On generating sets for ideals defining S-varieties

Evgeny Smirnov

Abstract

Let G be a semisimple algebraic group. Kostant's theorem describes the ideal defining the G-orbit closure of the sum of highest-weight vectors in a (reducible) G-module, such that the corresponding highest weights are linearly independent. This ideal is generated by quadratic polynomials.

In this paper we generalize this result, assuming that the highest weights can be linearly dependent. In this case the equations defining these varieties are not necessary quadratic.

1 Introduction

Let G be a semisimple algebraic group. We consider a (possibly reducible) finite-dimensional G-module, fix a highest weight vector in each of its irreducible submodules and consider their sum. The closure of the G-orbit of such a vector is said to be an *S-variety*. A classical example of an S-variety is provided by a Grassmann cone $\mathrm{Gr}(m, n)$, obtained as the closure of the highest weight vector orbit in the representation of SL_n acting on $\bigwedge^m k^n$. The Grassmann cone can be defined by a set of equations, known as *Plücker relations*; moreover, there equations generate its ideal. In the present paper, we obtain analogous relations for an arbitrary S-variety. Earlier it was done in the case of linearly independent highest weights (see [1]).

The author is grateful to Ernest B. Vinberg for the attention to this work, and to Dmitry Timashev for many useful remarks.

Notation

k	ground field, algebraically closed and of zero characteristics;
G	simply connected semisimple algebraic group;
B	a fixed Borel subgroup in G;
$\mathfrak{X}(T)$	the character group for a fixed maximal torus $T \subset B$;
$\mathfrak{X}_+(T)$	semigroup formed by the dominant characters for T;
V_λ	-the irreducible G-module with a highest weight $\lambda \in \mathfrak{X}_+(T)$
v_λ^+	highest weight vector in V_λ.

2 HV-varieties

First consider a particular case: the closures of the highest weight vector orbits. Such varieties are often called HV-varieties. In [2] it is proved that the closure X of the highest weight vector orbit in an irreducible module V_λ is defined by the following system of equations:

$$\Omega(v \otimes v) = (2\lambda + 2\rho, 2\lambda)(v \otimes v), \qquad (1)$$

where Ω stands for the Casimir operator, and ρ is equal to the half-sum of the positive roots.

(1) is an equality of elements from $S^2 V_\lambda$, so it can be considered as a system of $d_\lambda(d_\lambda + 1)/2$ quadratic equations on the coordinates of v, where $d_\lambda = \dim V_\lambda$. If $G = \mathrm{SL}_n$, and $\lambda = \pi_k$ is the k-th fundamental weight, what we get is exactly the set of Plücker relations.

Here is a streghtening of the main result from [2]:

Proposition 1. *The relations (1) generate the ideal of X.*

Proof. This follows from the Kostant theorem (Theorem 3 of this paper). Its proof will be given in the next section. □

3 S-varieties: Linearly independent weight case

Consider the S-variety which corresponds to dominant weights $\lambda_1, \dots, \lambda_k \in \mathfrak{X}_+(T)$, that is, the closure of the G-orbit of the sum of highest vectors $v_{\lambda_1}^+ + \dots + v_{\lambda_k}^+ \in V_{\lambda_1} \oplus \dots \oplus V_{\lambda_k} = V$. Denote it by $X = X(\lambda_1, \dots, \lambda_k)$. Also denote the set $\lambda_1, \dots, \lambda_k$ by $\mathfrak{E}_0$, and let $\mathfrak{E}$ be the semigroup in $\mathfrak{X}_+(T)$ generated by the weights

From [3], one knows that the coordinate ring $k[X]$ considered as a G-module is equal to

$$k[X] = \bigoplus_{\lambda \in \mathfrak{E}} S_\lambda \hookrightarrow \bigoplus_{\lambda \in \mathfrak{X}_+(T)} S_\lambda = k[G/U], \tag{2}$$

where

$$S_\lambda \cong \{f \in k[G] : f(gb) = \lambda(b) f(g) \ \forall g \in G, \ b \in B\},$$

(so, S_λ equals V_λ^* as a G-module), and the embedding $k[X] \hookrightarrow k[G/U]$ is G-equivariant. From the definition of S_λ we see that the rings $k[X]$ and $k[G/U]$ are graded by the elements of $\mathfrak{E}$, e.g., $S_\lambda S_\mu = S_{\lambda+\mu}$.

In [1], the following theorem, due to Kostant, is stated. It is a generalization of Prop. 1. Since our main result is based on its proof, we give this theorem with a sketch of the proof.

Theorem 2 (Kostant)**.** *Let the semigroup $\mathfrak{E} \subset \mathfrak{X}_+(T)$ be freely generated by a finite set of weights $\mathfrak{E}_0 = \{\lambda_1, \dots, \lambda_k\} \subset \mathfrak{E}$. Let I be an ideal in $k[V] = k[V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}]$, generated by the coordinates of tensors*

$$\Omega(v_\lambda \otimes v_\mu) - (\lambda + \mu + 2\rho, \lambda + \mu)(v_\lambda \otimes v_\mu), \tag{3}$$

where $v_\lambda \in V_\lambda$, $v_\mu \in V_\mu$, and λ and μ run over the set $\mathfrak{E}_0$. Then the coordinate algebra of the S-variety $X(\lambda_1, \dots, \lambda_k)$ is equal to $k[V]/I$.

Proof. The relations $\Omega(v_\lambda \otimes v_\mu) - (\lambda+\mu+2\rho, \lambda+\mu)(v_\lambda \otimes v_\mu) = 0$ hold for the sum of highest vectors $v^+_{\lambda_1} + \cdots + v^+_{\lambda_k}$, because $v^+_\lambda \otimes v^+_\mu = v^+_{\lambda+\mu}$. Further, they are G-invariant. This means that they hold along the whole orbit. This proves the inclusion $I \subseteq I(X)$.

Now let us prove the reverse inclusion. Let $n = \dim G$, and $\{x_1, \dots, x_n\}$ and $\{x_1^*, \dots, x_n^*\}$ be two bases of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, dual with respect to the Cartan-Killing form. Consider the Casimir operator

$$\Omega = \sum_{i=1}^{n} x_i x_i^* \in U(\mathfrak{g}).$$

Ω acts on V_λ as a scalar $(\lambda + 2\rho, \lambda)$. Denote this scalar by $\lambda[\Omega]$. If λ' is a dominant weight, such that $\lambda' = \lambda - \sum k_i \alpha_i$, where α_i are simple roots, and $k_i \geq 0$, with a strictly positive k_i among them, then $\Omega[\lambda] > \Omega[\lambda']$. Indeed,

$$(\lambda + 2\rho, \lambda) = (\lambda + 2\rho, \lambda') + \sum k_i (\lambda + 2\rho, \alpha_i) = (\lambda' + 2\rho, \lambda') + \\ \sum k_i ((\alpha_i, \lambda') + (\lambda + 2\rho, \alpha_i)) > (\lambda' + 2\rho, \lambda'). \tag{4}$$

Take $\lambda, \mu \in \mathfrak{E}_0$. Now let us rewrite the operator $\Omega - (\lambda + \mu)[\Omega]\mathrm{Id}$ acting on the tensor product $V_\lambda \otimes V_\mu$, in a slightly different form.

$$(\Omega - (\lambda+\mu)[\Omega]\mathrm{Id})(v_\lambda \otimes v_\mu) = \left(\sum_{i=1}^{n} (x_i \otimes x_i^* + x_i^* \otimes x_i) - 2(\lambda, \mu) \right) (v_\lambda \otimes v_\mu). \tag{5}$$

By assumption, the coordinates of all these expressions belong to I.

From the presentation (2) of the coordinate algebra $k[X]$ it follows that the ideal I is linearly generated by the kernels of projections onto the highest components

$$S^{n_1} V_{\lambda_1}^* \otimes \cdots \otimes S^{n_k} V_{\lambda_k}^* \to V_{n_1\lambda_1 + \cdots + n_k\lambda_k}^*$$

for $n_1, \ldots, n_m \in \mathbb{Z}_+$. Each such kernel is the image of a projection $P = \Omega - \lambda[\Omega]\mathrm{I}d \in \mathrm{E}nd S^{n_1} V_{\lambda_1} \otimes \cdots \otimes S^{n_k} V_{\lambda_k}$, where $\lambda = n_1\lambda_1 + \cdots + n_k\lambda_k$.

Let $v \in V$, and let v_j denote the projection of v onto V_{λ_j}. Then the coordinates of the tensor $P(v_1^{n_1} \otimes \cdots \otimes v_k^{n_k})$ can be considered as elements of $k[V]$.

A simple calculation shows that

$$P(v_1^{n_1} \otimes \cdots \otimes v_k^{n_k}) = \sum_{1 \le r < s \le k} (T_{rs} - 2\, n_r n_s (\lambda_r, \lambda_s)\mathrm{Id})\, (v_1^{n_1} \otimes \cdots \otimes v_k^{n_k}), \tag{6}$$

where

$$T_{rs}(v_1^{n_1} \otimes \cdots \otimes v_k^{n_k}) = \sum_{i=1}^{n} (v_1^{n_1} \otimes \cdots \otimes x_i v_r^{n_r} \otimes \cdots \otimes x_i^* v_s^{n_s} \otimes \cdots \otimes v_k + \\ + v_1^{n_1} \otimes \cdots \otimes x_i^* v_r^{n_r} \otimes \cdots \otimes x_i v_s^{n_s} \otimes \cdots \otimes v_k). \tag{7}$$

According to (5), the coordinates of each summand from (6) belong to I. So, the coordinates of $P(v_1^{n_1} \otimes \cdots \otimes v_k^{n_k})$ belong to I as well. $\square$

From the proof, we get the following corollary, also stated in [1].

Corollary 3. *Let $\mathfrak{E}_0$ be arbitrary (probably without the linear independency condition). Then the ideal I contains the coordinates of all the expressions of the form $(\Omega - (\lambda_{i_1} + \cdots + \lambda_{i_m})[\Omega])(v_{i_1} \otimes \cdots \otimes v_{i_m})$, where $v_{i_p} \in V_{\lambda_{i_p}}$, $\lambda_{i_p} \in \mathfrak{E}_0$.*

4 S-varieties: Linearly dependent weight case

Now let us find equations defining an S-variety in the case when the highest vectors λ_i are linearly dependent, i.e., when the semigroup $\mathfrak{E}$ admits relations on elements of $\mathfrak{E}_0$. It is clear that in this case the relations analogous to (3) hold. But the quotient of $k[V]$ over these relations is larger than the coordinate ring $k[X]$. The former ring may include more that one G-invariant components of the form V^*_λ; their number equals to the number of presentations of λ as $\mathbb{Z}_+$-linear combinations of elements from $\mathfrak{E}_0$. In this case we add an additional set of relations to the Kostant's one. To do this, let us scale the highest vectors of V_{λ_i} in the following way.

Considet the algebra $k[G/U] = \bigoplus S_\lambda \cong \bigoplus V^*_\lambda$. Let f_λ be the highest vector of S_λ satisfying the condition $f_\lambda(w_0) = 1$, where w_0 is the longest element of the Weyl group of G. So, the vectors f_λ form a multiplicative semigroup, that is isomorphic to $\mathfrak{X}_+(T)$.

Now consider a G-module embegging $V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k} \hookrightarrow k[G/U]$ mapping each $v^+_{\lambda_i}$ to $f_{\lambda^*_i}$. Each such embedding will be called a *canonical* one.

Now we state the main result of this paper.

Theorem 4. *Let $\mathfrak{E} \subset \mathfrak{X}_+$ be a semigroup generated by a finite set of weights $\mathfrak{E}_0 = \{\lambda_1, \dots, \lambda_k\} \subset \mathfrak{E}$ with defining relations of the form*

$$n_1\lambda_{i_1} + \cdots + n_r\lambda_{i_r} = m_1\lambda_{j_1} + \cdots + m_s\lambda_{j_s},$$

where $\lambda_{i_1}, \dots, \lambda_{i_r}, \mu_{j_1}, \dots, \mu_{j_r} \in \mathfrak{E}_0$, $\{i_1, \dots, i_r\} \cap \{j_1, \dots, j_s\} = \varnothing$, $n_i, m_j \in \mathbb{Z}_+$. Also suppose that the G-module $V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}$ is embedded into $k[G/U]$ canonically.

Then the ideal $I(X) \subset k[V]$ of $X = X(\lambda_1, \dots, \lambda_k)$ is generated by the coordinates of tensors

$$\Omega(v_\lambda \otimes v_\mu) - (\lambda + \mu + 2\rho, \lambda + \mu)(v_\lambda \otimes v_\mu), \tag{8}$$

where $v_\lambda \in V_\lambda$, $v_\mu \in V_\mu$, $\lambda, \mu \in \mathfrak{E}_0$, and

$$\pi_1(v^{n_1}_{\lambda_{i_1}} \otimes \cdots \otimes v^{n_r}_{\lambda_{i_r}}) - \pi_2(v^{m_1}_{\lambda_{m_1}} \otimes \cdots \otimes v^{m_s}_{\lambda_{j_s}}), \tag{9}$$

where $v^{n_1}_{\lambda_{i_1}} \otimes \cdots \otimes v^{n_r}_{\lambda_{i_r}} \in S^{n_1}V_{\lambda_{i_1}} \otimes \cdots \otimes S^{n_r}V_{\lambda_{i_r}}$, $v^{m_1}_{\lambda_{m_1}} \otimes \cdots \otimes v^{m_s}_{\lambda_{j_s}} \in S^{m_1}V_{\lambda_{j_1}} \otimes \cdots \otimes S^{m_s}V_{\lambda_{j_s}}$, and π_1 π_2 are the G-equivariant linear maps from $S^{n_1}V_{\lambda_{i_1}} \otimes \cdots \otimes S^{n_r}V_{\lambda_{i_r}}$ $S^{m_1}V_{\lambda_{j_1}} \otimes \cdots \otimes S^{m_s}V_{\lambda_{j_s}}$ to V_λ,

$\lambda = n_1\lambda_{i_1} + \cdots + n_r\lambda_{i_r} = m_1\lambda_{j_1} + \cdots + m_s\lambda_{j_s}$, *such that* $\pi_1((v^+_{\lambda i_1})^{n_1} \otimes \cdots \otimes (v^+_{\lambda_{i_r}})^{n_r}) = \pi_2((v^+_{\lambda j_1})^{m_1} \otimes \cdots \otimes (v^+_{\lambda_{j_s}})^{m_s})$.

Proof. Denote the ideal generated by the coordinates of (8) and (9), by I. These coordinates vanish on the sum of the highest vectors (this holds for (9), since the scaling of the highest vectors is canonical). They generate a G-invariant ideal. This means that $I \subseteq I(X)$.

The coordinate ring $k[V] = k[V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}]$ is isomorphic to $S(V^*) = \bigoplus\limits_{m\in\mathbb{Z}_+} S^m(V^*)$, and

$$S^m(V^*_{\lambda_1} \oplus \cdots \oplus V^*_{\lambda_k}) = \bigoplus_{i_1+\cdots+i_k=m} S^{i_1}V^*_{\lambda_1} \otimes \cdots \otimes S^{i_k}V^*_{\lambda_k}.$$

Each G-module $S^{i_1}V^*_{\lambda_1} \otimes \cdots \otimes S^{i_k}V^*_{\lambda_k}$ equals to the direct sum of the highest irreducible component $V^*_{i_1\lambda_1+\cdots+i_k\lambda_k}$ and the remaining (lower) components of the decomposition. According to Corollary 3, all the lower components are contained in the ideal I for each $(i_1, \ldots, i_k)$.

Roughly speaking, the generators of I that come from the relations (9), identify the highest components of the spaces $S^{n_1}V^*_{\lambda_{i_1}} \otimes \cdots \otimes S^{n_r}V^*_{\lambda_{i_r}}$ and $S^{m_1}V^*_{\lambda_{j_1}} \otimes \cdots \otimes S^{m_s}V^*_{\lambda_{j_s}}$. These relations vanish on the kernel Ker F of the operator $F \in \operatorname{End}(S^{n_1}V_{\lambda_{i_1}} \otimes \cdots \otimes S^{n_r}V_{\lambda_{i_r}} \oplus S^{m_1}V_{\lambda_{j_1}} \otimes \cdots \otimes S^{m_s}V_{\lambda_{j_s}})$, $F\colon v_1 + v_2 \mapsto \pi_1(v_1) - \pi_2(v_2)$, where $v_1 \in S^{n_1}V_{\lambda_{i_1}} \otimes \cdots \otimes S^{n_r}V_{\lambda_{i_r}}$, $v_2 \in S^{m_1}V_{\lambda_{j_1}} \otimes \cdots \otimes S^{m_s}V_{\lambda_{j_s}}$).

The space Ann Ker F of the linear functions that vanish on Ker F is G-invariant. As a G-module, it is isomorphic to the (indecomposable) representation $V^*_{n_1\lambda_{i_1}+\cdots+n_r\lambda_{i_r}}$. So, since the intersection of Ann Ker F with I is nontrivial, Ann Ker $F \subset I$.

Each relation in the semigroup $\mathfrak{E}$ is obtained as the sum of certain defining relations, i.e., can be presented in form $\sum_k(n_{k1}\lambda_{ki_1} + \cdots + n_{kr}\lambda_{ki_r}) = \sum_k(m_{k1}\lambda_{kj_1} + \cdots + m_{ks}\lambda_{kj_s})$. For each k, the highest vectors $v_k = (v^+_{\lambda^*_{ki_1}})^{n_{k1}} \otimes \cdots \otimes (v^+_{\lambda^*_{ki_r}})^{n_{kr}}$ and $\tilde{v}_k = (v^+_{\mu^*_{kj_1}})^{n_{kj}} \otimes \cdots \otimes (v^+_{\lambda^*_{kj_s}})^{n_{kr}}$ of the highest components of $S^{n_{k1}}V^*_{\lambda_{ki_1}} \otimes \cdots \otimes S^{n_{kr}}V^*_{\lambda_{ki_r}}$ $S^{m_{k1}}V^*_{\lambda_{kj_1}} \otimes \cdots \otimes S^{m_{ks}}V^*_{\lambda_{kj_s}}$ are equal modulo I. This means that the highest vectors $\bigotimes_k v_k$ and $\bigotimes_k \tilde{v}_k$ of the highest components of the modules $\bigotimes_k(S^{n_{k1}}V^*_{\lambda_{ki_1}} \otimes \cdots \otimes S^{n_{kr}}V^*_{\lambda_{ki_r}})$ $\bigotimes_k(S^{m_{k1}}V^*_{\lambda_{kj_1}} \otimes \cdots \otimes S^{m_{ks}}V^*_{\lambda_{kj_s}})$ are also equal modulo I.

Thus, the quotient of the ring $k[V]$ over the ideal I can be embedded into $\bigoplus\limits_{\lambda\in\mathfrak{E}} S_\lambda = k[X]$. Since $I \subset I(X)$, this ring equals $k[X]$. So, $I = I(X)$. □

References

[1] G. Lancaster, J. Towber, Representation-functors and flag-algebras for the classical groups I, *J.Algebra* **59** (1979), 16–38.

[2] W. Lichtenstein, A system of quadrics describing the orbit of the highest weight vector, *Proc. AMS* **84** (1982), No. 4, 605–608.

[3] È. B. Vinberg, V. L. Popov, On a class of quasihomogeneous affine varieties, *Izv. USSR Math.*, **6** (1972), 743–758.

Independent University of Moscow, Bolshoi Vlasievskii per., 11, 119002 Moscow, Russia

E-mail address: `smirnoff@mccme.ru`

Submitted: November 12, 2003.