Dubna 2018: lines on cubic surfaces

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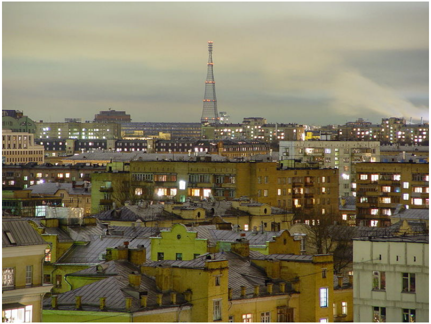
Lecture 2: surfaces and lines on them



Surfaces in complex projective space

Let S_d be a projective variety in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_d(x, y, z, t) = 0,$$

where $f_d(x, y, z, t)$ is an irreducible form of degree d.

Definition

We say that S_d is a surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree d.

- If $d = 1$, then S_1 is given by

  $$\boxed{Ax + By + Cz + Dt = 0}$$

  for some $[A : B : C : D] \in \mathbb{P}^3_{\mathbb{C}}$ and we call S_1 a plane.
- If $d = 2$, then S_2 is called a quadric surface and it is given by

$$\boxed{Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gxt + Hyt + Izt + Jt^2 = 0}$$

  for some $[A : B : C : D : E : F : G : H : I : J] \in \mathbb{P}^9_{\mathbb{C}}$.
- If $d = 3$, then S_3 is called a cubic surface.
- If $d = 4$, then S_4 is called a quartic surface.
- If $d = 5$, then S_5 is called a quintic surface.

Singular and non-singular surfaces

Let S_d be a projective variety $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_d\big(x,y,z,t\big) = 0$$

for an irreducible homogeneous $f_d \in \mathbb{C}[x,y,z,t]$ of degree d.

Definition

A point $[\alpha:\beta:\gamma:\delta] \in \mathbb{P}^3_{\mathbb{C}}$ is a singular point of S_d if

$$\boxed{\frac{\partial f(\alpha,\beta,\gamma,\delta)}{\partial x} = \frac{\partial f(\alpha,\beta,\gamma,\delta)}{\partial y} = \frac{\partial f(\alpha,\beta,\gamma,\delta)}{\partial z} = \frac{\partial f(\alpha,\beta,\gamma,\delta)}{\partial t} = 0.}$$

- Denote by $\mathrm{Sing}(S_d)$ the set of singular points of S_d.
- Non-singular points of S_d are called smooth or nonsingular.
- The surface S_d is smooth or nonsingular if $\mathrm{Sing}(S_d) = \varnothing$

Euler's formula for homogeneous polynomials gives

$$df_d = x\frac{\partial f_d(x,y,z,t)}{\partial x} + y\frac{\partial f_d(x,y,z,t)}{\partial y} + z\frac{\partial f_d(x,y,z,t)}{\partial z} + t\frac{\partial f_d(x,y,z,t)}{\partial t}.$$

Tangent planes

Let S_d be a projective variety $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$f_d\big(x, y, z, t\big) = 0$$

for an irreducible homogeneous $f_d \in \mathbb{C}[x, y, z, t]$ of degree d.

Remark

Almost all points of the surface S_d are smooth.

- Let $[\alpha : \beta : \gamma : \delta]$ be a smooth point of the surface S_d.

Definition

The plane in $\mathbb{P}^3_{\mathbb{C}}$ that is given by

$$\boxed{\frac{\partial f_d(\alpha, \beta, \gamma, \delta)}{\partial x} x + \frac{\partial f_d(\alpha, \beta, \gamma, \delta)}{\partial y} y + \frac{\partial f_d(\alpha, \beta, \gamma, \delta)}{\partial z} z + \frac{\partial f_d(\alpha, \beta, \gamma, \delta)}{\partial t} t = 0}$$

is the tangent plane to S_d at the point $[\alpha : \beta : \gamma : \delta]$.

Projective transformations in three dimensions

Fix a matrix with complex entries and non-zero determinant

$$\begin{pmatrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{01} & a_{11} & a_{21} & a_{31} \\ a_{02} & a_{12} & a_{22} & a_{32} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

A projective transformation $\mathbb{P}^3_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$ is a map given by

$$[x : y : z : t] \mapsto \Big[a_{00}x + a_{01}y + a_{02}z + a_{03}t : \\ : a_{10}x + a_{11}y + a_{12}z + a_{13}t : a_{20}x + a_{21}y + a_{22}z + a_{23}t : \\ : a_{30}x + a_{31}y + a_{32}z + a_{33}t\Big].$$

Subsets X and Y in $\mathbb{P}^3_{\mathbb{C}}$ are said to be projectively equivalent if

$$\phi(X) = Y$$

for some projective transformation $\phi \colon \mathbb{P}^3_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$.

- Projectively equivalent surfaces in $\mathbb{P}^3_{\mathbb{C}}$ have the same degree.

Singular quadric surfaces

Let S_2 be an irreducible quadric surface in $\mathbb{P}^3_{\mathbb{C}}$. Then it is given by

$$\begin{pmatrix} x & y & z & t \end{pmatrix} \begin{pmatrix} A & \frac{D}{2} & \frac{F}{2} & \frac{G}{2} \\ \frac{D}{2} & B & \frac{E}{2} & \frac{H}{2} \\ \frac{F}{2} & \frac{E}{2} & C & \frac{E}{2} \\ \frac{G}{2} & \frac{H}{2} & \frac{E}{2} & J \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 0$$

for some $[A : B : C : D : E : F : G : H : I : J] \in \mathbb{P}^9_{\mathbb{C}}$.

Lemma

The determinant of this matrix is zero $\iff$ S_2 is singular.

Lemma

If S_2 is singular, then it is projectively equivalent to $xy = z^2$.

Proof.

If S_2 is singular at $[0 : 0 : 0 : 1]$, then it is given by

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx = 0$$

for $[A : B : C : D : E : F] \in \mathbb{P}^5_{\mathbb{C}}$. Now use result for conics. □

Nonsingular quadric surfaces

Let S_2 be a nonsingular irreducible quadric surface in $\mathbb{P}^3_{\mathbb{C}}$.

Theorem

S_2 is projectively equivalent to the surface given by $xy = zt$.

Proof.

The surface S_2 is given by

$$\boxed{Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gxt + Hyt + Izt + Jt^2 = 0}$$

for some $[A : B : C : D : E : F : G : H : I : J] \in \mathbb{P}^9_{\mathbb{C}}$.
Apply projective transformation such that

- the surfaces S_2 contains the point $[0 : 0 : 0 : 1]$,
- the tangent plane to S_2 at the point P is $z = 0$.

Then $J = 0$, $G = 0$, $H = 0$, $I \neq 0$. We may assume $I = -1$.
Apply $[x : y : z : t] \mapsto [x : y : z : t - Ey - Cz]$.
Now $E = C = 0$, so that S_2 is given by $Ax^2 + By^2 + Dxy = zt$. $\square$

Lines on smooth quadric surfaces

- Let S_2 be the quadric surface in $\mathbb{P}^3_{\mathbb{C}}$ given by $xy = zt$.
- Let $P = [\alpha : \beta : \gamma : \delta]$ be a point in S_2, so that $\alpha\beta = \gamma\gamma$.

Let L be a line in S_2 such that $P \in L$. Then L can be given by

$$\lambda\big[\alpha : \beta : \gamma : \delta\big] + \mu\big[a : b : c : d\big],$$

where $Q = [a : b : c : d]$ is a point in L and $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$.
We may assume that P and L are not contained in the plane $t = 0$.
Then we can let $\delta = 1$ and $d = 0$. Then

$$\Big(\alpha + \mu a\Big)\Big(\beta + \mu b\Big) - \Big(\gamma + \mu c\Big) = 0$$

for every $\mu \in \mathbb{C}$. Then the polynomial

$$ab\mu^2 + \big(\alpha b + a\beta - c\big)\mu + \alpha\beta - \gamma$$

must be a zero polynomial. Then $ab = 0$ and $\alpha b + a\beta = c$.
Then either $\boxed{a = 0, b = 1, c = \alpha}$ or $\boxed{a = 1, b = 0, c = \beta}$.

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Lines on smooth quadric surfaces revisited

- Let S_2 be a smooth quadric surface in $\mathbb{P}^3_{\mathbb{C}}$.
- Take any point $P \in S_2$.
- Then S_2 contains contains 2 lines passing through P.

Applying projective transformations, we may assume that

- $P = [0 : 0 : 0 : 1]$,
- the tangent plane to S_2 at the point P is $t = 0$.

Then the quadric surface S_2 is given by

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx = zt$$

for some complex numbers A, B, C, D, E and F.
Now we can make $A = B = C = E = F = 0$ and $D = 1$ fixing P.
Let L be a line in S_2 such that $P \in L$. Then L can be given by

$$\lambda\Big[0 : 0 : 0 : 1\Big] + \mu\Big[a : b : c : 0\Big],$$

where $Q = [a : b : c : 0]$ is a point in L and $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$.
Then $ab\mu^2 - \mu c = 0$ for every $\mu \in \mathbb{C}$. Then $ab = 0$ and $c = 0$.

- Thus, either $Q = [0 : 1 : 0 : 0 :]$ or $Q = [1 : 0 : 0 : 1]$.

Smooth quadric surfaces are $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$

- The product $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ consists of all possible pairs

$$\Big(\big[\alpha : \beta \big] : \big[\gamma : \delta \big] \Big)$$

with $[\alpha : \beta]$ and $[\gamma : \delta]$ in $\mathbb{P}^1_{\mathbb{C}}$.

Define a map $\phi \colon \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}}$ by

$$\Big(\big[\alpha : \beta \big] : \big[\gamma : \delta \big] \Big) \to \Big[\alpha\gamma : \alpha\delta : \beta\gamma : \beta\delta \Big].$$

- The map ϕ is everywhere defined.
- The image of the map ϕ is the quadric $S_2 \subset \mathbb{P}^3_{\mathbb{C}}$ given by

$$xt = yz.$$

- The induced map $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \to S_2$ is a bijection.

For a point $P \in \mathbb{P}^1_{\mathbb{C}}$, both $\phi(P \times \mathbb{P}^1_{\mathbb{C}})$ and $\phi(\mathbb{P}^1_{\mathbb{C}} \times P)$ are lines.

Counting lines on surfaces

- Let S_d be an irreducible surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree d.
- Suppose that S_d does not have singular points.

Theorem (Segre, 1943)

If $d \geqslant 3$, then S_d contains at most $(d-2)(11d-6)$ lines.

- If S_d is general enough and $d \geqslant 4$, it contains no lines!

Example (Shioda)

Suppose that d is prime and $d \geqslant 5$. Then the surface given by

$$t^d + xy^{d-1} + yz^{d-1} + zx^{d-1} = 0$$

does not contain lines.

Theorem (Segre, 1943)

If $d = 4$, then S_4 contains at most 64 lines.

- The surface $x^4 + y^4 + z^4 + t^4 = 0$ contains 64 lines.
- The surface $xy^3 + yz^3 + zx^3 + t^4 = 0$ contain 0 lines.

Singular cubic surface containing one line

Let S_3 be the surface in $\mathbb{P}^3_{\mathbb{C}}$ given by $x^3 + y^2 z + z^2 t = 0$.

- Let L be a line in $\mathbb{P}^3_{\mathbb{C}}$ such that $L \subset S_3$.
- Let $P = [0 : a : b : c]$ be the intersection of L with $x = 0$.
- Let $Q = [\alpha : \beta : 0 : \gamma]$ be the intersection of L with $t = 0$.

We may assume that $P \neq Q$. Then L is given by

$$\left[\mu\alpha : \lambda a + \mu b : \lambda b : \lambda c + \mu\gamma\right],$$

where $[\lambda : \mu]$ runs through all points in $\mathbb{P}^1_{\mathbb{C}}$. Then

$$\mu^3\alpha^3 + \left(\lambda a + \mu b\right)^2 \lambda b + \lambda^2 b^2 \left(\lambda c + \mu\gamma\right)^2 = 0$$

for every $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$. This gives

$$\alpha^3\mu^3 + \beta^2 b\mu^2\lambda + \left(2a\beta b + b^2\gamma\right)\mu\lambda^2 + b^2 c\lambda^3$$

for every $[\lambda : \mu] \in \mathbb{P}^1_{\mathbb{C}}$. This gives

$$\boxed{\alpha^3 = \beta^2 b = 2a\beta b + b^2\gamma = b^2 c = 0.}$$

Then $\alpha = b = 0$, so that L is given by $x = z = 0$.

Blowing up the plane

Consider the product $\mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}}$. It consists of all pairs

$$\Big((x,y),[\alpha:\beta]\Big)$$

where $(x,y) \in \mathbb{C}^2$ and $[\alpha:\beta] \in \mathbb{P}^1_{\mathbb{C}}$.

- Let S be a subset in $\mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}}$ that is given by

$$x\beta = \alpha y.$$

- Let $\pi\colon S \to \mathbb{C}^2$ be the natural projection.
- Let E be the subset in S that is given by $x = y = 0$.

Definition

We say that π is a blow up of $\mathbb{C}^2$ at the point $(0,0)$.

We say that E is the exceptional curve of the blow up π. One has

$$S \setminus E \cong \mathbb{C}^2 \setminus (0,0).$$

Note that π is birational and $E \cong \mathbb{P}^1_{\mathbb{C}}$.

Blowing up the projective plane

Consider the product $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. It consists of all pairs

$$\Big([x:y:z],[\alpha:\beta]\Big)$$

where $[x:y:z] \in \mathbb{P}^2_{\mathbb{C}}$ and $[\alpha:\beta] \in \mathbb{P}^1_{\mathbb{C}}$.

- Let S be a subset in $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ that is given by

$$x\beta = \alpha y.$$

- Let $\pi\colon S \to \mathbb{P}^2_{\mathbb{C}}$ be the natural projection.
- Let E be the subset in S that is given by $x = y = 0$.

Definition

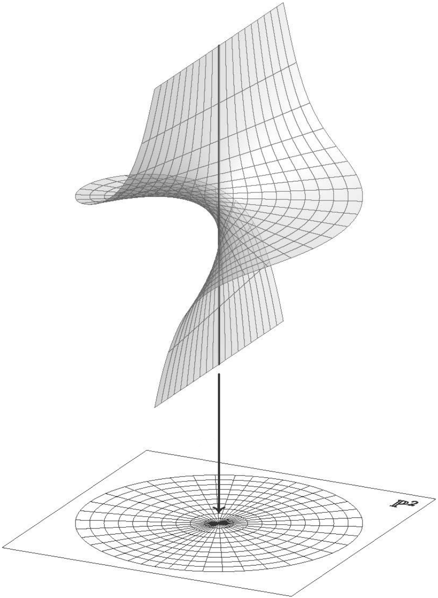
We say that π is a blow up of $\mathbb{P}^2_{\mathbb{C}}$ at the point $[0:0:1]$.

We say that E is the exceptional curve of the blow up π. One has

$$S \setminus E \cong \mathbb{C}^2 \setminus [0:0:1].$$

Note that π is birational and $E \cong \mathbb{P}^1_{\mathbb{C}}$.

Blow up and lines $\lambda x = \mu y$



Cuspidal cubic curve

Let C be the curve in $\mathbb{C}^2$ that is given by $x^3 = y^2$. Let

$$\widehat{C} = \overline{\pi^{-1}\Big(C \setminus (0,0)\Big)} \subset S.$$

Let U be the open subset in $\mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}}$ given by $\alpha \neq 0$. Let

$$z = \frac{\beta}{\alpha}.$$

Identify $U = \mathbb{C}^3$ with coordinates x, y, z. Then $S \cap U$ is given by

$$y = xz.$$

Outside of the curve $E \cap U$, the subset $\widehat{C} \cap U$ is given by

$$\begin{cases} y = xz, \\ x^3 = y^2. \end{cases}$$

Identify $S \cap U = \mathbb{C}^2$ with coordinates x and z. Then

$$\boxed{x = z^2}$$

defines $\widehat{C} \cap U$. Then $\widehat{C}$ is smooth and $|E \cap \widehat{C}| = 1$.

Nodal cubic curve

Let C be the curve in $\mathbb{C}^2$ that is given by $x^2 = y^2 + y^3$. Let

$$\widehat{C} = \overline{\pi^{-1}\Big(C \setminus (0,0)\Big)} \subset S.$$

Let W be the open subset in $\mathbb{C}^2 \times \mathbb{P}^1_{\mathbb{C}}$ given by $\beta \neq 0$. Let

$$z = \frac{\alpha}{\beta}.$$

Identify $W = \mathbb{C}^3$ with coordinates x, y, z. Then $S \cap W$ is given by

$$x = yz.$$

Outside of the curve $E \cap U$, the subset $\widehat{C} \cap U$ is given by

$$\begin{cases} x = yz, \\ x^2 = y^2 + y^3. \end{cases}$$

Identify $S \cap W = \mathbb{C}^2$ with coordinates y and z. Then

$$\boxed{z^2 = 1 + y}$$

defines $\widehat{C} \cap W$. Then $\widehat{C}$ is smooth and $|E \cap \widehat{C}| = 2$.

Blow ups and intersection multiplicities

- Let $\pi\colon S \to \mathbb{C}^2$ be the blow up of the point $(0,0)$.
- Let E be the exceptional curve of the blow up π.

Let C be the curve in $\mathbb{C}^2$ that contains $(0,0)$. Let $O=(0,0)$ and

$$\widehat{C} = \overline{\pi^{-1}\Big(C \setminus O\Big)} \subset S.$$

We say that $\widehat{C}$ is the proper transform of the curve C. Then

$$\boxed{\mathrm{mult}_O(C) = \sum_{P \in E}\Big(\widehat{C}\cdot E\Big)_P.}$$

- Let Z be the curve in $\mathbb{C}^2$ such that $O \in C$.
- Let $\widehat{Z}$ the proper transform of the curve Z.

If C and Z has no common components, then

$$\boxed{\Big(C\cdot Z\Big)_O = \mathrm{mult}_O(C)\mathrm{mult}_O(Z) + \sum_{P\in E}\Big(\widehat{C}\cdot\widehat{Z}\Big)_P.}$$

Stereographic projection

Let S_2 be the smooth quadric in $\mathbb{P}^3_{\mathbb{C}}$ given by $xy = zt$.

- Let $\phi \colon S_2 \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ be a map given by

$$[x : y : z : t] \mapsto [y : z : t].$$

- Let $\psi \colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^3_{\mathbb{C}}$ be a map given by

$$[x : y : z] \mapsto [yz : x^2 : xy : xz].$$

Then $\psi = \phi^{-1}$ and there is a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{f} & & \searrow^{g} & \\ \mathbb{P}^2_{\mathbb{C}} & & \overset{\psi}{\dashrightarrow} & & S_2 \end{array}$$

where f blows up $[0 : 0 : 1]$ and $[0 : 1 : 0]$, and g is a morphism.

- Let ℓ be the line in $\mathbb{P}^2_{\mathbb{C}}$ given by $x = 0$.
- Let $\widetilde{\ell}$ be the proper transform on X of the line ℓ.
- Then g blows down $\widetilde{\ell}$ to the point $[1 : 0 : 0 : 0]$.

What does f blow down?

Blowing up the space

Consider the product $\mathbb{C}^3 \times \mathbb{P}^2_{\mathbb{C}}$. It consists of all pairs

$$\Big((x, y, z), [\alpha : \beta : \gamma]\Big)$$

where $(x, y, z) \in \mathbb{C}^3$ and $[\alpha : \beta : \gamma] \in \mathbb{P}^2_{\mathbb{C}}$.

- Let V be a subset in $\mathbb{C}^3 \times \mathbb{P}^2_{\mathbb{C}}$ that is given by

$$\begin{cases} \alpha y = x\beta, \\ \alpha z = x\gamma, \\ \beta z = y\gamma. \end{cases}$$

- Let $\nu\colon V \to \mathbb{C}^3$ be the natural projection.
- Let E be the subset in V that is given by $x = y = z = 0$.

Definition

We say that ν is a blow up of $\mathbb{C}^3$ at the point $(0, 0, 0)$.

We say that E is the exceptional surface of the blow up η. One has

$$V \setminus E \cong \mathbb{C}^3 \setminus (0, 0, 0).$$

Note that $E \cong \mathbb{P}^2_{\mathbb{C}}$.

Singular cubic surface I

Let S_3 be the surface in $\mathbb{C}^3$ given by $x^3 + y^2 z + z^2 = 0$.

- Let $\chi\colon \mathbb{C}^3 \to \mathbb{C}^3$ be the map given by

$$\big(\widehat{x}, \widehat{y}, \widehat{z}\big) \mapsto \big(\widehat{x}\widehat{y}, \widehat{y}, \widehat{z}\widehat{y}, \big).$$

- Let E be the surface in $\mathbb{C}^3$ given by $\widehat{y} = 0$.
- Let $\widehat{S}_3$ be the proper transform on $\mathbb{C}^3$ of the surface S_3.

Then the surface $\widehat{S}_3$ is given by

$$\widehat{y}\widehat{x}^3 + \widehat{z}\widehat{y} + \widehat{z}^2 = 0.$$

It is singular at $(0,0,0)$. Note that $E \cap \widehat{S}_3$ is irreducible.

- We considered one chart of the blow up of $\mathbb{C}^3$ at $(0,0,0)$.
- The second chart of the blow up is given by

$$\big(\widehat{x}, \widehat{y}, \widehat{z}\big) \mapsto \big(\widehat{x}\widehat{z}, \widehat{y}\widehat{z}, \widehat{z}\big).$$

- The third chart of the blow up is given by

$$\big(\widehat{x}, \widehat{y}, \widehat{z}\big) \mapsto \big(\widehat{x}, \widehat{y}\widehat{x}, \widehat{z}\widehat{x}\big).$$

- In each of them, the proper transform of S_3 is smooth.

Singular cubic surface II

Let $\widehat{S}_3$ be the surface in $\mathbb{C}^3$ given by $\widehat{y}\widehat{x}^3 + \widehat{z}\widehat{y} + \widehat{z}^2 = 0$.

- Let $\rho\colon \mathbb{C}^3 \to \mathbb{C}^3$ be the map given by

$$\big(\overline{x}, \overline{y}, \overline{z}\big) \mapsto \big(\overline{x}, \overline{yz}, \overline{zx}\big).$$

- Let F be the surface in $\mathbb{C}^3$ given by $\overline{x} = 0$.
- Let $\overline{S}_3$ be the proper transform on $\mathbb{C}^3$ of the surface $\widehat{S}_3$.

Then $\overline{S}_3$ is given by $\overline{yx}^2 + \overline{zy} + \overline{z}^2 = 0$. It is singular.

- Let $\phi\colon \mathbb{C}^3 \to \mathbb{C}^3$ be the map given by

$$\big(\widetilde{x}, \widetilde{y}, \widetilde{z}\big) \mapsto \big(\widetilde{x}, \widetilde{y}\widetilde{z}, \widetilde{z}\widetilde{x}\big).$$

- Let G be the surface in $\mathbb{C}^3$ given by $\widetilde{x} = 0$.
- Let $\widetilde{S}_3$ be the proper transform on $\mathbb{C}^3$ of the surface $\overline{S}_3$.

Then $\widetilde{S}_3$ is given by $\widetilde{y}\widetilde{x} + \widetilde{z}\widetilde{y} + \widetilde{z}^2 = 0$. It is singular.

Singular cubic surface III

The surface $\widetilde{S}_3$ is given in $\mathbb{C}^3$ by $\widetilde{y}\widetilde{x} + \widetilde{z}\widetilde{y} + \widetilde{z}^2 = 0$.

- Let $\psi\colon \mathbb{C}^3 \to \mathbb{C}^3$ be the map given by

$$\big(\mathbf{x}, \mathbf{y}, \mathbf{z}\big) \mapsto \big(\mathbf{x}, \mathbf{yz}, \mathbf{zx}\big).$$

- Let H be the surface in $\mathbb{C}^3$ given by $\mathbf{x} = 0$.
- Let $\mathbf{S}_3$ be the proper transform on $\mathbb{C}^3$ of the surface $\widetilde{S}_3$.

We consecutively blew S_3 four times

$$\mathbf{S}_3 \xrightarrow{\quad\psi\quad} \widetilde{S}_3 \xrightarrow{\quad\phi\quad} \overline{S}_3 \xrightarrow{\quad\rho\quad} \widehat{S}_3 \xrightarrow{\quad\chi\quad} S_3$$

Then $\mathbf{S}_3$ is given by $\mathbf{y} + \mathbf{zy} + \mathbf{z}^2 = 0$. It is smooth.

- $E \cap \widehat{S}_3$ is a smooth irreducible curve.
- $F \cap \overline{S}_3$ is a union of two smooth irreducible curves.
- $G \cap \widetilde{S}_3$ is a union of two smooth irreducible curves.
- $H \cap \mathbf{S}_3$ is a smooth irreducible curve.

The constructed morphism $\mathbf{S}_3 \to S_3$ contract 6 irreducible curves.

- Their intersection graph is the Dynkin diagram $\mathbb{E}_6$.