^2 - xy + y^2 + 27\frac{x^2 + 2xy + y^2}{\lambda^3} - 3xy\right) = 0,$$

which implies that

$$(x+y)^3\Big(1 + \frac{27}{\lambda}^3\Big) = 0.$$

But $\lambda^3 \neq 27$. Then $x + y = 0$ and $z = 3\frac{x+y}{\lambda} = 0$, which implies that

$$[x : y : z] = [1 : -1 : 0],$$

so that $L_P \cap C = [1 : -1 : 0] = P$.

(c) Suppose that $\lambda^3 = -27$, so that $\lambda \in \{-3, -3\omega, -3\omega^2\}$. Let $[a : b : c]$ be a point in $\mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.$$

Arguing as in the case $\lambda^3 \neq -27$, we see that

$$\begin{cases} 3 + \lambda bc = 0, \\ 3b^2 + \lambda c = 0, \\ 3c^2 + \lambda b = 0, \end{cases}$$

which implies that $b^3 = 1$. This gives us three solutions in $\mathbb{P}^2_{\mathbb{C}}$. They are

$$\big[1 : 1 : -\frac{3}{\lambda}\big], \big[1 : \omega : -3\frac{\omega^2}{\lambda}\big], \big[1 : \omega^2 : -3\frac{\omega}{\lambda}\big].$$

Denote them by P_1, P_2 and P_3, respectively. Let L_{ij} be a line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through the point P_i and P_j for $i \neq j$. If $\lambda = -3$, then L_{12}, L_{13}, and L_{23} are given by

$$x + \omega y + \omega^2 z = 0, x + \omega^2 y + \omega z = 0, x + y + z = 0,$$

respectively. One can check that

$$x^3 + y^3 + z^3 + xyz = \big(x + \omega y + z\big)\big(x + \omega^2 y + \omega^2 z\big)\big(x + y + \omega z\big).$$

Similarly, if $\lambda = -3\omega$, then L_{12}, L_{13}, and L_{23} are given by

$$x + \omega y + z = 0, x + \omega^2 y + \omega^2 z = 0, x + y + \omega z = 0,$$

respectively. One can check that

$$x^3 + y^3 + z^3 - 3\omega xyz = \big(x + \omega y + z\big)\big(x + \omega^2 y + \omega^2 z\big)\big(x + y + \omega z\big).$$

Finally, if $\lambda = 3\omega^2$, then L_{12}, L_{13}, and L_{23} are given by

$$x + \omega y + \omega z = 0, x + \omega^2 y + z = 0, x + y + \omega^2 z = 0,$$

respectively. One can check that

$$x^3 + y^3 + z^3 - 3\omega^2 xyz = \big(x + \omega y + \omega z\big)\big(x + \omega^2 y + z\big)\big(x + y + \omega^2 z\big).$$

Thus, we see that

$$C = L_{12} \cup L_{13} \cup L_{23}.$$

Let us show this using Bezout theorem. Let $l_{ij}(x,y,z)$ be a homogeneous polynomial of degree 1 such that the equation $l_{ij}(x,y,z)=0$ defines the line L_{ij}. If $f(x,y,z)$ is not divisible by $l_{12}(x,y,z)$, which is equivalent to $L_{12}\not\subset C$ by Bezout theorem, then Bezout theorem (actually its refined simple case) implies that

$$
\begin{aligned}
3\geqslant L_{12}\cap C=\sum_{O\in C\cap L_{12}}\big(C\cdot L_{12}\big)_O\geqslant\big(C\cdot L_{12}\big)_{P_1}+\big(C\cdot L_{12}\big)_{P_2}\geqslant\\
\geqslant \mathrm{mult}_{P_1}\big(C\big)\mathrm{mult}_{P_1}\big(L_{12}\big)+\mathrm{mult}_{P_2}\big(C\big)\mathrm{mult}_{P_2}\big(L_{12}\big)=\\
=\mathrm{mult}_{P_1}\big(C\big)+\mathrm{mult}_{P_2}\big(C\big)\geqslant 2+2=4,
\end{aligned}
$$

which is absurd. Thus, we see that $f(x,y,z)$ is divisible by $l_{12}(x,y,z)$, so that the line L_{12} is contained in C. Similarly, we see that $f(x,y,z)$ is divisible by $l_{13}(x,y,z)$ and $l_{23}(x,y,z)$, so that the lines L_{13} and L_{23} are both contained in C.

□

Exercise 6. Let $\mathcal{C}$ be the conic in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$4x^2-4xy+y^2-4xz-13yz+12z^2=0.$$

Let $P_1=[0:1:1]$, $P_2=[-1:4:1]$, $P_3=[2:1:1]$. Then $\mathcal{C}$ contains the points P_1, P_2, P_3. Let $Q_1=[19:20:1]$, $Q_2=[1:2:0]$, $Q_3=[57:37:49]$. Then $\mathcal{C}$ contains Q_1, Q_2, Q_3.

(a) Show that $\mathcal{C}$ is irreducible. Find the intersection of the conic $\mathcal{C}$ and the line $z=0$.

(b) Find a projective transformation $\phi\colon\mathbb{P}^2_{\mathbb{C}}\to\mathbb{P}^2_{\mathbb{C}}$ such that $\phi(\mathcal{C})$ is given by

$$xz+y^2=0.$$

Compute $\phi(P_1)$, $\phi(P_2)$, $\phi(P_3)$, $\phi(Q_1)$, $\phi(Q_2)$ and $\phi(Q_3)$.

(c) Let L_{12}, L_{13}, L_{23}, L_{21}, L_{31}, L_{32} be the lines in $\mathbb{P}^2_{\mathbb{C}}$ defined as follows:
- L_{12} contains P_1 and Q_2; L_{13} contains P_1 and Q_3; L_{23} contains P_2 and Q_3;
- L_{21} contains P_2 and Q_1; L_{31} contains P_3 and Q_1; L_{32} contains P_3 and Q_2.

Find the defining equations of the lines L_{12}, L_{13}, L_{23}, L_{21}, L_{31} and L_{32}.
Show that the points $L_{12}\cap L_{21}$, $L_{13}\cap L_{31}$ and $L_{23}\cap L_{32}$ are collinear.

Solution. (a) To show that $\mathcal{C}$ is irreducible, rewrite the defining equation of the conic $\mathcal{C}$ as

$$\begin{pmatrix} x & y & z\end{pmatrix}\begin{pmatrix}4 & -2 & -2\\ -2 & 1 & -\frac{13}{2}\\ -2 & -\frac{13}{2} & 12\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0.$$

Then the determinant of the 3×3 matrix in this equation is 225. This implies that $\mathcal{C}$ is irreducible.

The intersection of the conic $\mathcal{C}$ and the line $z=0$ is given by

$$\begin{cases}4x^2-4xy+y^2-4xz-13yz+12z^2=0,\\ z=0.\end{cases}$$

Since $4x^2-4xy+y^2=(2x-y)^2$, the only solution in $\mathbb{P}^2_{\mathbb{C}}$ to this system of equations is the point $[1:2:0]$. This means that the line $z=0$ is tangent to the conic $\mathcal{C}$ at the point $[1:2:0]$.

(b) Note that the required projective transformation is not unique. To find one of them, we follow the algorithm described in lecture 2. Observe that $[0:1:1]\in\mathcal{C}$ and let x_1, y_1 and z_1 be new projective coordinates such that

$$\begin{cases}x_1=x,\\ y_1=y-z,\\ z_1=z.\end{cases}$$

In these coordinates our point $[0:1:1]$ is given $x_1=0$ and $y_1=0$. The meaning of this step is the following: we mapped the point $[0:1:1]$ to the point $[0:0:1]$. To find the equation of the conic $\mathcal{C}$ in new coordinates, we have to express the old coordinates x, y and z in terms of x_1, y_1 and z_1. This is done by

$$\begin{cases} x = x_1, \\ y = y_1 + z_1, \\ z = z_1. \end{cases}$$

Substituting this into $4x^2 - 4xy + y^2 - 4xz - 13yz + 12z^2$, we see that $\mathcal{C}$ is given by the equation

$$4x_1^2 - 4x_1y_1 + y_1^2 - (8x_1 + 11y_1)z_1 = 0.$$

Now we change projective coordinates as follows:

$$\begin{cases} x_2 = -8x_1 - 11y_1, \\ y_2 = y_1, \\ z_2 = z_1. \end{cases}$$

The geometrical meaning of this step is the following: we mapped the tangent line to $\mathcal{C}$ at the point $[0:1:1]$, which is given by $-8x_1 - 11y_1 = 0$, to the line $x_2 = 0$. Then $x_1 = -\frac{x_2+11y_2}{8}$, $y_1 = y_2$, $z_1 = z_2$. Substituting this into $4x_1^2 - 4x_1y_1 + y_1^2 - (8x_1 + 11y_1)z_1$, we see that $\mathcal{C}$ is given by

$$\frac{x_2^2}{16} + \frac{30}{16}x_2y_2 + z_2x_2 + \frac{225}{16}y_2^2 = 0.$$

In the next step, we let

$$\begin{cases} x_3 = x_2, \\ y_3 = y_2, \\ z_3 = z_2 + Ax_2 + By_2, \end{cases}$$

where A and B are some complex numbers to be chosen later. Then

$$\begin{cases} x_2 = x_3, \\ y_2 = y_3, \\ z_2 = z_3 - Ax_3 - By_3. \end{cases}$$

Substituting this into $\frac{x_2^2}{16} + \frac{30}{16}x_2y_2 + z_2x_2 + \frac{225}{16}y_2^2$, we see that $\mathcal{C}$ is given by the equation

$$\Big(-A + \frac{1}{16}\Big)x_3^2 + \Big(-B + \frac{30}{16}\Big)x_3y_3 + z_3x_3 + \frac{225}{16}y_3^2 = 0.$$

Now we let $A = \frac{1}{16}$ and $B = \frac{30}{16}$. Then $\mathcal{C}$ is given by

$$x_3z_3 + \left(\frac{15}{4}y_3\right)^2 = 0.$$

The geometrical meaning of this step is the following: we construct a projective transformation such that it changes the *infinite* line $z=0$, it fixes the point $[0:0:1]$, it maps the line $x=0$ to itself, and it simplifies the equation of the conic $\mathcal{C}$. Finally, we let

$$\begin{cases} x_4 = x_3, \\ y_4 = \dfrac{15}{4}y_3, \\ z_4 = z_3. \end{cases}$$

Then $x_3 = x_4$, $y_3 = \frac{4}{15}y_4$ and $z_3 = z_4$, so that $\mathcal{C}$ is given by $x_4z_4 + y_4^2 = 0$. This step does not have geometrical meaning: we just scale coordinates such that the equation of the conic $\mathcal{C}$ is as simple as it can be. Now we should combine all our coordinate changes together. We get

$$\begin{cases} x = -\dfrac{x_4}{8} - \dfrac{11}{30}y_4, \\ y = -\dfrac{x_4}{16} - \dfrac{7}{30}y_4 + z_4, \\ z = -\dfrac{x_4}{16} - \dfrac{y_4}{2} + z_4. \end{cases}$$

Substituting this into the polynomial $4x^2 - 4xy + y^2 - 4xz - 13yz + 12z^2$, we indeed get $x_4z_4 + y_4^2$. Similarly, we have

$$\begin{aligned} x_4 = x_3 = x_2 = -8x_1 - 11y_1 = -8x - 11(y - z) = -8x - 11y + 11z, \\ y_4 = \frac{15}{4}y_3 = \frac{15}{4}y_2 = \frac{15}{4}(y - z) = \frac{15}{4}y - \frac{15}{4}z, \\ z_4 = z_3 = z_2 + \frac{x_2}{16} + \frac{30}{16}y_2 = z_1 + \frac{-8x_1 - 11y_1}{16} + \frac{30}{16}y_1 = \\ = z + \frac{-8x - 11(y - z)}{16} + \frac{30}{16}(y - z) = -\frac{x}{2} + \frac{19}{16}y - \frac{3}{16}z. \end{aligned}$$

This gives

$$\begin{cases} x_4 = -8x - 11y + 11z, \\ y_4 = \dfrac{15}{4}y - \dfrac{15}{4}z, \\ z_4 = -\dfrac{x}{2} + \dfrac{19}{16}y - \dfrac{3}{16}z. \end{cases}$$

Substituting these expressions for x_4, y_4 and z_4 into to the polynomial $x_4z_4 + y_4^2$, we indeed get $4x^2 - 4xy + y^2 - 4xz - 13yz + 12z^2$. Let ϕ be the projective transformation of $\mathbb{P}^2$ that is given by

$$\phi\Big([x : y : z]\Big) = \Big[-8x - 11y + 11z : \frac{15}{4}y - \frac{15}{4}z : \frac{x}{2} + \frac{19}{16}y - \frac{3}{16}z\Big].$$

If you do not like denominators, you can rewrite this as

$$\phi\Big([x : y : z]\Big) = \big[-128x - 176y + 176z : 60y - 60z : -8x + 19y - 3z\big].$$

Then ϕ is the required projective transformation. It corresponds to the linear transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -8 & -11 & 11 \\ 0 & \frac{15}{4} & -\frac{15}{4} \\ -\frac{1}{2} & \frac{19}{16} & -\frac{3}{16} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Using this 3×3 matrix, we see that $\phi(P_1) = \phi([0 : 1 : 1]) = [0 : 0 : 1]$ and

$$\begin{aligned} \phi(P_2) = \phi([-1 : 4 : 1]) = \Big[-25 : \frac{45}{4} : \frac{81}{16}\Big] = \big[-400 : 180 : 81\big], \\ \phi(P_3) = \phi([2 : 1 : 1]) = \big[-16 : 0 : 0\big] = \big[1 : 0 : 0\big], \\ \phi(Q_1) = \phi([19 : 20 : 1]) = \Big[-361 : \frac{285}{4} : \frac{225}{16}\Big] = \big[-5776 : 1140 : 225\big], \\ \phi(Q_2) = \phi([1 : 2 : 0]) = \Big[-30 : \frac{15}{2} : \frac{15}{8}\Big] = \big[-16 : 4 : 1\big], \\ \phi(Q_3) = \phi([57 : 37 : 49]) = \Big[-324 : -45 : \frac{25}{4}\Big] = \big[-1296 : -180 : 25\big], \end{aligned}$$

We can double check that the points $\phi(P_1)$, $\phi(P_2)$, $\phi(P_3)$, $\phi(Q_1)$, $\phi(Q_2)$, $\phi(Q_3)$ are indeed contained in the conic $xz+y^2=0$. This confirms that the conic $\phi(\mathcal{C})$ is given by $xz+y^2=0$, because this conic is the unique conic in $\mathbb{P}^2$ that contains the points $\phi(P_1)$, $\phi(P_2)$, $\phi(P_3)$, $\phi(Q_1)$, $\phi(Q_2)$, $\phi(Q_3)$.

(c) The line L_{12} is the line in $\mathbb{P}^2$ that passes trough P_1 and Q_2. Its defining equation is

$$\det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ x & y & z \end{pmatrix} = 0.$$

The expanding this determinant, we see that L_{12} is given by $2x-y+z=0$. Similarly, we see that the line L_{13} is given by $12x+57y-57z=0$, the line L_{23} is given by $159x+106y-265z=0$, the line L_{21} is given by $16x-20y+96z=0$, the line L_{31} is given by $19x-17y-21z=0$, and the line L_{32} is given by $2x-y-3z=0$.

Let $O_{12}=L_{12}\cap L_{21}$, $O_{13}=L_{13}\cap L_{31}$, $O_{23}=L_{23}\cap L_{32}$. Then the projective coordinates of the point O_{12} are given by any non-zero solution to

$$\begin{cases} 2x-y+z=0=0, \\ 16x-20y+96z=0. \end{cases}$$

Solving this system, we see that $O_{12}=[19:44:6]$. Absolutely similarly, we see that find $O_{13}=[722:277:429]$ and $O_{23}=[11:1:7]$. To check that these three points are collinear, it is enough to check that

$$\det\begin{pmatrix} 19 & 44 & 6 \\ 722 & 277 & 429 \\ 11 & 1 & 7 \end{pmatrix} = 0.$$

This determinant is indeed 0. Thus, there is a line L in $\mathbb{P}^2$ that contains O_{12}, O_{13} and O_{23}. To find the equation of this line, we can use determinant formula we already used earlier. Namely, the line that contains O_{12} and O_{13} is given by

$$\det\begin{pmatrix} 19 & 44 & 6 \\ 722 & 277 & 429 \\ x & y & z \end{pmatrix} = 0.$$

Expanding this determinant and dividing it by 57, we see that the line L is given by the equation $302x-67y-465z=0$.

$\square$

Exercise 7. Put $f(x,y,z)=xy^3+yz^3+zx^3$. Let C be a subset in $\mathbb{P}^2_{\mathbb{C}}$ given by

$$f(x,y,z)=0.$$

(a) Show that there is no point $[a:b:c]\in\mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a,b,c)}{\partial x}=\frac{\partial f(a,b,c)}{\partial y}=\frac{\partial f(a,b,c)}{\partial z}=0.$$

Use Bezout theorem to show that $f(x,y,z)$ is irreducible.

(b) Let L be the tangent line to C at $[0:0:1]$. Find $L\cap C$.

(c) Denote by $g(x,y,z)$ the determinant of the matrix

$$\begin{pmatrix} \frac{\partial^2 f(x,y,z)}{\partial x\partial x} & \frac{\partial^2 f(x,y,z)}{\partial x\partial y} & \frac{\partial^2 f(x,y,z)}{\partial x\partial z} \\ \frac{\partial^2 f(x,y,z)}{\partial y\partial x} & \frac{\partial^2 f(x,y,z)}{\partial y\partial y} & \frac{\partial^2 f(x,y,z)}{\partial y\partial z} \\ \frac{\partial^2 f(x,y,z)}{\partial z\partial x} & \frac{\partial^2 f(x,y,z)}{\partial z\partial y} & \frac{\partial^2 f(x,y,z)}{\partial z\partial z} \end{pmatrix}.$$

Denote by Z the subset in $\mathbb{P}^2_{\mathbb{C}}$ given by $g(x,y,z)=0$. Show that $3\leqslant|C\cap Z|\leqslant 24$.

Solution. (a) Suppose that there is no point $[a : b : c] \in \mathbb{P}^2_{\mathbb{C}}$ such that

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.$$

Let us seek for a contradiction. We have

$$\begin{cases} \dfrac{\partial f(a,b,c)}{\partial x} = b^3 + 3a^2c = 0 \\ \dfrac{\partial f(a,b,c)}{\partial y} = c^3 + 3b^2a = 0 \\ \dfrac{\partial f(a,b,c)}{\partial z} = a^3 + 3c^2b = 0 \end{cases}$$

which implies that $a \neq 0$, $b \neq 0$ and $c \neq 0$, because $(a,b,c) \neq (0,0,0)$. In particular, dividing by c^3 and replacing a by $\frac{a}{c}$ and b by $\frac{b}{c}$, we may assume that $a = 1$. Then we have

$$\begin{cases} b^3 + 3a^2 = 0 \\ 1 + 3b^2a = 0 \\ a^3 + 3b = 0 \end{cases}$$

which gives $a^2 = \frac{a^9}{3^4}$ and $a^7 = -3$. This gives $-3 = 3^4$, which is absurd. Now we suppose that

$$f(x,y,z) = h(x,y,z)g(x,y,z),$$

for some homogeneous polynomials $h(x,y,z)$ and $h(x,y,z)$ of positive degrees. Then there exists a solution $[a : b : c] \in \mathbb{P}^2_{\mathbb{C}}$ to the system of equations

$$\begin{cases} h(x,y,z) = 0, \\ g(x,y,z) = 0. \end{cases}$$

Indeed, this follows from Bezout theorem. Thus, we have

$$\frac{\partial f(a,b,c)}{\partial x} = \frac{\partial h(a,b,c)}{\partial x} g(a,b,c) + h(a,b,c)\frac{\partial g(a,b,c)}{\partial x} = 0.$$

Similarly, we see that

$$\frac{\partial f(a,b,c)}{\partial y} = \frac{\partial f(a,b,c)}{\partial z} = 0.$$

But we already proved that this is impossible, so that the polynomial $f(x,y,z)$ is irreducible.

(b) For every point $[\alpha : \beta : \gamma] \in C$, the line

$$\frac{\partial f(\alpha,\beta,\gamma)}{\partial x} x + \frac{\partial f(\alpha,\beta,\gamma)}{\partial y} y + \frac{\partial f(\alpha,\beta,\gamma)}{\partial z} z = 0$$

is the line tangent to the curve C at the point $[\alpha : \beta : \gamma]$. For $[\alpha : \beta : \gamma] = [0 : 0 : 1]$, we get

$$\begin{cases} \dfrac{\partial f(0,0,1)}{\partial x} = 0 \\ \dfrac{\partial f(0,0,1)}{\partial y} = 1 \\ \dfrac{\partial f(0,0,1)}{\partial z} = 0 \end{cases}$$

so that the tangent line L to the curve C at the point $[0:0:1]$ is given by $y=0$. To find the intersection $C\cap L$, we have to solve the system of equations

$$\begin{cases} y=0, \\ xy^3+yz^3+zx^3. \end{cases}$$

This gives two points in $\mathbb{P}^2_{\mathbb{C}}$. One is $[0:0:1]$ and another is $[1:0:0]$. The first one is counted with multiplicity 3, i.e., $(C\cdot L)_P=3$ for $P=[0:0:1]$. Thus, $[0:0:1]$ is the inflection point of the curve C. The second one is counted with multiplicity 1.

(c) Note that $C\cap Z$ consists of all inflection points of the curve C, and the polynomial $g(x,y,z)$ is called the Hessian of the polynomial $f(x,y,z)$. We have

$$g(x,y,z)=\det\begin{pmatrix} 6zx & 3y^2 & 3x^2 \\ 3y^2 & 6xy & 3z^2 \\ 3x^2 & 3z^2 & 6yz \end{pmatrix}=3^3\Big(10x^2y^2z^2-2xz^5-2zy^5-2yx^5\Big).$$

This shows that $g(x,z,y)$ is not divisible by $f(x,y,z)$. Indeed, $[1:1:-\sqrt[3]{2}]\in C$, but

$$g(1:1:-\sqrt[3]{2})=3^3\Big(10\sqrt[3]{4}+2\sqrt[3]{32}+2\sqrt[3]{2}-2\Big)=3^3\Big(14\sqrt[3]{4}+2\sqrt[3]{2}-2\Big)\approx 614.$$

On the other hand, the set $C\cap Z$ is given by

$$f(x,y,z)=g(x,y,z)=0.$$

Thus, by the Bezout theorem, this system of equation has at most 24 solutions in $\mathbb{P}^2$, because $f(x,y,z)$ is irreducible and $g(x,z,y)$ is not divisible by $f(x,y,z)$. This shows that $|C\cap Z|\leqslant 24$. On the other hand, $C\cap Z$ contains the points $[0:0:1]$, $[0:1:0]$ and $[1:0:0]$, so that $|C\cap Z|\geqslant 3$. Note that C posses rather big group of symmetries. Namely, we can permute coordinates (x,y,z), which gives us 6 permutations. In fact, one can show that C is invariant with respect to a larger finite subgroup in $\mathrm{PGL}_3(\mathbb{C})$, which is classically known as the Klein simple group. It consists of 168 elements. This is the second smallest non-abelian simple group after A_5. The Klein simple group can be defined as $\mathrm{PSL}_2(\mathbb{F}_7)$. It has three-dimensional faithful representation. This representation gives the faithful action of the group $\mathrm{PSL}_2(\mathbb{F}_7)$ on $\mathbb{P}^2_{\mathbb{C}}$ such that the curve C is $\mathrm{PSL}_2(\mathbb{F}_7)$-invariant. Let G be the stabilizer of $[0:0:1]$ in $\mathrm{PSL}_2(\mathbb{F}_7)$, and let Σ be the $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit of $[0:0:1]$. Then $|\Sigma|$ is contained in $Z\cap C$, because Z is also $\mathrm{PSL}_2(\mathbb{F}_7)$-invariant. On the other hand, we have

$$24\geqslant|\Sigma|=\frac{168}{|G|},$$

which implies that $|G|\geqslant 7$. Moreover, the line tangent to C at $[0:0:1]$ is G-invariant. This implies that three-dimensional faithful representation of $\mathrm{PSL}_2(\mathbb{F}_7)$ restricted to G splits as a sum of one-dimensional representations. The same time this restriction must be faithful. This implies that G is abelian. Looking at the subgroups of the Klein group (see http://brauer.maths.qmul.ac.uk/Atlas/v3/lin/L27/), we see that $G\cong\mathbb{Z}_7$, so that $|\Sigma|=24$. Thus, we also have $|C\cap Z|=24$.

$\square$

Exercise 8. Let C_4 be an irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 4.

(a) Show that the curve C_4 has at most 3 singular points.

(b) Suppose that the curve C_4 has a singular point P such that

$$\mathrm{mult}_P\big(C_4\big)=3.$$

Show that the curve C_4 does not have other singular points.

(c) Give an example of a singular irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 4.

Solution. (a) Suppose that C_4 has at least 4 singular points. Denote any four of them by P_1, P_2, P_3, and P_4. Pick one more point Q in C_d that is different from these four points. There exists a non-zero homogeneous polynomial $f(x,y,z)$ of degree 2 such that

$$f(P_1) = f(P_2) = f(P_3) = f(P_4) = f(Q) = 0.$$

Let Z be the conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by $f(x,y,z)=0$. Since C_4 is assumed to be irreducible, we can apply Bezout theorem to C_4 and Z. This gives

$$\begin{aligned}8 \geqslant |C_4 \cap Z| = \sum_{O \in C_4 \cap Z} \big(C_4 \cdot Z\big)_O \geqslant \sum_{i=1}^{4} \big(C_4 \cdot Z\big)_{P_i} + \big(C_4 \cdot Z\big)_Q \geqslant \\ \geqslant \sum_{i=1}^{4} \mathrm{mult}_{P_i}\big(C_4\big)\mathrm{mult}_{P_i}\big(Z\big) + \mathrm{mult}_Q\big(C_4\big)\mathrm{mult}_Q\big(Z\big) \geqslant \\ \geqslant \sum_{i=1}^{4} 2\mathrm{mult}_{P_i}\big(Z\big) + 1 \geqslant \sum_{i=1}^{4} 2 + 1 = 9,\end{aligned}$$

which is absurd. This shows that the curve C_4 has at most 3 singular points.

(b) If C_4 has a singular point P of multiplicity 3 and another singular point Q, then Bezout theorem gives

$$4 \geqslant |C_4 \cap L| = \sum_{O \in C_4 \cap L} \big(C_4 \cdot L\big)_O \geqslant \big(C_4 \cdot L\big)_P + \big(C_4 \cdot L\big)_Q \geqslant 3 + 2 = 5,$$

where L is the line in $\mathbb{P}^2_{\mathbb{C}}$ that passes through P and Q.

(c) The easiest example of a singular irreducible curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree 4 is given by

$$zy^3 = x^4,$$

because the polynomial $zy^3 - x^4$ is irreducible (it can be considered as a polynomial in z of degree 1, which easily implies its irreducibility). However, let us consider a more interesting example that also shows that the bound in (a) is sharp. Namely, put $f(x,y,z) = x^2y^2 - 2x^2z^2 + y^2z^2$, and let C be the curve given by $f(x,y,z)=0$. Then C is singular at the points $[0:0:1]$, $[0:1:0]$, and $[1:0:0]$. Let us show that the polynomial $f(x,y,z)$ is irreducible. To simplify the proof a bit, let us de-homogenize this polynomial. Namely, put $g(x,y) = x^2y^2 - 2x^2 + y^2$. Then $f(x,y,z)$ is irreducible if and only if $g(x,y)$ is irreducible, because

$$f(x,y,z) = z^4 g\Big(\frac{x}{z}, \frac{y}{z}\Big)$$

and $g(x,y) = f(x,y,1)$. Let us show that $g(x,y)$ is irreducible. Rewrite $g(x,y)$ as $(y^2-2)x^2 + y^2$. Note that $g(x,y)$ can be considered as a polynomial in x with coefficients in $\mathbb{C}[y]$. Recall that $\mathbb{C}[y]$ is unique factorization domain. Suppose that $g(x,y)$ is not irreducible. Then either

$$(y^2-2)x^2 + y^2 = (Ax+B)(Cx+D)$$

for some polynomials A, B, C, D in $\mathbb{C}[y]$, or

$$(y^2-2)x^2 + y^2 = H(Ex^2 + Fx + G)$$

for some polynomials E, F, G, H in $\mathbb{C}[y]$ such that $H \notin \mathbb{C}$. In the former case we get

$$\begin{cases} AC = y^2 - 2, \\ AD + BC = 0, \\ BD = y^2. \end{cases}$$

In the latter case we get

$$\begin{cases} HE = y^2 - 2, \\ HF = 0, \\ HG = y^2, \end{cases}$$

so that $y^2 - 2$ must be divisible by y, which is absurd. Thus, we are in the former case, so that

$$AC = y^2 - 2, AD + BC = 0, BD = y^2.$$

Without loss of generality, we may assume that B is not in $\mathbb{C}$. Thus, it follows from $BD = y^2$ that B is either λy or λy^2 for some non-zero $\lambda \in \mathbb{C}$. Scaling $Ax + B$ by $\frac{1}{\lambda}$ and $Cx + D$ by λ, we may assume that either $B = y$ or $B = y^2$. If $B = y$, we get $D = y$ as well, so that $A = -C$ and

$$-A^2 = y^2 - 2,$$

which is absurd, because $2 - y^2$ is not a square in $\mathbb{C}[y]$. Thus, $B = y^2$. Ten $D = 1$, so that $A = -y^2C$ and

$$-y^2C^2 = y^2 - 2,$$

which is impossible, since $y^2 - 2$ is not divisible by y^2. This shows that $g(x, y)$ is irreducible, so that the polynomial $f(x, y, z)$ is also irreducible.

$\square$

Exercise 9. Let S_2 be an algebraic subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by $f_2(x, y, z, t) = 0$, where

$$f_2(x, y, z, t) = 2x^2 - 4tx - ty + xy + 2xz - y^2 + yz.$$

Put $P = [1 : -1 : 0 : 0]$.

(a) Show that $f_2(x, y, z, t)$ is irreducible. Prove that S_2 is smooth.
(b) Check that $P \in S_2$. Find all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2 and pass through P. Find $[A : B : C : D] \in \mathbb{P}^3_{\mathbb{C}}$ such that the equation

$$Ax + By + Cz + Dt = 0$$

defines a plane $\Pi \subset \mathbb{P}^3_{\mathbb{C}}$ that is tangent to S_2 at the point P. Describe $\Pi \cap S_2$.
(c) Find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(S_2)$ is given by $xy = zt$. Use this to describe all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2.

Solution. (a) Let us show that $f_2(x, y, z, t)$ is irreducible. There are many ways to do this. Suppose that $f_2(x, y, z, t)$ is reducible in $\mathbb{C}[x, y, z, t]$. Then it is a product of two non-constant polynomials. Since $f_2(x, y, z, t) = 2x^2 + x(y + 2z - 4t) - ty - y^2 + yz$ is a monic polynomial of degree 2 in t, we have

$$f_2(x, y, z, t) = \big(A(y, z, t)x + B(y, z, t)\big)\big(C(y, z, t)x + D(y, z, t)\big)$$

for some polynomials $A(y, z, t)$, $B(y, z, t)$ and $C(y, z, t)$ in $\mathbb{C}[y, z, t]$. Then

$$\begin{cases} A(y, z, t)C(y, z, t) = 2, \\ A(y, z, t)D(y, z, t) + B(y, z, t)C(y, z, t) = y + 2z - 4t, \\ B(y, z, t)D(y, z, t) = -ty - y^2 + yz. \end{cases}$$

This implies, in particular, that $A(x, y, z)$ and $C(x, y, z)$ are non-zero constant polynomials. Since $A(y, z, t)C(y, z, t) = 2$, we may assume that $A(x, y, z) = 1$ and $C(x, y, z) = 2$. Then

$$\begin{cases} D(y, z, t) + 2B(y, z, t) = y + 2z - 4t, \\ B(y, z, t)D(y, z, t) = yz - ty - y^2. \end{cases}$$

so that we have $D(x,y,z) = y + 2z - 4t - 2B(x,y,z)$ and

$$B(y,z,t)\Big(y + 2z - 4t - 2B(x,y,z)\Big) = -ty - y^2 + yz.$$

Completing the square, we see that

$$\frac{1}{2}\Big(B - \frac{y+2z-4t}{4}\Big)^2 = \frac{(y+2z-4t)^2}{16} - yz + ty + y^2 = \frac{16t^2 + 8ty - 16tz + 17y^2 - 12yz + 4z^2}{16}.$$

In particular, the polynomial $16t^2 + 8ty - 16tz + 17y^2 - 12yz + 4z^2$ is a square in $\mathbb{C}[y,z,t]$. Thus, we have $16t^2 + 8ty - 16tz + 17y^2 - 12yz + 4z^2 = (ay + bz + ct + d)^2$ for some complex numbers a, b, c and d. Then

$$16t^2 + 8ty - 16tz + 17y^2 - 12yz + 4z^2 = a^2y^2 + 2abyz + 2acyt + b^2z^2 + 2bczt + c^2t^2 + 2ady + 2bdz + 2cdt + d^2.$$

This is equality of polynomials. Thus, we have

$$\begin{cases} a^2 = 17, \\ 2ab = -12, \\ 3ac = 8, \\ b^2 = 4, \\ 2bc = -16, \\ c^2 = 16, \\ 2ad = 0, \\ 2bd = 0, \\ 2cd = 0, \\ d^2 = 0. \end{cases}$$

This system is inconsistent, which is a contradiction.

Now let us prove that S_2 is smooth. We have to show that $x = y = z = t = 0$ is the only solution to the system of equations

$$\begin{cases} \dfrac{\partial f_2(x,y,z,t)}{\partial x} = 0, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial y} = 0, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial z} = 0, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial t} = 0. \end{cases}$$

This is easy. Indeed, we have

$$\begin{cases} \dfrac{\partial f_2(x,y,z,t)}{\partial x} = -4t + 4x + y + 2z, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial y} = -t + x - 2y + z, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial z} = 2x + y, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial t} = -4x - y. \end{cases}$$

On the other hand, the system of linear equations

$$\begin{cases} -4t+4x+y+2z=0, \\ -t+x-2y+z=0, \\ 2x+y=0, \\ -4x-y=0. \end{cases}$$

does not have solutions except $x=y=z=t=0$, because

$$\begin{vmatrix} 4 & 1 & 2 & -4 \\ 1 & -2 & 1 & -1 \\ 2 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 \end{vmatrix} = 4 \neq 0.$$

(b) Since $f_2(1,-1,0,0)=0$, we see that $P \in S_2$. Let L be a line that passes through P. Let Q be the point of intersection of L and the plane $x=0$. Then $Q=[0:\alpha:\beta:\gamma]$ such that $f_2(0:\alpha:\beta:\gamma)=0$. Then L is given by

$$\lambda\big[1,-1,0,0\big]+\mu\big[0:\alpha:\beta:\gamma\big],$$

where $[\lambda:\mu]$ runs through $\mathbb{P}^1_{\mathbb{C}}$. Then $L \subset S_2$ if and only if

$$f_2(\lambda,-\lambda+\mu\alpha,\mu\beta,\mu\gamma)=0$$

for every $[\lambda:\mu] \in \mathbb{P}^1_{\mathbb{C}}$. Thus, $L \subset S_2$ if and only if

$$\Big(\alpha\beta-\alpha^2-\alpha\gamma\Big)\mu^2+\Big(3\alpha+\beta-3\gamma\Big)\mu\lambda=0$$

for every $[\lambda:\mu] \in \mathbb{P}^1_{\mathbb{C}}$. Hence, $L \subset S_2$ if and only if

$$\begin{cases} \alpha\beta-\alpha^2-\alpha\gamma=0, \\ 3\alpha+\beta-3\gamma=0. \end{cases}$$

This is gives us exactly two possibilities for the point Q: either $Q=[0:0:3:1]$ or $Q=[0:1:3:2]$. By construction, in both cases the line passing through P and Q is contained in S_2. Thus, there are exactly two lines lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2 and pass through P. They are the lines $[\lambda:-\lambda:3\mu:\mu]$ and $[\lambda:-\lambda+\mu:3\mu:2\mu]$, where $[\lambda:\mu] \in \mathbb{P}^1_{\mathbb{C}}$.

For every point $[a:b:c:d] \in S_2$, the plane in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$\frac{\partial f_2(a,b,c,d)}{\partial x}x+\frac{\partial f_2(a,b,c,d)}{\partial y}y+\frac{\partial f_2(a,b,c,d)}{\partial z}z+\frac{\partial f_2(a,b,c,d)}{\partial t}t=0$$

tangents the surface S_2 at the point $[a:b:c:d]$. Since

$$\begin{cases} \dfrac{\partial f_2(1,-1,0,0)}{\partial x}=3, \\ \dfrac{\partial f_2(1,-1,0,0)}{\partial y}=3, \\ \dfrac{\partial f_2(1,-1,0,0)}{\partial z}=1, \\ \dfrac{\partial f_2(1,-1,0,0)}{\partial t}=-3, \end{cases}$$

the plane $3x+3y+z-3t=0$ is the tangent plane to the surface S_2 at the point P, so that $[A:B:C:D]=[3:3:1:-3]$. Recall that we denoted this plane by Π. Then $\Pi \cap S_2$ is given by

$$\begin{cases} 2x^2+x(y+2z-4t)-ty-y^2+yz=0, \\ 3x+3y+z-3t=0. \end{cases}$$

Plugging in $z=3t-3x-3y$ into $2x^2+x(y+2z-4t)-ty-y^2+yz$, we get $2(x+y)(t-2x-2y)$. Thus, the intersection $\Pi\cap S_2$ consist of two lines: $x+y=3x+3y+z-3t=0$ and $t-2x-2y=3x+3y+z-3t=0$. These are exactly the lines we found earlier.

(c) Let us find a projective transformation $\phi\colon \mathbb{P}^2_{\mathbb{C}}\to\mathbb{P}^2_{\mathbb{C}}$ such that $\phi(S_2)$ is given by $xy=zt$. Recall that S_2 is given by $f_2(x,y,z,t)=0$, where $f_2(x,y,z,t)=2x^2-4tx-ty+xy+2xz-y^2+yz$. Observe that S_2 and the surface $xy=zt$ both contain the point $[0:0:0:1]$. On the other hand, the plane in $\mathbb{P}^3_{\mathbb{C}}$ that is tangent to S_2 at the point $[0:0:0:1]$ is given by

$$4x+y=0,$$

while the plane in $\mathbb{P}^3_{\mathbb{C}}$ that is tangent to $xy=zt$ at the point $[0:0:0:1]$ is given by $z=0$. Let us introduce new coordinates $\overline{x}=x$, $\overline{y}=\overline{z}$, $\overline{z}=4x+y$ and $\overline{t}=t$. Plugging $x=\overline{x}$, $y=\overline{z}-4\overline{x}$, $z=\overline{y}$, $t=\overline{t}$ into $f_2(x,y,z,t)$, we obtain the polynomial

$$\overline{f}_2(\overline{x},\overline{y},\overline{z},\overline{t})=-\overline{t}\overline{z}-18\overline{x}^2-2\overline{xy}+9\overline{xz}+\overline{yz}-\overline{z}^2$$

It gives us the defining equation of S_2 in this new homogeneous coordinates. Now we put $\widehat{x}=\overline{x}$, $\widehat{y}=\overline{y}$, $\widehat{z}=\overline{z}$ and $\widehat{t}=\overline{t}-9\overline{x}-\overline{y}+\overline{z}$. Plugging $\overline{x}=\widehat{x}$, $\overline{y}=\widehat{y}$, $\overline{z}=\widehat{z}$ and $\overline{t}=\widehat{t}+9\widehat{x}+\widehat{y}-\widehat{z}$ into $\overline{f}_2(\overline{x},\overline{y},\overline{z},\overline{t})$, we obtain the polynomial

$$\widehat{f}_2(\widehat{x},\widehat{y},\widehat{z},\widehat{t})=-\widehat{t}\widehat{z}-18\widehat{x}^2-2\widehat{xy}.$$

This is the defining equation of S_2 in new homogeneous coordinates $\widehat{x}$, $\widehat{y}$, $\widehat{z}$, $\widehat{t}$. Now we put $\widetilde{x}=2\widehat{x}$, $\widetilde{y}=-\widehat{y}-9\widehat{x}$, $\widetilde{z}=\widehat{z}$ and $\widetilde{t}=\widehat{t}$. Then S_2 is given by $\widetilde{x}\widetilde{y}=\widetilde{z}\widetilde{t}$. Since

$$\begin{cases}\widetilde{x}=2x,\\ \widetilde{y}=-9x-z,\\ \widetilde{z}=y+4x,\\ \widetilde{t}=-5x+y-z+t,\end{cases}$$

the required projective transformation ϕ is given by

$$[x:y:z:t]\mapsto\big[2x:-9x-z:y+4x:-5x+y-z+t\big].$$

One can double check that

$$\big(2x\big)\big(-9x-z\big)-\big(y+4x\big)\big(-5x+y-z+t\big)=2x^2-4tx-ty+xy+2xz-y^2+yz,$$

so that $\phi(S_2)$ is indeed given by $xy=zt$.

Now let us describe all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2. To do this, let us recall the description of all lines in the quadric surface $xy=zt$. Recall that the quadric $xy=zt$ can be identified with $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ via the map $\upsilon\colon\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\to\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$\Big(\big[u_1:v_1\big],\big[u_2:v_2\big]\Big)\mapsto\Big[u_1u_2:v_1v_2:u_1v_2:v_1u_2\Big].$$

Check that the image of υ is indeed contained in the quadric $xy=zt$. For every *fixed* point $[u_1:v_1]\in\mathbb{P}^1_{\mathbb{C}}$, the set

$$\big[u_1u_2:v_1v_2:u_1v_2:v_1u_2\big]$$

gives us a line in the quadric $xy=zt$ when $[u_2:v_2]$ runs through $\mathbb{P}^1_{\mathbb{C}}$. Vice versa, for every *fixed* point $[u_2:v_2]\in\mathbb{P}^1_{\mathbb{C}}$, the set

$$\big[u_1u_2:v_1v_2:u_1v_2:v_1u_2\big]$$

also gives us a line in the quadric $xy=zt$ when $[u_1:v_1]$ runs through $\mathbb{P}^1_{\mathbb{C}}$. This gives us all lines in the quadric in $xy=zt$.

Let $\psi\colon\mathbb{P}^3_{\mathbb{C}}\to\mathbb{P}^3_{\mathbb{C}}$ be the projective transformation that is the inverse of ϕ. Then ψ maps lines to lines, so that

$$\psi\Big(\big[u_1u_2:v_1v_2:u_1v_2:v_1u_2\big]\Big)$$

gives us all lines in S_2 when we *fix* $[u_1 : v_1] \in \mathbb{P}^1_{\mathbb{C}}$ or $[u_2 : v_2] \in \mathbb{P}^1_{\mathbb{C}}$. Namely, the map ψ is given by

$$[x : y : z : t] \mapsto \Big[\frac{x}{2} : -2x + z : -\frac{9x}{2} - y : t - y - z\Big].$$

Thus, the composition $\psi \circ \upsilon$ is given by

$$\Big([u_1 : v_1], [u_2 : v_2]\Big) \mapsto \Big[\frac{u_1u_2}{2} : -2u_1u_2 + u_1v_2 : -\frac{9u_1u_2}{2} - v_1v_2 : v_1u_2 - v_1v_2 - u_1v_2\Big].$$

This gives us the description of all lines in S_2 as

$$\Big[\frac{u_1u_2}{2} : -2u_1u_2 + u_1v_2 : -\frac{9u_1u_2}{2} - v_1v_2 : v_1u_2 - v_1v_2 - u_1v_2\Big]$$

when we fix $[u_1 : v_1] \in \mathbb{P}^1_{\mathbb{C}}$ or $[u_2 : v_2] \in \mathbb{P}^1_{\mathbb{C}}$. For example, $\phi(P) = [2 : -9 : 3 : -6]$ and

$$\upsilon\Big(([-1 : 3], [2 : 3])\Big) = [2 : -9 : 3 : -6].$$

Thus, the above description gives us two lines in S_2 that passes through P. The first line is given by

$$\Big[-\frac{u_2}{2} : 2u_2 - v_2 : \frac{9u_2}{2} - 3v_2 : 3u_2 - 2v_2\Big]$$

where $[u_2 : v_2] \in \mathbb{P}^1_{\mathbb{C}}$. The second line is given by

$$\Big[u_1 : -u_1 : -9u_1 - 3v_1 : -v_1 - 3u_1\Big]$$

where $[u_1 : v_1] \in \mathbb{P}^1_{\mathbb{C}}$. We already found these two lines in the solution to (a) twice. □

Exercise 10. Let S_2 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by $f_2(x, y, z, t) = 0$, where

$$f_2(x, y, z, t) = t^2 + tx - 2ty + tz + xy + xz - y^2 + yz.$$

Put $P = [1 : -2 : 1 : 1]$.

(a) Show that $f_2(x, y, z, t)$ is irreducible. Prove that S_2 is smooth.
(b) Check that $P \in S_2$. Find all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2 and pass through P. Find $[A : B : C : D] \in \mathbb{P}^3_{\mathbb{C}}$ such that the equation

$$Ax + By + Cz + Dt = 0$$

defines a plane $\Pi \subset \mathbb{P}^3_{\mathbb{C}}$ that is tangent to S_2 at the point P. Describe $\Pi \cap S_2$.
(c) Find a projective transformation $\phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(S_2)$ is given by $xy = zt$. Use this to describe all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2.

Solution. (a) Let us show that $f_2(x, y, z, t)$ is irreducible. There are many ways to do this. Suppose that $f_2(x, y, z, t)$ is reducible in $\mathbb{C}[x, y, z, t]$. Then it is a product of two non-constant polynomials. Since $f_2(x, y, z, t) = t^2 + (x - 2y + z)t + xy + xz - y^2 + yz$ is a monic polynomial of degree 2 in t, we have

$$f_2(x, y, z, t) = \big(A(x, y, z)t + B(x, y, z)\big)\big(C(x, y, z)t + D(x, y, z)\big)$$

for some polynomials $A(x, y, z)$, $B(x, y, z)$ and $C(x, y, z)$ in $\mathbb{C}[x, y, z]$. Then

$$\begin{cases} A(x, y, z)C(x, y, z) = 1, \\ A(x, y, z)D(x, y, z) + B(x, y, z)C(x, y, z) = x - 2y + z, \\ B(x, y, z)D(x, y, z) = xy + xz - y^2 + yz. \end{cases}$$

This implies, in particular, that $A(x, y, z)$ and $C(x, y, z)$ are non-zero constant polynomials. Since $A(x, y, z)C(x, y, z) = 1$, we may assume that $A(x, y, z) = 1$ and

$C(x,y,z) = 1$, because we can scale $A(x,y,z)t + B(x,y,z)$ by $C(x,y,z)$ and we can scale $C(x,y,z)t + D(x,y,z)$ by $\frac{1}{C(x,y,z)}$. Then

$$\begin{cases} D(x,y,z) + B(x,y,z) = x - 2y + z, \\ B(x,y,z)D(x,y,z) = xy + xz - y^2 + yz. \end{cases}$$

so that we have $D(x,y,z) = x - 2y + z - B(x,y,z)$ and

$$(x - 2y + z)B(x,y,z) - B^2(x,y,z) = xy + xz - y^2 + yz.$$

Completing the square, we see that

$$\Big(B - \frac{x - 2y + z}{2}\Big)^2 = \frac{(x - 2y + z)^2}{4} - xy + xz - y^2 + yz = \frac{x^2 - 8xy + 6xz + z^2}{4}.$$

In particular, the polynomial $x^2 - 8xy + 6xz + z^2$ is a square in $\mathbb{C}[x,y,z]$. Thus, we have $x^2 - 8xy + 6xz + z^2 = (ax + by + cz + d)^2$ for some complex numbers a, b, c and d. Then

$$x^2 - 8xy + 6xz + z^2 = a^2x^2 + 2abxy + 2acxz + b^2y^2 + 2bcyz + c^2z^2 + 2adx + 2bdy + 2cdz + d^2.$$

This is equality of polynomials. Thus, we have

$$\begin{cases} a^2 = 1, \\ 2ab = -8, \\ 3ac = 6, \\ b^2 = 0, \\ 2bc = 0, \\ c^2 = 1, \\ 2ad = 0, \\ 2bd = 0, \\ 2cd = 0, \\ d^2 = 0. \end{cases}$$

This system is inconsistent, which is a contradiction.

Now let us prove that S_2 is smooth. We have to show that $x = y = z = t = 0$ is the only solution to the system of equations

$$\begin{cases} \dfrac{\partial f_2(x,y,z,t)}{\partial x} = 0, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial y} = 0, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial z} = 0, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial t} = 0. \end{cases}$$

This is easy. Indeed, we have

$$\begin{cases} \dfrac{\partial f_2(x,y,z,t)}{\partial x} = t+y+z, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial y} = -2t+x-2y+z, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial z} = t+x+y, \\ \dfrac{\partial f_2(x,y,z,t)}{\partial t} = 2t+x-2y+z. \end{cases}$$

On the other hand, the system of linear equations

$$\begin{cases} t+y+z=0, \\ -2t+x-2y+z=0, \\ t+x+y=0, \\ 2t+x-2y+z=0, \end{cases}$$

does not have solutions except $x=y=z=t=0$, because

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 1 & -2 \\ 1 & 1 & 0 & 1 \\ 1 & -2 & 1 & 2 \end{vmatrix} = 16 \neq 0.$$

Note that this also implies that $f_2(x,y,z,t)$ is irreducible, which we already proved by *brute force*. Indeed, if $f_2(x,y,z,t)$ is a product of two non-constant polynomials $g(x,y,z,t)$ and $h(x,y,z,t)$, then they must be homogeneous, and Bezout theorem (actually, its new born baby version) implies that there is $[\alpha:\beta:\gamma] \in \mathbb{P}^2_{\mathbb{C}}$ such that $g(\alpha,\beta,\gamma,0) = h(\alpha,\beta,\gamma,0) = 0$, which implies that

$$\begin{cases} \dfrac{\partial f_2(\alpha,\beta,\gamma,0)}{\partial x} = g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial x} + h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial x} = 0, \\ \dfrac{\partial f_2(\alpha,\beta,\gamma,0)}{\partial y} = g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial y} + h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial y} = 0, \\ \dfrac{\partial f_2(\alpha,\beta,\gamma,0)}{\partial z} = g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial z} + h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial z} = 0, \\ \dfrac{\partial f_2(\alpha,\beta,\gamma,0)}{\partial t} = g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial t} + h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial t} = 0, \end{cases}$$

which contradicts to what we just proved above.

(b) Since $f_2(1,-2,1,1) = 0$, we see that $P \in S_2$. Let L be a line that passes through P. Let Q be the point of intersection of L and the plane $t=0$. Then $Q = [\alpha:\beta:\gamma:0]$ such that $f_2(\alpha:\beta:\gamma:0) = 0$. Then L is given by

$$\lambda\big[1:-2:1:1\big] + \mu\big[\alpha:\beta:\gamma:0\big],$$

where $[\lambda:\mu]$ runs through $\mathbb{P}^1_{\mathbb{C}}$. Then $L \subset S_2$ if and only if

$$f_2(\lambda+\mu\alpha, -2\lambda+\mu\beta, \lambda+\mu\gamma, \lambda) = 0$$

for every $[\lambda:\mu] \in \mathbb{P}^1_{\mathbb{C}}$. Thus, $L \subset S_2$ if and only if

$$\big(\alpha\beta + \alpha\gamma - \beta^2 + \beta\gamma\big)\mu^2 + 4\beta\mu\lambda = 0$$

for every $[\lambda:\mu] \in \mathbb{P}^1_{\mathbb{C}}$. Hence, $L \subset S_2$ if and only if $\beta = 0$ and $\alpha\gamma = 0$. This gives us exactly two possibilities for the point Q: either $Q = [1:0:0:0]$ or $Q = [0:0:1:0]$. Moreover, in both cases the line passing through P and Q is contained in S_2. Thus, there are exactly two lines lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2 and pass through P. They are the lines $[\lambda+\mu : -\lambda : \lambda : \lambda]$ and $[\lambda : -\lambda : \lambda+\mu : \lambda]$, where $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$.

For every point $[a : b : c : d] \in S_2$, the plane in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$\frac{\partial f_2(a,b,c,d)}{\partial x}x + \frac{\partial f_2(a,b,c,d)}{\partial y}y + \frac{\partial f_2(a,b,c,d)}{\partial z}z + \frac{\partial f_2(a,b,c,d)}{\partial t}t = 0$$

tangents the surface S_2 at the point $[a : b : c : d]$. Since

$$\begin{cases} \dfrac{\partial f_2(1:-2:1:1)}{\partial x} = 0, \\ \dfrac{\partial f_2(1:-2:1:1)}{\partial y} = 4, \\ \dfrac{\partial f_2(1:-2:1:1)}{\partial z} = 0, \\ \dfrac{\partial f_2(1:-2:1:1)}{\partial t} = 8, \end{cases}$$

the plane $y + 2t = 0$ is the tangent plane to the surface S_2 at the point P, so that $[A : B : C : D] = [0 : 1 : 0 : 2]$. Recall that we denoted this plane by Π. Then $\Pi \cap S_2$ is given by

$$\begin{cases} t^2 + tx - 2ty + tz + xy + xz - y^2 + yz = 0, \\ y + 2t = 0. \end{cases}$$

Plugging in $y = -2t$ into $t^2 + tx - 2ty + tz + xy + xz - y^2 + yz$, we get $t^2 - tx - tz + xz$. Thus, $\Pi \cap S_2$ is given by $y + 2t = t^2 - tx - tz + xz = 0$. On the other hand, we have

$$t^2 - tx - tz + xz = (t - z)(t - x).$$

Thus, the intersection $\Pi \cap S_2$ consist of two lines: $y+2t = z-t = 0$ and $y+2t = x-t = 0$. These are exactly the lines we found earlier.

(c) Let us find a projective transformation $\phi \colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$ such that $\phi(S_2)$ is given by $xy = zt$. Note that such transformation is not unique: we have a lot of freedom in choosing it. Observe that

$$f_2(x,y,z,t) = (t + y + z)x + t^2 - 2ty + tz - y^2 + yz.$$

Let us introduce new coordinates $\overline{x} = x$, $\overline{y} = t + y + z$, $\overline{z} = z$ and $\overline{t} = t$. Plugging $x = \overline{x}$, $y = \overline{y} - \overline{t} - \overline{z}$, $z = \overline{z}$, $t = \overline{t}$ into $f_2(x,y,z,t)$, we obtain the polynomial

$$\overline{f}_2(\overline{x},\overline{y},\overline{z},\overline{t}) = \overline{xy} + 2\overline{t}^2 - \overline{y}^2 + 3\overline{yz} - 2\overline{z}^2.$$

This is the defining equation of S_2 in this new homogeneous coordinates. It can be rewritten as

$$(\overline{x} - \overline{y} + 3\overline{z})\overline{y} + 2\overline{t}^2 - 2\overline{z}^2 = 0.$$

Put $\widehat{x} = \overline{x} - \overline{y} + 3\overline{z}$, $\widehat{y} = \overline{y}$, $\widehat{z} = \overline{z}$ and $\widehat{t} = \overline{t}$. Plugging $\overline{x} = \widehat{x} + \widehat{y} - 3\widehat{z}$, $\overline{y} = \widehat{y}$, $\overline{z} = \widehat{z}$, $\overline{t} = \widehat{t}$ into $\overline{f}_2(\overline{x},\overline{y},\overline{z},\overline{t})$, we obtain the polynomial

$$\widehat{f}_2(\widehat{x},\widehat{y},\widehat{z},\widehat{t}) = \widehat{x}\widehat{y} + 2\widehat{t}^2 - 2\widehat{z}^2.$$

This is the defining equation of S_2 in new homogeneous coordinates $\widehat{x}$, $\widehat{y}$, $\widehat{z}$, $\widehat{t}$. Now we put $\widetilde{x} = \widehat{x}$, $\widetilde{y} = \widehat{y}$, $\widetilde{z} = 2\widehat{z} - 2\widehat{t}$ and $\widetilde{t} = \widehat{z} + \widehat{t}$. Then S_2 is given by $\widetilde{x}\widetilde{y} = \widetilde{z}\widetilde{t}$. Since

$$\begin{cases} \widetilde{x} = x - y + 2z - t, \\ \widetilde{y} = y + z + t, \\ \widetilde{z} = 2z - 2t, \\ \widetilde{t} = z + t, \end{cases}$$

the required projective transformation ϕ is given by

$$[x:y:z:t] \mapsto \big[x-y+2z-t : y+z+t : 2z-2t : z+t\big].$$

One can double check that

$$\big(x-y+2z-t\big)\big(y+z+t\big) - \big(2z-2t\big)\big(z+t\big) = t^2+tx-2ty+tz+xy+xz-y^2+yz,$$

so that $\phi(S_2)$ is indeed given by $xy=zt$.

Now let us describe all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_2. To do this, let us recall the description of all lines in the quadric surface $xy=zt$. Recall that the quadric $xy=zt$ can be identified with $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$ via the map $\upsilon\colon \mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\to\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$\Big(\big[u_1:v_1\big],\big[u_2:v_2\big]\Big) \mapsto \Big[u_1u_2 : v_1v_2 : u_1v_2 : v_1u_2\Big].$$

Check that the image of υ is indeed contained in the quadric $xy=zt$. For every *fixed* point $[u_1:v_1]\in\mathbb{P}^1_{\mathbb{C}}$, the set

$$\big[u_1u_2 : v_1v_2 : u_1v_2 : v_1u_2\big]$$

gives us a line in the quadric $xy=zt$ when $[u_2:v_2]$ runs through $\mathbb{P}^1_{\mathbb{C}}$. Vice versa, for every *fixed* point $[u_2:v_2]\in\mathbb{P}^1_{\mathbb{C}}$, the set

$$\big[u_1u_2 : v_1v_2 : u_1v_2 : v_1u_2\big]$$

also gives us a line in the quadric $xy=zt$ when $[u_1:v_1]$ runs through $\mathbb{P}^1_{\mathbb{C}}$. This gives us all lines in the quadric in $xy=zt$.

Let $\psi\colon\mathbb{P}^3_{\mathbb{C}}\to\mathbb{P}^3_{\mathbb{C}}$ be the projective transformation that is the inverse of ϕ. Then ψ maps lines to lines, so that

$$\psi\Big(\big[u_1u_2 : v_1v_2 : u_1v_2 : v_1u_2\big]\Big)$$

gives us all lines in S_2 when we *fix* $[u_1:v_1]\in\mathbb{P}^1_{\mathbb{C}}$ or $[u_2:v_2]\in\mathbb{P}^1_{\mathbb{C}}$. Namely, the map ψ is given by

$$[x:y:z:t] \mapsto \big[4x+4y-3z-6t : 4y-4t : z+2t : 2t-z\big].$$

Thus, the composition $\psi\circ\upsilon$ is given by

$$\Big(\big[u_1:v_1\big],\big[u_2:v_2\big]\Big) \mapsto \Big[4u_1u_2+4v_1v_2-3u_1v_2-6v_1u_2 : 4v_1v_2-4v_1u_2 : u_1v_2+2v_1u_2 : 2v_1u_2-u_1v_2\Big].$$

This gives us the description of all lines in S_2 as

$$\Big[4u_1u_2+4v_1v_2-3u_1v_2-6v_1u_2 : 4v_1v_2-4v_1u_2 : u_1v_2+2v_1u_2 : 2v_1u_2-u_1v_2\Big]$$

when we fix $[u_1:v_1]\in\mathbb{P}^1_{\mathbb{C}}$ or $[u_2:v_2]\in\mathbb{P}^1_{\mathbb{C}}$. For example, $\phi(P)=[2:0:0:1]$ and

$$\upsilon\Big(\big([2:1],[1:0]\big)\Big) = [2:0:0:1].$$

Thus, the above description gives us two lines in S_2 that passes through P. The first line is given by

$$\Big[2u_2-2v_2 : 4v_2-4u_2 : 2v_2+2u_2 : 2u_2-2v_2\Big]$$

where $[u_2:v_2]\in\mathbb{P}^1_{\mathbb{C}}$. The second line is given by

$$\Big[4u_1-6v_1 : -4v_1 : 2v_1 : 2v_1\Big]$$

where $[u_1:v_1]\in\mathbb{P}^1_{\mathbb{C}}$. We already found these two lines in the solution to (a) twice.

$\square$

Exercise 11. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x,y,z,t)=0,$$

where $f_3(x,y,z,t)=txz+xy^2+y^3$. Do the following.

(a) Show that $f_3(x,y,z,t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Solution. Let us show that the polynomial $f_3(x,y,z,t)=txz+xy^2+y^3$ is irreducible. Observe that $f_3(x,y,z,t)=(tz+y^2)x+y^3$. Suppose that it is not irreducible. Then

$$(tz+y^2)x+y^3=\Big(A(y,z,t)x+B(y,z,t)\Big)C(y,z,t)$$

for some polynomials $A(y,z,t)$, $B(y,z,t)$ and $C(y,z,t)$. Then

$$\begin{cases} A(y,z,t)C(y,z,t)=tz+y^2,\\ B(y,z,t)C(y,z,t)=y^3. \end{cases}$$

Since $\mathbb{C}[y,z,t]$ is a unique factorization domain, we see that $C(y,z,t)$ is divisible by y. Thus, $tz+y^2$ is divisible by y, so that tz is also divisible by y, which is absurd. This shows that $f_3(x,y,z,t)$ is irreducible.

Let us find singular points of S_3. We have

$$\begin{cases} \dfrac{\partial f_3(x,y,z,t)}{\partial x}=tz+y^2,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial y}=2xy+3y^2,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial z}=tx,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial t}=xz. \end{cases}$$

Thus, the singular points of S_3 are given by

$$tz+y^2=2xy+3y^2=tx=xz=0,$$

which gives us the points $[1:0:0:0]$, $[0:0:1:0]$ and $[0:0:0:1]$. Thus, the singular points of the surface S_3 are $[1:0:0:0]$, $[0:0:1:0]$ and $[0:0:0:1]$. Note that these points are different in nature. You do not need to care about this at the moment, but this is a good thing to know. Namely, the point $[1:0:0:0]$ is an *ordinary double point* of the surface S_3, which is also denoted by $\mathbb{A}_1$. The remaining two singular points of the surface S_3 are singular points of type $\mathbb{A}_2$, which means that up to an *analytic* change of coordinates, the surface S_3 is given by

$$xy+z^3=0$$

in a neighborhood of any of these two points. These are the basic examples of the so-called Du Val singularities, which are also known by other names: rational double points, simple surface singularities, Kleinian singularities, two-dimensional canonical singularities, two-dimensional rational Gorenstein singularities etc.

Let us find all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_3. Observe that S_3 contains the following five lines: $y=x=0$, $y=z=0$, $y=t=0$, $z=x+y=0$ and $t=x+y=0$. Let us show that these 5 lines are all lines contained in S_3.

Let L be a line in S_3. Denote by Q a point in the intersection of this line with the plane $y=0$. Then $Q=[\alpha:0:\beta:\gamma]$, where at least one number among α, β, γ is not zero. Let us choose the second point on the line L. If $\alpha\neq 0$, let P be a point in the intersection of L with the plane $x=0$. If $\beta\neq 0$, let P be a point in the intersection of L with the plane $z=0$. If $\gamma \neq 0$, let P be a point in the intersection of L with the plane $t=0$. Then $P \neq Q$, so that L is uniquely determined by these two points.

If P is contained in the plane $y=0$, then the whole line L is contained in this plane, because Q is contained in the plane $y=0$ by construction. On the other hand, all lines in S_3 that are contained in the plane $y=0$ are the lines $y=x=0$, $y=z=0$, $y=t=0$. Thus, to complete the solution, we may assume that L is not one of these three lines, so that P is not contained in the plane $y=0$. Then $P=[a:1:b:c]$ for some complex numbers a, b and c. Note that at least one number among a, b, c is zero by construction.

For every $s \in \mathbb{C}$, the point $[a+s\alpha : 1 : b+s\beta : c+s\gamma]$ is contained in S_3. This means that

$$\big(\gamma s+c)(\alpha s+a)(\beta s+b)+\alpha s+a+1=0$$

for every $s \in \mathbb{C}$. Thus, we see that

$$\alpha\beta\gamma s^3+(\alpha\beta c+\beta\gamma a+\alpha\gamma b)s^2+(ac\beta+\alpha bc+\gamma ab+\alpha)s+abc+a+1.$$

for every $s \in \mathbb{C}$. Thus, this polynomial in s must be a zero polynomial. This gives us

$$\begin{cases} \alpha\beta\gamma=0, \\ \alpha\beta c+\beta\gamma a+\alpha\gamma b=0, \\ ac\beta+\alpha bc+\gamma ab+\alpha=0, \\ abc+a+1=0. \end{cases}$$

Recall that at least one number among a, b, c is zero. Since $abc+a+1=0$, $a \neq 0$. Thus, either $b=0$ or $c=0$ (or both).

Suppose that $\alpha=0$. Then

$$\beta\gamma a=ac\beta+\gamma ab=abc+a+1=0.$$

If $b=0$, this gives $\beta\gamma=c\beta=0$ and $a=-1$, so that either $P=[-1:1:0:0]$ and $Q=[0:0:1:0]$, or $P=[-1:1:0:c]$ and $Q=[0:0:0:1]$. In the former case, the line L is $t=x+y=0$. In the latter case, the line L is $z=x+y=0$. Similarly, if $c=0$ and $b \neq 0$, then $\gamma=0$ and $a=-1$, so that $P=[-1:1:b:0]$ and $Q=[0:0:1:0]$, which implies that L is the line $t=x+y=0$.

We may assume that $\alpha \neq 0$. Then we may assume that $\alpha=1$, so that we have

$$\begin{cases} \beta\gamma=0, \\ \beta c+\beta\gamma a+\gamma b=0, \\ ac\beta+bc+\gamma ab+1=0, \\ abc+a+1=0. \end{cases}$$

If $b=0$, then

$$\begin{cases} \beta\gamma=0, \\ \beta c+\beta\gamma a=0, \\ ac\beta+1=0, \\ a+1=0. \end{cases}$$

This linear system is inconsistent, so that $b \neq 0$. Then $c=0$, so that

$$\begin{cases} \beta\gamma=0, \\ \beta\gamma a+\gamma b=0, \\ \gamma ab+1=0, \\ a+1=0, \end{cases}$$

which does not have solutions as well. This shows that the only lines contained in S_3 are the five lines given by $y=x=0$, $y=z=0$, $y=t=0$, $z=x+y=0$ and $t=x+y=0$. □

Exercise 12. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x,y,z,t)=0,$$

where $f_3(x,y,z,t)=xyz+xyt+xzt+yzt$. Do the following.

(a) Show that $f_3(x,y,z,t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Solution. Observe that $f_3(x,y,z,t)=(xy+xz+yz)t+xyz$. Suppose that it is not irreducible. Then

$$(xy+xz+yz)t+xyz=\Big(A(x,y,z)t+B(x,y,z)\Big)C(x,y,z)$$

for some polynomials $A(x,y,z)$, $B(x,y,z)$ and $C(x,y,z)$. Then

$$\begin{cases} A(x,y,z)C(x,y,z)=xy+xz+yz,\\ B(x,y,z)C(x,y,z)=xyz.\end{cases}$$

Since $\mathbb{C}[x,y,z]$ is a unique factorization domain, and x, y and z are irreducible polynomials, we deduce that $C(x,y,z)$ is divisible by one of them. Thus, $xy+xz+yz$ is divisible by one polynomial among x, y and z, which is absurd. This shows that $f_3(x,y,z,t)$ is irreducible.

Let us find singular points of S_3. We have

$$\begin{cases} \dfrac{\partial f_3(x,y,z,t)}{\partial x}=ty+tz+yz,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial y}=tx+tz+xz,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial z}=tx+ty+xy,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial t}=xy+xz+yz.\end{cases}$$

Thus, we have to find all $[x:y:z:t]\in\mathbb{P}^3_{\mathbb{C}}$ such that

$$\begin{cases} ty+tz+yz=0,\\ tx+tz+xz=0,\\ tx+ty+xy=0,\\ xy+xz+yz=0.\end{cases}$$

This is easy. Observe that the points $[1:0:0:0]$, $[0:1:0:0]$, $[0:0:1:0]$, $[0:0:0:1]$ are solutions. We claim that these four points are the only solutions to this system of equations. Indeed, if $x=0$, then this system gives

$$tz=ty=yz=0,$$

which gives us the points $[0:1:0:0]$, $[0:0:1:0]$, $[0:0:0:1]$. Thus, we may assume that $x\neq 0$, so that we may assume that $x=1$. Then we have to solve

$$\begin{cases} ty+tz+yz=0,\\ t+tz+z=0,\\ t+ty+y=0,\\ y+z+yz=0.\end{cases}$$

Adding the last three equations together and using the first one, we get $t + y + z = 0$. Now we can plug in $t = -y - z$ into $t + tz + z = 0$ and $t + ty + y = 0$. This gives us

$$\begin{cases} yz + z^2 + y = 0, \\ y^2 + yz + z = 0, \\ y + z + yz = 0. \end{cases}$$

In particular, we have $z \neq -1$, since $y + z + yz = 0$. Then $y = -\frac{1}{z+1}$. Plugging this into $yz + z^2 + y = 0$ and $y^2 + yz + z = 0$, we obtain

$$(z-1)z = \frac{z(2z+1)}{(z+1)^2} = 0,$$

which implies that $z = 0$, so that $y = t = 0$ as well. This gives us the point $[1 : 0 : 0 : 0]$. Therefore, the points $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$, $[0 : 0 : 0 : 1]$ are the only singular points of the surface S_3.

Now it is time to find all lines in S_3. We already met three of them: $y = z = 0$, $y = t = 0$ and $z = t = 0$. Similarly, we get three more lines: $x = y = 0$, $x = z = 0$ and $x = t = 0$. Let us try to show that these 6 lines are all lines contained in S_3.

Let L be a line in S_3. Denote by Q a point in the intersection of this line with a plane $t = 0$. Then $Q = [\alpha : \beta : \gamma : 0]$. Let P_x, P_y and P_z be points in the intersections of L with the planes $x = 0$, $y = 0$ and $z = 0$, respectively. At least one of them should be different from Q, because at least one number among α, β, γ is not zero. Denote this point (the one which is not Q) by P. Put $P = [a : b : c : d]$. Then at least one number among a, b, c is zero by construction. Moreover, if $d = 0$, then L is contained in the plane $t = 0$. However, this plane intersects the surface S_3 by the lines $x = t = 0$, $y = t = 0$ and $z = t = 0$. Thus, if L is not one of them, then $d \neq 0$. Hence, we may assume that $d \neq 0$, so that we can put $d = 1$.

The line L consists of all points

$$\Big[ra + s\alpha : rb + s\beta : rc + s\gamma : r\Big]$$

where $[r : s]$ runs through $\mathbb{P}^1_{\mathbb{C}}$. For simplicity we may ignore the point $[0 : 1] \in \mathbb{P}^1_{\mathbb{C}}$. Thus, for every $s \in \mathbb{C}$, the point $[a + s\alpha : b + s\beta : c + s\gamma : 1]$ is contained in S_3. This means that

$$(\alpha s + a)(\beta s + b)(\gamma s + c) + (\alpha s + a)(\beta s + b) + (\alpha s + a)(\gamma s + c) + (\beta s + b)(\gamma s + c) = 0$$

for every $s \in \mathbb{C}$. Thus, we see that

$$\alpha\beta\gamma s^3 + \Big(\alpha\gamma b + \beta\gamma a + \alpha\beta c + \alpha\beta + \alpha\gamma + \beta\gamma\Big)s^2 + \\ + \Big(ab\gamma + \alpha bc + \beta ac + \alpha b + \beta a + \alpha c + a\gamma + \beta c + b\gamma\Big)s + abc + ab + ac + bc = 0$$

for every $s \in \mathbb{C}$. Thus, this polynomial in s must be a zero polynomial. This gives us

$$\begin{cases} \alpha\beta\gamma = 0, \\ \alpha\gamma b + \beta\gamma a + \alpha\beta c + \alpha\beta + \alpha\gamma + \beta\gamma = 0, \\ ab\gamma + \alpha bc + \beta ac + \alpha b + \beta a + \alpha c + a\gamma + \beta c + b\gamma = 0, \\ abc + ab + ac + bc = 0. \end{cases}$$

Recall that at least one number among a, b, c is zero. Actually, at least two numbers among a, b, c must be zero, because we have $abc + ab + ac + bc = 0$. Thus, it is enough to consider the following four cases: $a = b = c = 0$, $a = b = 0 \neq c$, $a = c = 0 \neq b$, $b = c = 0 \neq a$. Let us do this separately case by case.

Suppose that $a=b=c=0$, so that $P=[0:0:0:1]$. Then

$$\begin{cases} \alpha\beta\gamma=0, \\ \alpha\beta+\alpha\gamma+\beta\gamma=0. \end{cases}$$

This gives exactly three possibilities for the point Q. Namely, either $Q=[0:0:1:0]$, or $Q=[0:1:0:0]$, or $Q=[1:0:0:0]$, so that either L is the line $x=y=0$, or L is the line $x=z=0$, or L is the line $y=z=0$, respectively.

Suppose that $a=b=0\neq c$. Then $P=[0:0:c:1]$ and

$$\begin{cases} \alpha\beta\gamma=0, \\ \alpha\beta c+\alpha\beta+\alpha\gamma+\beta\gamma=0, \\ \alpha+\beta=0. \end{cases}$$

Thus, at least one of the numbers α, β, γ is zero. If $\alpha=0$, then we have

$$\begin{cases} \beta\gamma=0, \\ \beta=0, \end{cases}$$

which gives $Q=[0:0:1:0]$, so that L is the line $x=y=0$. If $\beta=0$, then we have

$$\begin{cases} \alpha\gamma=0, \\ \alpha=0. \end{cases}$$

which also gives $Q=[0:0:1:0]$, so that L is the line $x=y=0$ as before. Finally, if $\gamma=0$, then

$$\begin{cases} \alpha\beta c+\alpha\beta=0, \\ \alpha+\beta=0. \end{cases}$$

which gives $\alpha=-\beta\neq 0$ and $c=-1$, because at least one number among α, β and γ is not zero. Thus, if $\gamma=0$, then $P=[0:0:-1:1]$ and $Q=[1:-1:0:0]$. Actually the line that passes through these two points is different from any line among $y=z=0$, $y=t=0$, $z=t=0$, $x=y=0$, $x=z=0$ and $x=t=0$. Indeed, none of these six lines contains both points $[0:0:-1:1]$ and $[1:-1:0:0]$. This shows that our original guess was wrong! We found seventh line on S_3. This line is given by $x+y=z+t=0$. OK, lets continue.

Now we consider the case $a=c=0\neq b$. In this case, we have $P=[0:b:0:1]$ and

$$\begin{cases} \alpha\beta\gamma=0, \\ \alpha\gamma b+\alpha\beta+\alpha\gamma+\beta\gamma=0, \\ \alpha+\gamma=0. \end{cases}$$

If $\alpha=0$, then

$$\begin{cases} \beta\gamma=0, \\ \gamma=0, \end{cases}$$

so that $Q=[0:1:0:0]$, which implies that L is the line $x=z=0$. If $\gamma=0$, then

$$\begin{cases} \alpha\beta=0, \\ \gamma=0, \end{cases}$$

so that $Q=[0:1:0:0]$ again, which again implies that L is the line $x=z=0$. However, if $\beta=0$, then

$$\begin{cases} \alpha\gamma b+\alpha\gamma=0, \\ \alpha+\gamma=0, \end{cases}$$

which implies that $Q=[1:0:-1:0]$ and $P=[0:-1:0:1]$, so that L is given by $x+z=y+t=0$. This is new line! This line is different from the lines we found so far, and it is contained in S_3. So, it total we found eight lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_3.

Finally, we consider the case $b = c = 0 \neq a$. Then $P = [a : 0 : 0 : 1]$ and

$$\begin{cases} \alpha\beta\gamma = 0, \\ \beta\gamma a + \alpha\beta + \alpha\gamma + \beta\gamma = 0, \\ \beta + \gamma = 0. \end{cases}$$

If $\beta = 0$ or $\gamma = 0$, then $Q = [1 : 0 : 0 : 0]$, so that L is given by $y = z = 0$. On the other hand, it $\alpha = 0$, then

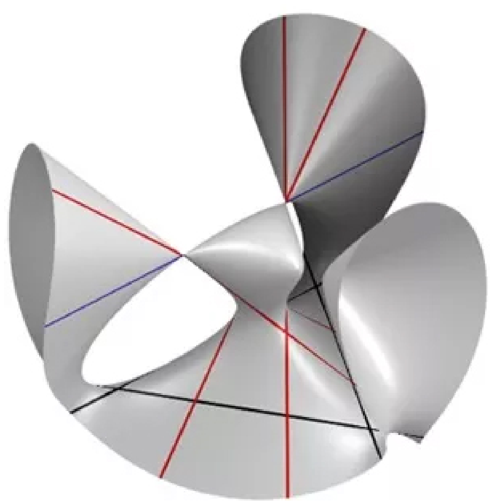
$$\begin{cases} \beta\gamma a + \beta\gamma = 0, \\ \beta + \gamma = 0, \end{cases}$$

which implies that $Q = [0 : 1 : -1 : 0]$ and $P = [-1 : 0 : 0 : 1]$, so that L is given by $x + t = y + z = 0$. This line is also different from the lines we found so far, and it is contained in S_3.

Thus, we found nine lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_3. These lines are $y = z = 0$, $y = t = 0$, $z = t = 0$, $x = y = 0$, $x = z = 0$, $x = t = 0$, $x + y = z + t = 0$, $x + z = y + t = 0$ and $x + t = y + z = 0$. Each of the first six lines passes through pair of singular points, so all of them forms something that looks like tetrahedron with vertices in $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$, $[0 : 0 : 0 : 1]$. Actually, these is a tetrahedron, and the lines $y = z = 0$, $y = t = 0$, $z = t = 0$, $x = y = 0$, $x = z = 0$ and $x = t = 0$ are just the lines that passes through its edges. However, the lines $x + y = z + t = 0$, $x + z = y + t = 0$ and $x + t = y + z = 0$ lie in the smooth locus of the surface S_3, i.e. they do not contain singular points of the surface S_3. In fact, they also lie in one plane. This plane is given, what a surprise, by $x+y+z+t = 0$. We could find these three lines in the very beginning of our hunt if we pugged $t = -x - y - z$ into $f_3(x, y, z, t)$ and get

$$f_3\big(x, y, z, -x-y-z\big) = -x^2y - x^2z - xy^2 - 2xyz - xz^2 - y^2z - yz^2 = -(y+z)(x+z)(x+y),$$

which basically implies that the hyperplane section of S_3 by the plane $x + y + z + t = 0$ splits as a union of three lines $x + y = z + t = 0$, $x + z = y + t = 0$ and $x + t = y + z = 0$. Alternatively, we could google the equation $xyz + xyt + xzt + yzt = 0$ of the surface S_3 or google "cubic surface with four singular points" to find out that our cubic surface S_3 actually has a name: it is called Cayley cubic surface. Some web pages about Cayley cubic surface mention that it contains nine lines or contains a picture like this



where you can see 3 lines that do not pass through singular points, so that it is not hard to guess their equations from there. Of course, the way we found the missing three lines on S_3 is more fun, because it gave us a feeling of discovery. $\square$

Exercise 13. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x,y,z,t)=0,$$

where $f_3(x,y,z,t)=txz+y^2z+x^3+\lambda z^3$ for some complex number λ. Do the following.

(a) Show that $f_3(x,y,z,t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Solution. Suppose that $f_3(x,y,z,t)$ is not irreducible. Then

$$txz+y^2z+x^3+\lambda z^3=\Big(A(x,y,z)x+B(x,y,z)\Big)C(x,y,z)$$

for some polynomials $A(x,y,z)$, $B(x,y,z)$ and $C(x,y,z)$ in $\mathbb{C}[x,y,z]$ such that $C(x,y,z)$ is a non-constant polynomial. Then

$$\begin{cases}A(x,y,z)C(x,y,z)=xz,\\ B(x,y,z)C(x,y,z)=y^2z+x^3+\lambda z^3.\end{cases}$$

Since $\mathbb{C}[x,y,z]$ is a unique factorization domain, we see that $C(x,y,z)$ is divisible either by y or by x (or both). Thus, $y^2z+x^3+\lambda z^3$ is divisible either by y or by x (or both), which is not true. This shows that $f_3(x,y,z,t)$ is irreducible.

If $\lambda\neq 0$, then S_3 is projectively equivalent to the cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$txz+y^2z+x^3-z^3=0$$

Indeed, let ω be a complex number such that $\omega^6=\lambda$. If $\lambda\neq 0$, then $\phi(S_3)$ is given by $txz+y^2z+x^3+z^3=0$, where $\phi\colon\mathbb{P}^3_{\mathbb{C}}\to\mathbb{P}^3_{\mathbb{C}}$ is a projective transformation given by

$$[x:y:z:t]\mapsto\Big[x:\frac{i}{\omega}y:-\omega^2z:-\frac{1}{\omega^2}t\Big].$$

Thus, we have to consider two cases here: $\lambda=0$ and $\lambda=-1$.

To find all singular points (if any) of the surface S_3, observe that

$$\begin{cases}\dfrac{\partial f_3(x,y,z,t)}{\partial x}=tz+3x^2,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial y}=2yz,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial z}=tx+y^2+3az^2,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial t}=xz.\end{cases}$$

The singular points of S are $[x:y:z:t]\in\mathbb{P}^3_{\mathbb{C}}$ given by

$$\begin{cases}tz+3x^2=0,\\ 2yz=0,\\ tx+y^2+3\lambda z^2=0,\\ xz=0.\end{cases}$$

If $z=0$, then this system gives

$$\begin{cases}3x^2=0,\\ tx+y^2=0.\end{cases}$$

so that $x = y = 0$, which gives us the point $[0 : 0 : 0 : 1]$. Hence, the surface S is singular at the point $[0 : 0 : 0 : 1]$ regardless of what λ is. If $z \neq 0$, then we have

$$\begin{cases} tz = 0, \\ y = 0, \\ 3\lambda z^2 = 0, \\ x = 0, \end{cases}$$

so that $\lambda = 0$ and $x = y = t = 0$. This shows that if $\lambda \neq 0$, then the only singular point of the surface S_3 is the point $[0 : 0 : 0 : 1]$. Moreover, if $\lambda = 0$, then the surface S_3 is also singular at the point $[0 : 0 : 1 : 0]$.

The point $[0 : 0 : 0 : 1]$ is a singular point of S_3 of type $\mathbb{A}_5$. This means that there exists an analytical change of coordinates of $\frac{x}{t}$, $\frac{y}{t}$, $\frac{z}{t}$ such that S is given by

$$\mathbf{x}\mathbf{y} + \mathbf{z}^6 = 0$$

in a neighborhood of this point. If $\lambda = 0$, then $[0 : 0 : 1 : 0]$ is a singular point of S_3 of type $\mathbb{A}_1$, which is also known as the simplest isolated double point or ordinary double point. This is the simplest singularity a surface can have.

Let us find all lines in the surface S_3. Recall that we assume that either $\lambda = 0$ or $\lambda = -1$. If $\lambda = 0$, then S_3 contains the lines $x = y = 0$ and $x = z = 0$. If $\lambda = -1$, then S_3 contains the lines $x = z = 0$, $x = y - z = 0$ and $x = y + z = 0$. Note that these are all lines that are contained in the plane $x = 0$. Let us show that S_3 does not contain other lines.

Let L be a line in $S - 3$. Suppose that L is not one of the lines described above. Then L is not contained in the plane $x = 0$, so that this plane has unique common point with the line L. Denote this point by $Q = [0 : \alpha : \beta : \gamma]$, where at least one number among α, β, γ is not zero. Let us choose another point in the line L. If $\alpha \neq 0$, let P be a point in the intersection of L with the plane $y = 0$. If $\alpha = 0$ and $\beta \neq 0$, let P be a point in the intersection of L with the plane $z = 0$. If $\alpha = \beta = 0$ and $\gamma \neq 0$, let P be a point in the intersection of L with the plane $t = 0$. Then $P \neq Q$, so that L is uniquely determined by these two points.

If P is contained in the plane $x = 0$, then the whole line L is contained in this plane, because Q is contained in the plane $x = 0$ by construction. Since we assumed that L is not contained in the plane $x = 0$, we see that P is not contained in the plane $x = 0$ either. Then $P = [1 : a : b : c]$ for some complex numbers a, b and c.

Recall that at least one number among a, b, c is zero by construction. In fact, the construction of the point P gives slightly more. If $\alpha \neq 0$, then $a = 0$. If $\alpha = 0$ and $\beta \neq 0$, then $b = 0$. If $\alpha = \beta = 0$ and $\gamma \neq 0$, then $c = 0$.

For every $s \in \mathbb{C}$, the point $[1 : a + s\alpha : b + s\beta : c + s\gamma]$ is contained in S_3. This means that

$$(\gamma s + c)(\beta s + b) + (\alpha s + a)^2(\beta s + b) + 1 + \lambda(\beta s + b)^3 = 0$$

for every $s \in \mathbb{C}$. Thus, we see that

$$\big(\beta^3\lambda+\alpha^2\beta\big)s^3+(3\beta^2 b\lambda+\alpha^2 b+2\alpha\beta a+\beta\gamma)s^2+(3\beta b^2\lambda+2\alpha ab+\beta a^2+\beta c+\gamma b)s+\lambda b^3+a^2b+cb+1 = 0$$

for every $s \in \mathbb{C}$. Thus, this polynomial in s must be a zero polynomial. This gives us

$$\begin{cases} \beta^3\lambda + \alpha^2\beta = 0, \\ 3\beta^2 b\lambda + \alpha^2 b + 2\alpha\beta a + \beta\gamma = 0, \\ 3\beta b^2\lambda + 2\alpha ab + \beta a^2 + \beta c + \gamma b = 0, \\ \lambda b^3 + a^2 b + cb + 1 = 0. \end{cases}$$

This implies, in particular, that $b \neq 0$.

Suppose that $\alpha \neq 0$. Then we may assume that $\alpha = 1$. Moreover, we have $a = 0$ by assumption. Then

$$\begin{cases} \beta^3\lambda + \beta = 0, \\ 3\beta^2 b\lambda + b + \beta\gamma = 0, \\ 3\beta b^2\lambda + \beta c + \gamma b = 0, \\ \lambda b^3 + cb + 1 = 0. \end{cases}$$

If $\beta = 0$, we get $b = 0$, $\gamma b = 0$, and $\lambda b^3 + cb + 1 = 0$, which is a contradiction. Thus, we have $\beta \neq 0$. Then

$$\begin{cases} \beta^2\lambda + 1 = 0, \\ 3\beta^2 b\lambda + b + \beta\gamma = 0, \\ 3\beta b^2\lambda + \beta c + \gamma b = 0, \\ \lambda b^3 + cb + 1 = 0. \end{cases}$$

This implies that $\lambda \neq 0$, so that $\lambda = -1$ by our assumption. Hence, we have

$$\begin{cases} \beta^2 = 1, \\ 3\beta^2 b - b - \beta\gamma = 0, \\ 3\beta b^2 - \beta c - \gamma b = 0, \\ b^3 - cb - 1 = 0, \end{cases}$$

which implies that either $\beta = 1$ or $\beta = -1$. If $\beta = 1$, we get

$$\begin{cases} 3b - b - \gamma = 0, \\ 3b^2 - c - \gamma b = 0, \\ b^3 - cb - 1 = 0, \end{cases}$$

Multiplying the first equality by b and subtracting the resulting equality from the second equality, we get $b^2 = c$. Then the third equality gives $0 = 1$, which is absurd. Similarly, if $\beta = -1$, then

$$\begin{cases} 3b - b + \gamma = 0, \\ 3b^2 - c + \gamma b = 0, \\ b^3 - cb - 1 = 0, \end{cases}$$

This system is inconsistent as well. Thus, we see that $\alpha \neq 0$.

If $\alpha = 0$ and $\beta \neq 0$, then $b = 0$ by assumption, which we already see not to be the case. Thus, we see that $\alpha = \beta = 0$. Then $\gamma \neq 0$, so that $c = 0$ by assumption. Now our main system of equation gives us $\gamma b = 0$ and $\lambda b^3 + a^2 b + cb + 1 = 0$, which is impossible, since $\gamma \neq 0$. The obtained contradiction completes the solution. □

Exercise 14. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x, y, z, t) = 0,$$

where $f_3(x, y, z, t) = tz^2 + zx^2 + y^2x + \lambda t^3$ for some complex number λ. Do the following.

(a) Show that $f_3(x, y, z, t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Solution. Arguing as in the solution to Exercises 13, we see that $f_3(x, y, z, t)$ is irreducible. Likewise, if $\lambda \neq 0$, then S_3 is projectively equivalent to the surface in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$tz^2 + zx^2 + y^2x - t^3 = 0,$$

Indeed, let ω be a complex number such that $\omega^{24} = -\lambda$. If $\lambda \neq 0$, then $\phi(S_3)$ is given by $txz + y^2z + x^3 + z^3 = 0$, where $\phi\colon \mathbb{P}^3_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$ is a projective transformation given by

$$[x : y : z : t] \mapsto \Big[\omega^2 x : \frac{y}{\omega} : \frac{z}{\omega^4} : \omega^8 t\Big].$$

Thus, we have to consider two cases here: $\lambda = 0$ and $\lambda = -1$.

Let us find all singular points (if any) of the cubic surface S_3. We have

$$\begin{cases} \dfrac{\partial f_3(x,y,z,t)}{\partial x} = 2xz + y^2, \\ \dfrac{\partial f_3(x,y,z,t)}{\partial y} = 2xy, \\ \dfrac{\partial f_3(x,y,z,t)}{\partial z} = 2tz + x^2, \\ \dfrac{\partial f_3(x,y,z,t)}{\partial t} = 3\lambda t^2 + z^2. \end{cases}$$

We have to find all points $[x : y : z : t] \in \mathbb{P}^3_{\mathbb{C}}$ given by

$$\begin{cases} 2xz + y^2 = 0, \\ 2xy = 0, \\ 2tz + x^2 = 0, \\ 3\lambda t^2 + z^2 = 0. \end{cases}$$

Thus, either $x = 0$ or $y = 0$. If $x = 0$, then this system gives $y = tz = 3\lambda t^2 + z^2 = 0$, which gives $\lambda = y = z = 0$, so that $\lambda = 0$ and $[x : y : z : t] = [0 : 0 : 0 : 1]$. If $x \neq 0$, then $y = 0$, so that

$$\begin{cases} 2xz = 0, \\ 2tz + x^2 = 0, \\ 3\lambda t^2 + z^2 = 0, \end{cases}$$

which gives $z = 0$ and $x = 0$, which is a contradiction. Thus, we see that S_3 is smooth if $\lambda \neq 0$, and S_3 has unique singular point $[0 : 0 : 0 : 1]$ if $\lambda = 0$. In the latter case, $[0 : 0 : 0 : 1]$ is a singular point of S of type $\mathbb{D}_5$ (google it).

Note that our computations also implies that $f_3(x,y,z,t)$ is irreducible, which we already know. Indeed, suppose that $f_3(x,y,z,t)$ is a product of two non-constant polynomials $g(x,y,z,t)$ and $h(x,y,z,t)$. Multiplying homogeneous parts of $g(x,y,z,t)$ and $h(x,y,z,t)$ and comparing the result to $f_3(x,y,z,t)$, we see that both $g(x,y,z,t)$ and $h(x,y,z,t)$ are also homogeneous. Then there is $[\alpha : \beta : \gamma : 0] \in \mathbb{P}^3_{\mathbb{C}}$ (why?) such that $g(\alpha,\beta,\gamma,0) = 0$ and $h(\alpha,\beta,\gamma,0) = 0$. This gives

$$\begin{cases} \dfrac{\partial f_3(\alpha,\beta,\gamma,0)}{\partial x} = g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial x} + h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial x} = 0, \\ \dfrac{\partial f_3(\alpha,\beta,\gamma,0)}{\partial y} = g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial y} + h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial y} = 0, \\ \dfrac{\partial f_3(\alpha,\beta,\gamma,0)}{\partial z} = g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial z} + h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial z} = 0, \\ \dfrac{\partial f_3(\alpha,\beta,\gamma,0)}{\partial t} = g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial t} + h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial t} = 0. \end{cases}$$

However, we just proved that $[0 : 0 : 0 : 1] \neq [\alpha : \beta : \gamma : 0]$ is only point that satisfies this system of equations. Thus, we see that $f_3(x,y,z,t)$ is irreducible.

To find all lines on S_3, we first consider the case $\lambda = 0$. In this case, the surface S_3 contains the lines $x = z = 0$, $x = t = 0$ and $y = z = 0$. Moreover, the lines $x = z = 0$

and $x = t = 0$ are the only lines in S that are contained in the plane $x = 0$. Similarly, the lines $x = z = 0$ and $y = z = 0$ are the only lines in S that are contained in the plane $z = 0$. Let us show that S_3 does not contain other lines except these three.

Let L be a line in S_3. Suppose that L is neither the line $x = z = 0$ nor the line $y = z = 0$. Thus, in particular, L is not contained in the plane $z = 0$. Let us show that L is the line $x = t = 0$.

Since L is not contained in the plane $z = 0$, it intersects this plane by a point. Denote this point by Q. Then $Q = [\alpha : \beta : 0 : \gamma]$ for some complex number α, β and γ such that at least one of them is not zero. Note that L is uniquely determined by Q and a point in L that is different from Q. Let us choose this point is a good way. Namely, if $\alpha \neq 0$, let P be the intersection point of the line L and the plane $x = 0$. Similarly, if $\alpha = 0$ and $\beta \neq 0$, let P be the intersection point of the line L and the plane $y = 0$. Finally, if both α and β vanish, then $\gamma \neq 0$, so we choose P to be the intersection point of the line L and the plane $t = 0$. Then $P \neq Q$ by construction.

If P is contained in the plane $z = 0$, then the whole line L is contained in this plane, because Q is contained in the plane $z = 0$ by construction. Thus, P is not contained in the plane $z = 0$, because L is not contained in the plane $z = 0$. In particular, we have $P = [a : b : 1 : c]$ for some complex numbers a, b and c. Moreover, if $\alpha \neq 0$, then $a = 0$. Similarly, if $\alpha = 0$ and $\beta \neq 0$, then $b = 0$. Finally, if $\alpha = \beta = 0$ and $\gamma \neq 0$, then $c = 0$.

For every $s \in \mathbb{C}$, the point $[a + s\alpha : b + s\beta : 1 : c + s\gamma]$ is contained in S_3. This means that

$$(c + s\gamma) + (a + s\alpha)^2 + (b + s\beta)^2(a + s\alpha) = 0$$

for every $s \in \mathbb{C}$. Thus, we see that

$$\alpha\beta^2 s^3 + \big(\alpha^2 + a\beta^2 + 2b\alpha\beta\big)s^2 + \big(\gamma + 2a\alpha + b^2\beta + 2ab\beta\big)s + c + a^2 + b^2 a = 0$$

for every $s \in \mathbb{C}$. Thus, this polynomial in s must be a zero polynomial. This gives us

$$\begin{cases} \alpha\beta^2 = 0, \\ \alpha^2 + a\beta^2 + 2b\alpha\beta = 0, \\ \gamma + 2a\alpha + b^2\beta + 2ab\beta = 0, \\ c + a^2 + b^2 a = 0. \end{cases}$$

This implies, in particular, that either $\alpha = 0$ or $\beta = 0$ (or both). Thus, if $\alpha \neq 0$, then $\beta = 0$, which implies that $a = 0$ by the construction of the point P, so that the second equation of the system above gives us $1 = 0$, which is absurd. Thus, we have $\alpha = 0$. If $\beta \neq 0$, then $b = 0$ by the construction of the point P, so that we have

$$\begin{cases} a\beta^2 = 0, \\ \gamma = 0, \\ c + a^2 = 0, \end{cases}$$

which implies that $P = [0 : 0 : 1 : 0]$ and $Q = [0 : 1 : 0 : 0]$, so that L is the line $x = t = 0$. If $\alpha = \beta = 0$, the third equation of our system gives $\gamma = 0$, which is a contradiction. This shows that L is the line $x = t = 0$.

Now we will find all lines on S_3 in the case when $\lambda \neq 0$. Then $\lambda = -1$, so that S_3 is given by

$$tz^2 + zx^2 + y^2x - t^3 = 0.$$

This surface is smooth, so that it contains 27 lines. Let us find these 27 lines and (for consistency) prove that these are all lines contained in S_3. First of all, let us spot three of them. This is easy: the intersection of S_3 and the plane $x = 0$ splits as a union of three lines: $x = t = 0$, $x = z - t = 0$ and $x = z + t = 0$. Denote them by L_1, L_2 and L_3, respectively.

Let Π_1 be the plane in $\mathbb{P}^3_{\mathbb{C}}$ that is given by $x = \mu t$, where $\mu \in \mathbb{C}$. When μ runs through $\mathbb{C}$, the plane Π_1 runs through all planes in $\mathbb{P}^3_{\mathbb{C}}$ that contains L_1 except the plane $t = 0$. By construction, the intersection $\Pi_1 \cap S$ contains L_1. Thus, it splits as a union of the line L_1 and a (possibly) conic C_1. Moreover, the intersection of the surface S and the plane $t = 0$ is a union of the line L_1 and an irreducible conic that is given by $t = zx + y^2 = 0$. Thus, every line in S_3 that intersects L_1 must be an irreducible component of the conic C_1 for some complex number μ. Let us find all such μ.

The intersection $\Pi_1 \cap S_3$ is given by

$$\begin{cases} tz^2 + zx^2 + y^2x - t^3 = 0, \\ x = \mu t. \end{cases}$$

We can rewrite it as

$$\begin{cases} tz^2 + \mu^2 zt^2 + \mu y^2 t - t^3 = 0, \\ x = \mu t. \end{cases}$$

Thus, the conic C_1 is given by

$$\begin{cases} z^2 + \mu^2 zt + \mu y^2 - t^2 = 0, \\ x = \mu t. \end{cases}$$

It is isomorphic to a conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$z^2 + \mu^2 zt + \mu y^2 - t^2 = 0,$$

where y, z, t are homogeneous coordinates on $\mathbb{P}^2_{\mathbb{C}}$. If this conic C_1 is reducible, it must split as a union of two lines. Let us find all $\mu \in \mathbb{C}$ such that this happens.

We can rewrite the polynomial $z^2 + \mu^2 zt + \mu y^2 - t^2$ in the matrix form as

$$\begin{pmatrix} y & z & t \end{pmatrix} \begin{pmatrix} \mu & 0 & 0 \\ 0 & 1 & \frac{\mu^2}{2} \\ 0 & \frac{\mu^2}{2} & -1 \end{pmatrix} \begin{pmatrix} y \\ z \\ t \end{pmatrix}.$$

This implies that $z^2 + \mu^2 zt + \mu y^2 - t^2$ is reducible if and only if the rank of this matrix is one, which simply means that its determinant is zero. On the other hand, we have

$$\begin{vmatrix} \mu & 0 & 0 \\ 0 & 1 & \frac{\mu^2}{2} \\ 0 & \frac{\mu^2}{2} & -1 \end{vmatrix} = -\frac{\mu(\mu^4 + 4)}{4}.$$

Thus, C_1 is reducible $\iff$ μ is one of the following numbers: 0, $1-i$, $1+i$, $-1-i$, $-1+i$. Moreover, if $\mu = 0$, then C_1 splits as a union of the lines $x = z - t = 0$ nd $x = z + t = 0$. We already know these two lines. Let us describe the irreducible components of C_1 when μ is $1 - i$, $1 + i$, $-1 - i$ or $-1 + i$.

Suppose that $\mu = 1 - i$. Then C_1 is given by

$$x - (1 - i)t = (1 - i)y^2 + z^2 - 2itz - t^2.$$

We already know that the polynomial $(1 - i)y^2 + z^2 - 2itz - t^2$ must splits as a product of two linear polynomials $l_1(y, z, t)$ and $l_2(y, z, t)$. Thus, the conic C_1 splits as a union of two lines ℓ_1 and ℓ_2 that are given by

$$x - (1 - i)t = l_1(y, z, t) = 0$$

and $x - (1 - i)t = l_2(y, z, t) = 0$, respectively. Let us find these polynomials $l_1(y, z, t)$ and $l_2(y, z, t)$. Taking the partial derivatives of the polynomial $(1 - i)y^2 + z^2 - 2itz - t^2$, we see that their vanish only when $y=0$, $z=1$, $t=-i$. Thus, the point $\ell_1\cap\ell_2$ is the point $[1+i:0:1:-i]$. Thus, we can put $\overline{t}=t+iz$, $\overline{y}=y$ and $\overline{z}=z$. Then

$$(1-i)y^2+z^2-2itz-t^2=(1-i)\overline{y}^2-\overline{t}^2=(\sqrt{1-i}\overline{y}-\overline{t})(\sqrt{1-i}\overline{y}+\overline{t}),$$

where $\sqrt{1-i}$ is one of the complex square roots of $1-i$. So that

$$(1-i)y^2+z^2-2itz-t^2=(\sqrt{1-i}y-t-iz)(\sqrt{1-i}y+t+iz).$$

Thus, we may assume that ℓ_1 is given by $x-(1-i)t=\sqrt{1-i}y-t-iz=0$, and ℓ_2 is given by $x-(1-i)t=\sqrt{1-i}y+t+iz=0$, where we choose

$$\sqrt{1-i}=-\frac{\sqrt{1+\sqrt{2}}}{\sqrt{2}}+\frac{\sqrt{\sqrt{2}-1}}{\sqrt{2}}i.$$

We can rewrite these equations as

$$x-(1-i)t=\Big(\sqrt{1+\sqrt{2}}-\sqrt{\sqrt{2}-1}i\Big)y+\sqrt{2}t+i\sqrt{2}z=0$$

and $x-(1-i)t=(\sqrt{1+\sqrt{2}}-\sqrt{\sqrt{2}-1}i)y-\sqrt{2}t-i\sqrt{2}z=0$, respectively.

Similarly, if $\mu=1+i$, then C_1 splits as a union of two lines

$$x-(1+i)t=\Big(\sqrt{1+\sqrt{2}}+\sqrt{\sqrt{2}-1}i\Big)y+\sqrt{2}t-i\sqrt{2}z=0$$

and $x-(1+i)t=(\sqrt{1+\sqrt{2}}+\sqrt{\sqrt{2}-1}i)y-\sqrt{2}t+i\sqrt{2}z=0$, If $\mu=-(1+i)$, then C_1 splits as a union of two lines

$$x+(1+i)t=\Big(\sqrt{1+\sqrt{2}}+\sqrt{\sqrt{2}-1}i\Big)y+\sqrt{2}it+\sqrt{2}z=0$$

and $x+(1+i)t=(\sqrt{1+\sqrt{2}}+\sqrt{\sqrt{2}-1}i)y-\sqrt{2}it-\sqrt{2}z=0$, Finally, if $\mu=-1+i$, then C_1 splits as a union of two lines

$$x+(1-i)t=\Big(\sqrt{1+\sqrt{2}}-\sqrt{\sqrt{2}-1}i\Big)y-\sqrt{2}it+\sqrt{2}z=0$$

and $x+(1-i)t=(\sqrt{1+\sqrt{2}}-\sqrt{\sqrt{2}-1}i)y+\sqrt{2}it-\sqrt{2}z=0$,

Thus, we found $3+8=11$ lines among 27 lines on S_3. Now let us do the same trick with the line $x=z-t=0$. Let Π_2 be the plane in $\mathbb{P}^3_{\mathbb{C}}$ that is given by $x=\mu(z-t)$, where $\mu\in\mathbb{C}$. Then

$$\Pi_2\cap S=L_2\cup C_2,$$

where C_2 is a conic in the plane Π_2. Then C_2 is given by

$$\begin{cases} t(z+t)+\mu^2(z-t)z+\mu y^2=0,\\ x=\mu(z-t).\end{cases}$$

It is isomorphic to a conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$\mu^2z^2+\mu y^2+t^2+(1-\mu^2)tz=0,$$

where y, z, t are homogeneous coordinates on $\mathbb{P}^2_{\mathbb{C}}$. Let us find all $\mu\in\mathbb{C}$ such that C_2 is reducible. Rewrite the last equation as

$$\begin{pmatrix} y & z & t\end{pmatrix}\begin{pmatrix} \mu & 0 & 0\\ 0 & \mu^2 & \frac{1-\mu^2}{2}\\ 0 & \frac{1-\mu^2}{2} & 1\end{pmatrix}\begin{pmatrix} y\\ z\\ t\end{pmatrix}=0.$$

This implies that C_2 is reducible if and only if

$$\begin{vmatrix} \mu & 0 & 0 \\ 0 & \mu^2 & \frac{1-\mu^2}{2} \\ 0 & \frac{1-\mu^2}{2} & 1 \end{vmatrix} = -\frac{\mu(\mu^4-6\mu^2+1)}{4} = 0.$$

Thus, C_2 is reducible $\iff$ μ is one of the following numbers: 0, $\sqrt{2}-1$, $-1-\sqrt{2}$, $1-\sqrt{2}$ and $1+\sqrt{2}$. If $\mu=0$, then C_2 splits as a union of the lines $x=t=0$ and $x=z-t=0$. If $\mu=\sqrt{2}-1$, then C_2 splits as a union of the line

$$x-(\sqrt{2}-1)(z-t)=\sqrt{\sqrt{2}-1}y-i(1-\sqrt{2})z+it=0$$

and the line $x-(\sqrt{2}-1)(z-t)=\sqrt{\sqrt{2}-1}y+i(1-\sqrt{2})z-it=0$. If $\mu=-1-\sqrt{2}$, then C_2 splits as a union of the line

$$x+(\sqrt{2}+1)(z-t)=\sqrt{\sqrt{2}+1}y+(1+\sqrt{2})z-t=0$$

and the line $x+(\sqrt{2}+1)(z-t)=\sqrt{\sqrt{2}+1}y-(1+\sqrt{2})z+t=0$. If $\mu=1-\sqrt{2}$, then C_2 splits as a union of the line

$$x-(1-\sqrt{2})(z-t)=\sqrt{\sqrt{2}-1}y-(1-\sqrt{2})z+t=0$$

and the line $x-(1-\sqrt{2})(z-t)=\sqrt{\sqrt{2}-1}y+(1-\sqrt{2})z-t=0$. Finally, if $\mu=1+\sqrt{2}$, then C_2 splits as a union of the line

$$x-(1+\sqrt{2}+1)(z-t)=\sqrt{\sqrt{2}+1}y-i(1+\sqrt{2})z+it=0$$

and the line $x-(1+\sqrt{2}+1)(z-t)=\sqrt{\sqrt{2}+1}y+i(1+\sqrt{2})z-it=0$.

Therefore, we found $3+8+8=19$ lines among 27 lines on S_3. Let us find the remaining 8 lines on the surface S_3.

Let Π_3 be the plane in $\mathbb{P}^3_{\mathbb{C}}$ that is given by $x=\mu(z+t)$, where $\mu\in\mathbb{C}$. Then

$$\Pi_3\cap S=L_3\cup C_3,$$

where C_3 is a conic in the plane Π_3. Then the conic C_3 is given by

$$\begin{cases} t(z-t)+\mu^2(z+t)z+\mu y^2=0, \\ x=\mu(z+t). \end{cases}$$

It is isomorphic to a conic in $\mathbb{P}^2_{\mathbb{C}}$ that is given by

$$\mu y^2+\mu^2z^2-t^2+(1+\mu^2)tz=0,$$

where y, z, t are homogeneous coordinates on $\mathbb{P}^2_{\mathbb{C}}$. Then C_3 is reducible if and only if

$$\begin{vmatrix} \mu & 0 & 0 \\ 0 & \mu^2 & \frac{1+\mu^2}{2} \\ 0 & \frac{1+\mu^2}{2} & -1 \end{vmatrix} = -\frac{\mu(\mu^4-6\mu^2+1)}{4} = 0.$$

Thus, C_3 is reducible $\iff$ μ is one of the following numbers: 0, $i(1+\sqrt{2})$, $-i(1+\sqrt{2})$, $i(1-\sqrt{2})$ and $i(\sqrt{2}-1)$. If $\mu=0$, then C_3 splits as a union of the lines $x=t=0$ and $x=z+t=0$. If $\mu=i(1+\sqrt{2})$, then C_3 splits as a union of the line

$$x-i(1+\sqrt{2})(z-t)=\sqrt{\sqrt{2}+1}(1+i)y+(\sqrt{2}+2)z+\sqrt{2}t=0$$

and the line $x-i(1+\sqrt{2})(z-t)=\sqrt{\sqrt{2}+1}(1+i)y-(\sqrt{2}+2)z-\sqrt{2}t=0$. If $\mu=-i(1+\sqrt{2})$, then C_3 splits as a union of the line

$$x+i(1+\sqrt{2})(z-t)=\sqrt{\sqrt{2}+1}(1-i)y+(\sqrt{2}+2)z+\sqrt{2}t=0$$

and the line $x+i(1+\sqrt{2})(z-t)=\sqrt{\sqrt{2}+1}(1-i)y-(\sqrt{2}+2)z-\sqrt{2}t=0$.

If $\mu=i(1-\sqrt{2})$, then C_3 splits as a union of the line

$$x-i(1-\sqrt{2})(z-t)=\sqrt{\sqrt{2}-1}(1+i)y+(\sqrt{2}-2)z+\sqrt{2}t=0$$

and the line $x-i(1-\sqrt{2})(z-t)=\sqrt{\sqrt{2}-1}(1+i)y-(\sqrt{2}-2)z-\sqrt{2}t=0$. If $\mu=i(\sqrt{2}-1)$, then C_3 splits as a union of the line

$$x-i(\sqrt{2}-1)(z-t)=\sqrt{\sqrt{2}-1}(1-i)y+(\sqrt{2}-2)z+\sqrt{2}t=0$$

and the line $x-i(\sqrt{2}-1)(z-t)=\sqrt{\sqrt{2}-1}(1-i)y-(\sqrt{2}-2)z-\sqrt{2}t=0$.

Thus, we found 27 lines. We claim that these are all lines contained in S_3. Indeed, let L be a line in S_3. If L is contained in the plane $x=0$, then L is one of the lines L_1, L_2 or L_3. Thus, we may assume that L is not contained in this plane. Then the intersection of L and the plane $x=0$ consists of a single point. Let us call this point P. Then $P\in L_1\cup L_2\cup L_3$. Thus, the line L intersects at least one of the lines L_1, L_2, L_3. However, we already found all lines that intersect these lines (these are the last 24 lines that we found). Thus, L is one of them. $\square$

Exercise 15. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x,y,z,t)=0,$$

where $f_3(x,y,z,t)=x^3+y^2z+z^2t$. Do the following.

(a) Show that $f_3(x,y,z,t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Solution. Let us show that the polynomial $f_3(x,y,z,t)$ is irreducible. Suppose that it is not irreducible. Then

$$x^3+y^2z+z^2t=\Big(A(x,y,z)t+B(x,y,z)\Big)C(x,y,z)$$

for some polynomials $A(x,y,z)$, $B(x,y,z)$ and $C(x,y,z)$. Then

$$\begin{cases} A(x,y,z)C(x,y,z)=z^2,\\ B(x,y,z)C(x,y,z)=x^3+y^2z.\end{cases}$$

Since $\mathbb{C}[x,y,z]$ is a unique factorization domain, we see that $C(x,y,z)$ is divisible by z. Thus, x^3+y^2z is divisible by z as well, so that x^3 is also divisible by z, which is absurd. This shows that $f_3(x,y,z,t)$ is irreducible.

Let us find singular points of S_3. We have

$$\begin{cases} \dfrac{\partial f_3(x,y,z,t)}{\partial x}=3x^2,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial y}=2yz,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial z}=y^2+2zt,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial t}=z^2.\end{cases}$$

Thus, the singular points of S_3 are given by

$$3x^2 = 2yz = y^2 + 2zt = z^2 = 0,$$

which gives us the point $[0 : 0 : 0 : 1]$. Thus, the surface S_3 has unique singular point, which is the point $[0 : 0 : 0 : 1]$. This singular point is known as the singular point of type $\mathbb{E}_6$. This is the worst singularity that cubic surface can have if it has finalely many points and it is not a cone.

Let us find all lines in $\mathbb{P}^3_{\mathbb{C}}$ that are contained in S_3. Observe that S_3 contains the line $x = z = 0$. Let us show that this line is the only line contained in S_3.

Let L be a line in S_3. Denote by Q a point in the intersection of this line with the plane $x = 0$. Then $Q = [0 : \alpha : \beta : \gamma]$, where at least one number among α, β, γ is not zero. Let us choose the second point on the line L. If $\alpha \neq 0$, let P be a point in the intersection of L with the plane $y = 0$. If $\alpha = 0$ and $\beta \neq 0$, let P be a point in the intersection of L with the plane $z = 0$. If $\alpha = \beta = 0$ and $\gamma \neq 0$, let P be a point in the intersection of L with the plane $t = 0$. Then $P \neq Q$, so that L is uniquely determined by these two points.

If P is contained in the plane $x = 0$, then the whole line L is contained in this plane, because Q is contained in the plane $x = 0$ by construction. On the other hand, the intersection of the surface S_3 with the plane $x = 0$ is given by

$$x = z(y^2 + zt) = 0,$$

so that it consists of the line $x = z = 0$ and an irreducible conic $x = y^2 + zt = 0$. Thus, to complete the solution, we may assume that L is not the line $x = y = 0$. Then P is not contained in the plane $x = 0$, so that $P = [a : 1 : b : c]$ for some complex numbers a, b and c. Let us seek for a contradiction.

For every $s \in \mathbb{C}$, the point $[1 : a + s\alpha : b + s\beta : c + s\gamma]$ is contained in S_3. This means that

$$(\gamma s + c)(\beta s + b)^2 + (\alpha s + a)^2(\beta s + b) + 1 = 0$$

for every $s \in \mathbb{C}$. Thus, we see that

$$\Big(\alpha^2\beta+\beta^2\gamma\Big)s^3+\Big(\alpha^2 b+2\alpha\beta a+\beta^2 c+2\beta\gamma b\Big)s^2+\Big(2\alpha ab+\beta a^2+2\beta bc+\gamma b^2\Big)s+a^2b+cb^2+1 = 0$$

for every $s \in \mathbb{C}$. Thus, this polynomial in s must be a zero polynomial. This gives us

$$\begin{cases} \alpha^2\beta + \beta^2\gamma = 0, \\ \alpha^2 b + 2\alpha\beta a + \beta^2 c + 2\beta\gamma b = 0, \\ 2\alpha ab + \beta a^2 + 2\beta bc + \gamma b^2 = 0, \\ a^2 b + cb^2 + 1 = 0. \end{cases}$$

Recall that at least one number among a, b, c is zero. Since $a^2b + cb^2 + 1 = 0$, $b \neq 0$. Thus, either $a = 0$ or $c = 0$ (or both).

If $\alpha = \beta = 0$, then $\gamma \neq 0$ and $c = 0$, so that $\gamma b^2 = 0$ and $a^2 b + 1 = 0$, which is absurd. If $\alpha = 0$ and $\beta \neq 0$, then $b = 0$ and we may assume that $\beta = 1$, so that the system above becomes

$$\begin{cases} \gamma = 0, \\ c = 0, \\ a^2 = 0, \\ cb^2 + 1 = 0, \end{cases}$$

which is inconsistent. Thus, we see that $\alpha \neq 0$. Then we may assume that $\alpha = 1$. By construction of the point P, we have $a = 0$. Then

$$\begin{cases} \beta + \beta^2\gamma = 0, \\ b + \beta^2 c + 2\beta\gamma b = 0, \\ 2\beta bc + \gamma b^2 = 0, \\ cb^2 + 1 = 0. \end{cases}$$

If $\beta = 0$, then the second equation of this system gives $b = 0$, which contradicts to its third equation. Thus, $\beta \neq 0$. Then the first equation gives $\gamma = -\frac{1}{\beta}$. Thus, the third equation gives $2\beta^2 bc + b^2 = 0$, so that $\beta^2 c = -\frac{b}{2}$, because $b \neq 0$ (this follows from $cb^2 + 1 = 0$). Now using $b + \beta^2 c + 2\beta\gamma b = 0$, we obtain $b - \frac{b}{2} + 2b = 0$, which implies that $b = 0$. This is a contradiction. It shows that the only line contained in S_3 is the line $x = z = 0$. □

Exercise 16. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x, y, z, t) = 0,$$

where $f_3(x, y, z, t) = x^3 + y^3 + z^3 + t^3 - (x + y + z + t)^3$. Do the following.
(a) Show that $f_3(x, y, z, t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Solution. Suppose that there is a point $[x : y : z : t] \in \mathbb{P}^3_{\mathbb{C}}$ such that

$$\frac{\partial f_3(x, y, z, t)}{\partial x} = \frac{\partial f_3(x, y, z, t)}{\partial y} = \frac{\partial f_3(x, y, z, t)}{\partial z} = \frac{\partial f_3(x, y, z, t)}{\partial t} = 0.$$

We have

$$\begin{cases} \dfrac{\partial f_3(x, y, z, t)}{\partial x} = -3(t + y + z)(t + 2x + y + z), \\ \dfrac{\partial f_3(x, y, z, t)}{\partial y} = -3(t + x + z)(t + x + 2y + z), \\ \dfrac{\partial f_3(x, y, z, t)}{\partial z} = -3(t + x + y)(t + x + y + 2z), \\ \dfrac{\partial f_3(x, y, z, t)}{\partial t} = -3(x + y + z)(x + 2t + y + z) \end{cases}$$

so that

$$\begin{cases} -3(t + y + z)(t + 2x + y + z) = 0, \\ -3(t + x + z)(t + x + 2y + z) = 0, \\ -3(t + x + y)(t + x + y + 2z) = 0, \\ -3(x + y + z)(x + 2t + y + z) = 0. \end{cases}$$

Permuting coordinates x, y, z, t, we may assume that $t \neq 0$. Then we can put $t = 1$. Then

$$\begin{cases} -3(1 + y + z)(1 + 2x + y + z) = 0, \\ -3(1 + x + z)(1 + x + 2y + z) = 0, \\ -3(1 + x + y)(1 + x + y + 2z) = 0, \\ -3(x + y + z)(x + 2 + y + z) = 0. \end{cases}$$

If $1 + y + z = 0$, then

$$\begin{cases} -3(x - y)(x + y) = 0, \\ -3(1 + x + y)(-1 + x - y) = 0, \\ -3(x - 1)(x + 1) = 0, \end{cases}$$

which is inconsistent. Similarly, if $1+2x+y+z=0$, then

$$\begin{cases} (x+y)(y-x)=0, \\ (1+x+y)(1+3x+y)=0, \\ (x+1)(x-1)=0, \end{cases}$$

which is also inconsistent. This is a contradiction. This shows that $f_3(x,y,z,t)$ is irreducible and S_3 is smooth. Indeed, if $f_3(x,y,z,t)$ is a product of two non-constant polynomials $g(x,y,z,t)$ and $h(x,y,z,t)$, then both of them must be homogeneous, so that there is $[\alpha:\beta:\gamma]\in\mathbb{P}^2_{\mathbb{C}}$ such that $g(\alpha,\beta,\gamma,0)=h(\alpha,\beta,\gamma,0)=0$, which implies that

$$\begin{cases} \dfrac{\partial f_3(\alpha,\beta,\gamma,0)}{\partial x}=g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial x}+h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial x}=0, \\ \dfrac{\partial f_3(\alpha,\beta,\gamma,0)}{\partial y}=g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial y}+h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial y}=0, \\ \dfrac{\partial f_3(\alpha,\beta,\gamma,0)}{\partial z}=g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial z}+h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial z}=0, \\ \dfrac{\partial f_3(\alpha,\beta,\gamma,0)}{\partial t}=g(\alpha,\beta,\gamma,0)\dfrac{\partial h(\alpha,\beta,\gamma,0)}{\partial t}+h(\alpha,\beta,\gamma,0)\dfrac{\partial g(\alpha,\beta,\gamma,0)}{\partial t}=0, \end{cases}$$

which contradicts to what we just proved. Thus, we see that $f_3(x,y,z,t)$ is irreducible and the surface S_3 is smooth.

Note that S_3 is acted on by the group S_5. This determines the cubic surface S_3 uniquely. This surface is known as Clebsch cubic surface.

We know that S_3 contains 27 lines by the theorem of Cayley and Salmon. Using symmetries of the surface S_3, it is not hard to find them all. Instead of doing this, let us find all lines on S_3 using *brute force* without guessing anything and without using the theorem of Cayley and Salmon.

Let L be a line in S_3. Denote by Q a point in the intersection of this line with a plane $t=0$. Then $Q=[\alpha:\beta:\gamma:0]$. Let us choose the second point on the line L. If $\alpha\neq 0$, let P be a point in the intersection of L with the plane $x=0$. If $\alpha=0\neq\beta$, let P be a point in the intersection of L with the plane $y=0$. If $\alpha=\beta=0\neq\gamma$, let P be a point in the intersection of L with the plane $z=0$. Then $P\neq Q$, so that L is uniquely determined by these two points.

If P is contained in the plane $t=0$, then L is contained in this plane as well. On the other hand, the intersection of the surface S_3 and the plane $t=0$ is given by

$$t=(y+z)(x+z)(x+y)=0.$$

This gives us 3 lines $t=y+z=0$, $t=x+z=0$ and $t=x+y=0$. To find the remaining 24 lines, we may assume that L is not one of them, so that P is not contained in the plane $t=0$. Then $P=[a:b:c:1]$ for some complex numbers a, b and c, so that L consists of all points $[ra+s\alpha:rb+s\beta:rc+s\gamma:r]$ where $[r:s]$ runs through $\mathbb{P}^1_{\mathbb{C}}$. In particular, the point $[a+s\alpha:b+s\beta:c+s\gamma:1]$ is contained in S_3 for every $s\in\mathbb{C}$. This means that

$$(\alpha s+a)^3+(\beta s+b)^3+(\gamma s+c)^3+1-(\alpha s+\beta s+\gamma s+a+b+c+1)^3=0$$

for every $s \in \mathbb{C}$. Then

$$(\beta+\gamma)(\alpha+\gamma)(\alpha+\beta)s^3+\Big(3\alpha^2b+3\alpha^2c+6\alpha\beta a+6\alpha\beta b+6\alpha\beta c+6\alpha\gamma a+6\alpha\gamma b+6\alpha\gamma c+$$
$$+3\beta^2a+3\beta^2c+6\beta\gamma a+6\beta\gamma b+6\beta\gamma c+3\gamma^2a+3\gamma^2b+3\alpha^2+6\alpha\beta+6\alpha\gamma+3\beta^2+6\beta\gamma+3\gamma^2\Big)s^2+$$
$$+\Big(6\alpha ab+6\alpha ac+3\alpha b^2+6\alpha bc+3\alpha c^2+3\beta a^2+6\beta ab+6\beta ac+6\beta bc+3\beta c^2+3\gamma a^2+6\gamma ab+6\gamma ac+3\gamma b^2+$$
$$+6\gamma bc+6\alpha a+6\alpha b+6\alpha c+6\beta a+6\beta b+6\beta c+6\gamma a+6\gamma b+6\gamma c+3\alpha+3\beta+3\gamma\Big)s+$$
$$+3a^2b+3a^2c+3ab^2+6abc+3ac^2+3b^2c+3bc^2+3a^2+6ab+6ac+3b^2+6bc+3c^2+3a+3b+3c=0$$

for every complex number s. This gives us four equations for a, b, c, α, β, γ. The first equation is $(\beta+\gamma)(\alpha+\gamma)(\alpha+\beta)=0$. The second equation is

$$3\alpha^2b+3\alpha^2c+6\alpha\beta a+6\alpha\beta b+6\alpha\beta c+6\alpha\gamma a+6\alpha\gamma b+6\alpha\gamma c+3\beta^2a+3\beta^2c+$$
$$+6\beta\gamma a+6\beta\gamma b+6\beta\gamma c+3\gamma^2a+3\gamma^2b+3\alpha^2+6\alpha\beta+6\alpha\gamma+3\beta^2+6\beta\gamma+3\gamma^2=0.$$

The third equation is

$$6\alpha ab+6\alpha ac+3\alpha b^2+6\alpha bc+3\alpha c^2+3\beta a^2+6\beta ab+6\beta ac+6\beta bc+3\beta c^2+3\gamma a^2+6\gamma ab+6\gamma ac+$$
$$+3\gamma b^2+6\gamma bc+6\alpha a+6\alpha b+6\alpha c+6\beta a+6\beta b+6\beta c+6\gamma a+6\gamma b+6\gamma c+3\alpha+3\beta+3\gamma=0.$$

The fourth equation is

$$3a^2b+3a^2c+3ab^2+6abc+3ac^2+3b^2c+3bc^2+3a^2+6ab+6ac+3b^2+6bc+3c^2+3a+3b+3c=0.$$

They look pretty ugly. But we also know that at least one of the numbers a, b and c is zero. This simplifies these equations quite a lot.

Suppose that $\alpha\neq 0$. Then $a=0$ and we may assume that $\alpha=1$. Then

$$\begin{cases}(\gamma+1)(\beta+1)(\beta+\gamma)=0,\\ 3\beta^2c+6\beta\gamma b+6\beta\gamma c+3\gamma^2b+3\beta^2+6\beta\gamma+6\beta b+6\beta c+3\gamma^2+6\gamma b+6\gamma c+6\beta+6\gamma+3b+3c+3=0,\\ 6\beta bc+3\beta c^2+3\gamma b^2+6\gamma bc+6\beta b+6\beta c+6\gamma b+6\gamma c+3b^2+6bc+3c^2+3\beta+3\gamma+6b+6c+3=0,\\ (c+1)(b+1)(b+c)=0.\end{cases}$$

Then either $\beta=-1$ or $\gamma=-1$ or $\beta=-\gamma$. Let us consider these subcases separately.

Suppose that $\beta=-1$. Then

$$\begin{cases}3\gamma^2b+3\gamma^2-3b=0,\\ 3\gamma b^2+6\gamma bc+6\gamma b+6\gamma c+3b^2+3\gamma=0,\\ (c+1)(b+1)(b+c)=0.\end{cases}$$

Then $b\neq -1$, since $-3\gamma^2b-3\gamma^2+3b=0$. Then $c=-1$ or $b=-c$. If $c=-1$, then

$$\begin{cases}3\gamma^2b+3\gamma^2-3b=0,\\ 3\gamma b^2+3b^2-3\gamma=0,\end{cases}$$

so that either $\gamma=b=0$ or $\gamma\neq 0\neq b$. In the former case, we have $Q=[1:-1:0:0]$ and $P=[0:0:-1:1]$, so that L is the line $x+y=z+t=0$. In the latter case, the equation $3\gamma^2b+3\gamma^2-3b=0$ gives $b=-\frac{\gamma^2}{\gamma^2-1}$, so that the equation $3\gamma b^2+3b^2-3\gamma=0$ gives $\gamma^2+\gamma-1=0$, which implies that $\gamma=b=\frac{\pm\sqrt{5}-1}{2}$. In this case we have $Q=[1:-1:\frac{\pm\sqrt{5}-1}{2}:0]$ and $P=[0:\frac{\pm\sqrt{5}-1}{2}:-1:1]$, so that either L is the line $2x+2y+(1-\sqrt{5})t=(1-\sqrt{5})x+2z+2t=0$, or L is the line

$$2x+2y+(1+\sqrt{5})t=(1+\sqrt{5})x+2z+2t=0,$$

Similarly, if $c=-b$, then

$$\begin{cases} 3\gamma^2 b+3\gamma^2-3b=0,\\ 3\gamma b^2-3b^2-3\gamma=0, \end{cases}$$

so that either $\gamma=b=0$ or $\gamma\neq 0\neq b$. In the former case, we have $Q=[1:-1:0:0]$ and $P=[0:0:0:1]$, so that L is the line $z=x+y=0$. In the latter case, the equation $3\gamma^2b+3\gamma^2-3b=0$ gives $b=-\frac{\gamma^2}{\gamma^2-1}$, so that the equation $3\gamma b^2-3b^2-3\gamma=0$ gives $\gamma^2-\gamma-1=0$, which gives $b=-\gamma$ and $\gamma=\frac{\pm\sqrt{5}+1}{2}$. In this case we have $Q=[1:-1:\frac{\pm\sqrt{5}+1}{2}:0]$ and $P=[0:\frac{\mp\sqrt{5}-1}{2}:\frac{\pm\sqrt{5}+1}{2}:1]$, so that either L is the line

$$2x+2y+(1+\sqrt{5})t=(1+\sqrt{5})y+2z+2t=0,$$

or L is the line $2x+2y+(1-\sqrt{5})t=(1-\sqrt{5})y+2z+2t=0$.

Therefore, we found 6 lines on the surface S_3 in the case when $\alpha\neq 0$ and the subcase when $\beta=-1$. We have to consider the remaining subcases: $\gamma=-1$ and $\beta=-\gamma$.

Suppose that $\gamma=-1$. Then

$$\begin{cases} 3\beta^2 c+3\beta^2-3c=0,\\ 3\beta c^2+6\beta bc+6\beta b+6\beta c+3c^2+3\beta=0,\\ (c+1)(b+1)(b+c)=0. \end{cases}$$

Since $3\beta^2c+3\beta^2-3c=0$, we have $c\neq -1$, so that $b=-1$ or $b=-c$. If $b=-1$, then

$$\begin{cases} 3\beta^2 c+3\beta^2-3c=0,\\ 3\beta c^2+3c^2-3\beta=0, \end{cases}$$

so that either $\beta=c=0$ or $\beta\neq 0\neq c$. In the former case, we have $Q=[1:0:-1:0]$ and $P=[0:-1:0:1]$, so that L is the line $x+z=y+t=0$. In the latter case, the equation $3\beta^2c+3\beta^2-3c=0$ gives $c=-\frac{\beta^2}{\beta^2-1}$, so that the equation $3\beta c^2+3c^2-3\beta=0$ gives $\beta^2+\beta-1=0$, which implies that $\beta=c=\frac{\pm\sqrt{5}-1}{2}$. In this case we have $Q=[1:\frac{\pm\sqrt{5}-1}{2}:-1:0]$ and $P=[0:-1:\frac{\pm\sqrt{5}-1}{2}:1]$, so that either L is the line $2x+2z+(1-\sqrt{5})t=(1-\sqrt{5})x+2y+2t=0$, or L is the line

$$2x+2z+(1+\sqrt{5})t=(1+\sqrt{5})x+2y+2t=0,$$

Similarly, if $b=-c$, then

$$\begin{cases} 3\beta^2 c+3\beta^2-3c=0,\\ 3\beta c^2-3c^2-3\beta=0, \end{cases}$$

so that either $\beta=c=0$ or $\beta\neq 0\neq c$. In the former case, we have $Q=[1:0:-1:0]$ and $P=[0:0:0:1]$, so that L is the line $y=x+z=0$. In the latter case, the equation $3\beta^2c+3\beta^2-3c=0$ gives $c=-\frac{\beta^2}{\beta^2-1}$, so that the equation $3\beta c^2-3c^2-3\beta=0$ gives $\beta^2-\beta-1=0$, which implies that $c=-\beta$ and $\beta=\frac{\pm\sqrt{5}+1}{2}$. In this case we have $Q=[1:\frac{\pm\sqrt{5}+1}{2}:-1:0]$ and $P=[0:\frac{\pm\sqrt{5}+1}{2}:\frac{\mp\sqrt{5}-1}{2}:1]$, so that either L is the line

$$2x+2z+(1+\sqrt{5})t=2y+(1+\sqrt{5})z+2t=0,$$

or L is the line $2x+2z+(1-\sqrt{5})t=2y+(1-\sqrt{5})z+2t=0$.

Now we consider the subcase $\gamma=-\beta$. Then

$$\begin{cases} 3\gamma^2 b+3\gamma^2 c-3b-3c-3=0,\\ -3\gamma b^2+3\gamma c^2-3b^2-6bc-3c^2-6b-6c-3=0,\\ (c+1)(b+1)(b+c)=0. \end{cases}$$

Then $3\gamma^2 b+3\gamma^2 c-3b-3c-3=0$ implies that $b\neq c$, so that either $b=-1$ or $c=-1$. If $b=-1$, then

$$\begin{cases} 3\gamma^2 c-3\gamma^2-3c=0, \\ 3\gamma c^2-3c^2-3\gamma=0, \end{cases}$$

so that either $\gamma=c=0$ or $\gamma\neq 0\neq c$. In the former case, we have $Q=[1:0:0:0]$ and $P=[0:-1:0:1]$, so that L is the line $z=y+t=0$. In the latter case, the equation $3\gamma^2 c-3\gamma^2-3c=0$ gives $c=\frac{\gamma^2}{\gamma^2-1}$, so that the equation $3\gamma c^2-3c^2-3\gamma=0$ gives $\gamma^2-\gamma-1=0$, which implies that $\gamma=c=\frac{\pm\sqrt{5}+1}{2}$. In this case we have $Q=[1:\frac{\mp\sqrt{5}-1}{2}:\frac{\pm\sqrt{5}+1}{2}:0]$ and $P=[0:-1:\frac{\pm\sqrt{5}+1}{2}:1]$, so that either L is the line $2x+(1+\sqrt{5})y+2z=(1+\sqrt{5})x+2y+2t=0$ or L is the line

$$2x+(1-\sqrt{5})y+2z=(1-\sqrt{5})x+2y+2t=0.$$

Similarly, if $c=-1$, then

$$\begin{cases} 3\gamma^2 b-3\gamma^2-3b=0, \\ 3\gamma b^2+3b^2-3\gamma=0, \end{cases}$$

so that either $\gamma=b=0$ or $\gamma\neq 0\neq b$. In the former case, we have $Q=[1:0:0:0]$ and $P=[0:0:-1:1]$, so that L is the line $y=z+t=0$. In the latter case, the equation $3\gamma^2 b-3\gamma^2-3b=0$ gives $b=\frac{\gamma^2}{\gamma^2-1}$, so that the equation $3\gamma b^2+3b^2-3\gamma=0$ gives $\gamma^2+\gamma-1=0$, which gives $b=-\gamma$ and $\gamma=\frac{\pm\sqrt{5}-1}{2}$. In this case we have $Q=[1:\frac{\mp\sqrt{5}+1}{2}:\frac{\pm\sqrt{5}-1}{2}:0]$ and $P=[0:\frac{\mp\sqrt{5}+1}{2}:-1:1]$, so that either L is the line

$$2y+2z+(1+\sqrt{5})t=2x+(1+\sqrt{5})y+2t=0,$$

or L is the line $2y+2z+(1-\sqrt{5})t=2x+(1-\sqrt{5})y+2t=0$.

Let us summarize what we did so far. We found 3 lines $t=y+z=0$, $t=x+z=0$, $t=x+y=0$ contained in the plane $t=0$, and then we found 18 lines in the case when $\alpha\neq 1$. Altogether, this gives us 21 lines among 27 lines we are looking for.

Now we consider the case when $\alpha=0$ and $\beta\neq 0$. Then $b=0$ and we may assume that $\beta=1$. Then

$$\begin{cases} \gamma(\gamma+1)=0, \\ 3\gamma^2 a+3\gamma^2+6\gamma a+6\gamma c+6\gamma+3a+3c+3=0, \\ 3\gamma a^2+6\gamma ac+6\gamma a+6\gamma c+3a^2-6ac+3c^2+3\gamma+6a+6c+3=0, \\ (c+1)(a+1)(a+c)=0. \end{cases}$$

Thus, either $\gamma=0$ or $\gamma=-1$. If $\gamma=0$, then $Q=[0:1:0:0]$ and

$$a+c+1=(c+1)(a+1)(a+c)=0,$$

so that either $a=-1$ and $c=0$, or $a=0$ and $c=-1$. In the former case, we have $P=[-1:0:0:1]$, so that L is the line $z=x+t=0$. In the latter case, we have $P=[0:0:-1:1]$, so that L is the line $x=z+t=0$. Similarly, if $\gamma=-1$, then $Q=[0:1:-1:0]$ and $c=(c+1)(a+1)(a+c)=0$, so that $c=0$ and either $a=-1$ or $a=0$. In the former case, we have $P=[-1:0:0:1]$, so that L is the line $x+t=y+z=0$. In the latter case, $P=[0:0:0:1]$ and L is the line $x=y+z=0$.

Finally, we consider the case when $\alpha=\beta=0$ and $\gamma\neq 0$. Then $c=0$ and we may assume that $\gamma=1$. Thus, we have $Q=[0:0:1:0]$. Then our four equations give

$$a+b+1=(b+1)(a+1)(a+b)=0,$$

so that either $a = -1$ and $b = 0$, or $b = -1$ and $a = 0$. In the former case, we have $P = [-1 : 0 : 0 : 1]$, so that L is the line $y = x + t = 0$. In the latter case, we have $P = [0 : -1 : 0 : 1]$, so that L is the line $x = y + t = 0$.

Therefore, we found 27 lines on the surface S_3. All these lines are real. In fact, 15 of them are defined over $\mathbb{Q}$. They are the lines $t = y + z = 0$, $t = x + z = 0$, $t = x + y = 0$, $x + y = z + t = 0$, $z = x + y = 0$, $x + z = y + t = 0$, $y = x + z = 0$, $z = y + t = 0$, $y = z + t = 0$, $z = x + t = 0$, $x = z + t = 0$, $x + t = y + z = 0$, $x = y + z = 0$, $y = x+t = 0$, $x = y+t = 0$. The remaining 12 lines are defined over $\mathbb{Q}(\sqrt{5})$. They are the lines $2x+2y+(1-\sqrt{5})t = (1-\sqrt{5})x+2z+2t = 0$, $2x+2y+(1+\sqrt{5})t = (1+\sqrt{5})x+2z+2t = 0$, $2x+2y+(1+\sqrt{5})t = (1+\sqrt{5})y+2z+2t = 0$, $2x+2y+(1-\sqrt{5})t = (1-\sqrt{5})y+2z+2t = 0$, $2x+2z+(1-\sqrt{5})t = (1-\sqrt{5})x+2y+2t = 0$, $2x+2z+(1+\sqrt{5})t = (1+\sqrt{5})x+2y+2t = 0$, $2x+2z+(1+\sqrt{5})t = 2y+(1+\sqrt{5})z+2t = 0$, $2x+2z+(1-\sqrt{5})t = 2y+(1-\sqrt{5})z+2t = 0$, $2x+(1+\sqrt{5})y+2z = (1+\sqrt{5})x+2y+2t = 0$, $2x+(1-\sqrt{5})y+2z = (1-\sqrt{5})x+2y+2t = 0$, $2y+2z+(1+\sqrt{5})t = 2x+(1+\sqrt{5})y+2t = 0$, $2y+2z+(1-\sqrt{5})t = 2x+(1-\sqrt{5})y+2t = 0$. We also proved that S_3 does not contain other lines. You can see these lines on the plaster model of this cubic surface

□

Exercise 17. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x, y, z, t) = 0,$$

where $f_3(x, y, z, t) = txz + y^2z + x^3$. Do the following.

(a) Show that $f_3(x, y, z, t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Solution. Arguing as in the solution to Exercise 17, we see that $f_3(x,y,z,t)$ is irreducible. Likewise, if $[x:y:z:t]$ is a singular point of the surface S_3, then

$$\frac{\partial f_3(x,y,z,t)}{\partial x}=\frac{\partial f_3(x,y,z,t)}{\partial y}=\frac{\partial f_3(x,y,z,t)}{\partial z}=\frac{\partial f_3(x,y,z,t)}{\partial t}=0.$$

On the other hand, we have

$$\begin{cases}\dfrac{\partial f_3(x,y,z,t)}{\partial x}=tz+3x^2,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial y}=2yz,\\ \dfrac{\partial f_3(x,y,z,t)}{\partial z}=tx+y^2.\\ \dfrac{\partial f_3(x,y,z,t)}{\partial z}=xz.\end{cases}$$

Thus, we have $[x:y:z:t]\in\mathrm{Sing}(S_3)\iff x=y=z=0$ or $x=y=t=0$. Therefore, the only singular points of S_3 are $[0:0:0:1]$ and $[0:0:1:0]$.

Observe that S_3 contains the lines $x=y=0$ and $x=z=0$. Let L be a line in $\mathbb{P}^3_{\mathbb{C}}$ that is contained in the cubic surface S_3. Let us show that L is one of the lines $x=y=0$ or $x=z=0$. Suppose that this is not the case. Let us seek for a contradiction.

The only lines in S contained in $x=0$ are $x=y=0$ and $x=z=0$. Since L is not one of them, the plane $x=0$ intersects L by a single point. Denote this point by P. Then

$$P=[0:b:c:d]$$

for some complex numbers b, c and d such that $(b,c,d)\neq(0,0,0)$.

If $b\neq0$, let Q be the intersection point of L and the plane $y=0$. If $b=0$ and $c\neq0$, let Q be the intersection point of L and $z=0$. If $b=c=0$, let Q be the intersection point of L and $t=0$. Then $P\neq Q$ and

$$Q=[A:B:C:D]$$

for some complex numbers A, B, C, D such that $(A,B,C,D)\neq(0,0,0,0)$. Moreover, by construction, if $b\neq0$, then $B=0$. Similarly, if $b=0$ and $c\neq0$, then $C=0$. Finally, if $b=c=0$, then $D=0$.

The only lines in S contained in $x=0$ are $x=z=0$ and $x=z=0$. Since L is not one of them, the plane $x=0$ does not contain the point Q. Thus, we have $A\neq0$. Therefore, we may assume that $A=1$.

The points in the line L are given by

$$\big[r:Br+bs:Cr+cs:Dr+ds\big]$$

when $[r:s]$ runs through all points in $\mathbb{P}^1_{\mathbb{C}}$. Plugging $[r:Br+bs:Cr+cs:Dr+ds]$ into $f(x,y,z,t)$, we see that

$$\Big(B^2C+CD+1\Big)r^3+\Big(B^2c+2BCb+Cd+Dc\Big)sr^2+\Big(2Bbc+Cb^2+cd\Big)s^2r+b^2cs^3=0$$

for every $[r:s]\in\mathbb{P}^1_{\mathbb{C}}$. Thus, we have

$$\begin{cases}B^2C+CD+1=0,\\ B^2c+2BCb+Cd+Dc=0,\\ 2Bbc+Cb^2+cd=0,\\ b^2c=0.\end{cases}$$

Note that the equation $b^2c=0$ simply means that $P\in S_3$. Similarly, the equation $B^2C+CD+1=0$ means that $Q\in S_3$.

If $b \neq 0$, then $B = 0$ and we may assume that $b = 1$, so that we get

$$\begin{cases} CD + 1 = 0, \\ Cd + Dc = 0, \\ C + cd = 0, \\ c = 0. \end{cases}$$

This system of equations is inconsistent.

If $b = 0$ and $c \neq 0$, then $C = 0$, so that we get $1 = 0$, which is absurd. If $b = c = 0$ and $d \neq 0$, then $D = 0$, so that we get $1 = 0$ again. The obtained contradiction implies that S_3 contains exactly two lines. $\square$

Exercise 18. Let S_3 be a subset in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_3(x, y, z, t) = 0,$$

where $f_3(x, y, z, t) = xyz - t^3$. Do the following.

(a) Show that $f_3(x, y, z, t)$ is irreducible.
(b) Find all singular points (if any) of the cubic surface S_3.
(c) Find all lines on S_3.

Solution. Let us show that $f_3(x, y, z, t)$ is irreducible. This polynomial is a polynomial of degree 1 in x with coefficients in $\mathbb{C}[y, z, t]$. If it is not irreducible, then

$$xyz - t^3 = \Big(A(x, y, t)x + B(y, z, t)\Big)C(y, z, t)$$

for some polynomials $A(y, z, t)$, $B(y, z, t)$ and $C(y, z, t)$ such that $C(y, z, t) \notin \mathbb{C}$, so that

$$\begin{cases} A(y, z, t)C(y, z, t) = yz, \\ B(y, z, t)C(y, z, t) = -t^3, \end{cases}$$

which implies that $C(y, z, t)$ is divisible by t, which is impossible, since $A(y, z, t)C(y, z, t) = yz$. Thus, we see that $f_3(x, y, z, t)$ is irreducible.

Let us find singular points of S_3. We have

$$\begin{cases} \dfrac{\partial f_3(x, y, z, t)}{\partial x} = yz, \\ \dfrac{\partial f_3(x, y, z, t)}{\partial y} = xz, \\ \dfrac{\partial f_3(x, y, z, t)}{\partial z} = xy, \\ \dfrac{\partial f_3(x, y, z, t)}{\partial t} = -3t^2. \end{cases}$$

Thus, the point $[x : y : z : t] \in \mathbb{P}^3_{\mathbb{C}}$ is singular point of S_3 if and only if $yz = xz = xy = -3t^2 = 0$. This implies that the only singular points of the surface S_3 are the points $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$. These are the singular points of type $\mathbb{A}_2$. In these case, the surface S_3 is a global quotient of $\mathbb{P}^2_{\mathbb{C}}$ by the action of the cyclic group $\mathbb{Z}_3$ that fixes 3 points in $\mathbb{P}^2_{\mathbb{C}}$. The images of these points are the points $[1 : 0 : 0 : 0]$, $[0 : 1 : 0 : 0]$, $[0 : 0 : 1 : 0]$.

Now it is time to find all lines in S_3. Note that S_3 contains the lines $y = t = 0$, $z = t = 0$ and $x = t = 0$. Let us show that these 3 lines are all lines contained in S_3.

Let L be a line in S_3. Denote by Q a point in the intersection of this line with a plane $t = 0$. Then $Q = [\alpha : \beta : \gamma : 0]$. Let us choose the second point on the line L. If $\alpha \neq 0$, let P be a point in the intersection of L with the plane $x = 0$. If $\alpha = 0$ and $\beta \neq 0$, let P be a point in the intersection of L with the plane $y = 0$. If $\alpha = \beta = 0$ and $\gamma \neq 0$, let P be a point in the intersection of L with the plane $z = 0$. Then $P \neq Q$, so that L is uniquely determined by these two points.

If P is contained in the plane $t = 0$, then L is contained in this plane as well. In this case, L is one of the lines $x = t = 0$, $y = t = 0$ and $z = t = 0$, because the plant $t = 0$ intersects the surface S_3 by these three lines.

Suppose that L is not one of these lines. Then P is not contained in the plane $t = 0$. Thus, we have $P = [a : b : c : 1]$ for some complex numbers a, b and c. Moreover, at least one number among a, b, c is zero by construction. Furthermore, the line L consists of all points

$$\Big[ra + s\alpha : rb + s\beta : rc + s\gamma : r\Big]$$

where $[r : s]$ runs through $\mathbb{P}^1_{\mathbb{C}}$. In particular, for every $s \in \mathbb{C}$, the point $[a + s\alpha : b + s\beta : c + s\gamma : 1]$ is contained in S_3. This means that

$$(\alpha s + a)(\beta s + b)(\gamma s + c) - 1 = 0$$

for every $s \in \mathbb{C}$. Thus, we see that

$$\alpha\beta\gamma s^3 + \Big(\alpha\gamma b + \beta\gamma a + \alpha\beta c\Big)s^2 + \Big(ab\gamma + \alpha bc + \beta ac\gamma\Big)s + abc - 1 = 0$$

for every $s \in \mathbb{C}$. Thus, this polynomial in s must be a zero polynomial. This gives us

$$\begin{cases} \alpha\beta\gamma = 0, \\ \alpha\gamma b + \beta\gamma a + \alpha\beta c = 0, \\ ab\gamma + \alpha bc + \beta ac\gamma = 0, \\ abc - 1 = 0. \end{cases}$$

On the other hand, at least one number among a, b, c is zero. This contradicts to $abc-1 = 0$. Thus, the only lines contained in S_3 are the lines $x = t = 0$, $y = t = 0$ and $z = t = 0$. $\square$