Multiple flag varieties

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Abstract. This is a survey of results on multiple flag varieties, i.e. varieties of the form $G/P_1 \times \cdots \times G/P_r$. We provide a classification of multiple flag varieties of complexity 0 and 1 and results on the combinatorics and geometry of B-orbits and their closures in double cominuscule flag varieties. We also discuss questions of finiteness for the number of G-orbits and existence of an open G-orbits on a multiple flag variety.

1. Introduction

1.1. **Multiple flag varieties.** Grassmann varieties and flag varieties first appeared at the end of the 19th century in the papers of Hermann Günter Grassmann, Julius Plücker, Hermann Schubert and other mathematicians. These varieties are homogeneous spaces of the group $G = \mathrm{GL}(V)$, with a parabolic subgroup P as the stabilizer of a point. One can also consider homogeneous spaces G/P not only for $\mathrm{GL}(V)$, but also for other connected reductive group.

In this survey we consider *multiple flag varieties*: these are direct products of several flag varieties, i.e. varieties of the form $G/P_1 \times \cdots \times G/P_r$ (for $r = 2$ they are called *double flag varieties*). Each multiple flag variety can be viewed as a G-variety for the diagonal action of the group G.

Among these, *spherical multiple flag varieties* are of particular interest. They are given by the following property: the algebra of regular functions on them as a G-module is decomposed into the sum of irreducible G-modules with multiplicity zero or one. In other words, spherical varieties are G-varieties with an open orbit of a Borel subgroup $B \subset G$. It is not hard to see that if a multiple flag variety is spherical, then the number of its factors G/P does not exceed two.

1.2. **Double flag varieties, unipotent invariants of Cox rings, and representation theory.** Multiple (in particular double) flag varieties appear in one of the fundamental problems of representation theory: in the problem of decomposition of the tensor product of two irreducible representations of a group G into irreducible summands.

The description of irreducible representations of a connected reductive algebraic group G dates back to Hermann Weyl. They are indexed

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by the dominant weights. There are many formulas for decomposing the tensor product of two irreducible G-modules into the direct sum of irreducible modules. In the general case (for an arbitrary group G) this problem can be solved by means of the Weyl character formula (see, for instance, [20]), but this approach has a serious disadvantage: it requires lengthy computations. For various concrete situations there are simpler and more explicit approaches which work in given particular cases: Steinberg's formula, Parthasarathy–Ranga Rao–Varadarajan's formula [39] etc.

Moreover, there are formulas using properties of concrete groups. The first and the most important of these results concerns representations of groups of type A, that is, $\mathrm{GL}(n)$. This is *the Littlewood–Richardson rule*, stated in 1934 [32] and proven by M.-P. Schützenberger [49] three decades later (since then a number of simpler proofs appeared: see, for example, [28], [14], to name just a few). Its main frature is an explicit, although quite involved, combinatorial description of irreducible components appearing in the decomposition of the product of two representations. An important (and the simplest) particular case of the Littlewood–Richardson rule is the *Pieri rule*: it describes the situation when one of the representations is a symmetric or an exterior power of the tautological representation. There is also a generalization of the Littlewood–Richardson rule for the case of other groups, due to P. Littelmann [30].

Double flag varieties appear in yet another approach to the problem of the decomposition of tensor products. This approach is based on a geometric realization of irreducible representations in the spaces of sections of line bundles on flag varieties.

According to the Borel–Weil theorem, every irreducible G-module can be realized as the space of global sections $H^0(G/P, \mathcal{L})$ of a line bundle $\mathcal{L}$ on G/P. The tensor product of the spaces of global sections $H^0(G/P, \mathcal{L}) \otimes H^0(G/Q, \mathcal{M})$ can be viewed as the space of sections $H^0(G/P \times G/Q, \mathcal{L} \boxtimes \mathcal{M})$ of the line bundle $\mathcal{L} \boxtimes \mathcal{M}$ on the direct product $G/P \times G/Q$. Here $\mathcal{L} \boxtimes \mathcal{M}$ is the line bundle whose fibers are the tensor products of fibers over the corresponding points: $(\mathcal{L} \boxtimes \mathcal{M})_{(x,y)} = \mathcal{L}_x \otimes \mathcal{M}_y$, where $x \in G/P$, $y \in G/Q$.

If the variety $X = G/P \times G/Q$ has complexity 0 (i.e., is spherical) or 1, there is an efficient way of decomposing the space of sections into the direct sum of irreducible G-modules. It will be described below in Subsection 3.3. This motivates the question of classifying all double flag varieties of complexity 0 and 1.

The relation between spherical double flag varieties and the problem of tensor product decomposition was first observed by Peter Littelmann in [31]. He classified all spherical varieties of the form $G/P \times G/Q$, where P, Q are *maximal* parabolic subgroups in G. Dmitry Panyushev in [38] computed the complexity of all varieties $G/P \times G/Q$ with

P, Q being maximal. The classification of spherical double flag varieties (with P, Q not necessarily maximal) was obtained by John Stembridge [55]. Finally, Elizaveta Ponomareva [41] obtained a full classification of all double flag varieties of complexity 0 and 1 by a uniform method, thus generalizing all the previous results. In Section 3.1 we give this classification.

A convenient tool for solving the tensor product decomposition problem is the notion of the *Cox ring* of a double flag variety. Consider the direct sum $R(X) = \bigoplus H^0(X, \mathcal{L})$ of the spaces of global sections of line bundles on X; if X satisfies some mild restrictions, this space can be equipped with a ring structure, called the Cox ring of X (this ring can be viewed as an analogue of the ring of regular functions $\mathbb{C}[X]$ on X). The problem of decomposition of the spaces $H^0(X, \mathcal{L})$ into irreducible G-modules is reduced to the description of the unipotent invariant algebra R^U of the Cox ring, where $U \subset B$ is the maximal unipotent subgroup.

Thus we need to describe the unipotent invariant algebras $R(X)^U$ in the Cox rings of double flag varieties X. Littelmann [31] showed that for X spherical this algebra is free. If X is of complexity 1, this algebra is either free or isomorphic to the quotient of a free algebra modulo a single relation (i.e. a hypersurface). Panyushev showed this for $X = G/P \times G/Q$ with P and Q maximal parabolic subgroups; the general case was considered by Ponomareva in [42] and [43]. We discuss these results in Subsection 3.4.

1.3. **Geometry of B-orbits on spherical double flag varieties.** Section 4 is devoted to the study of geometric and combinatorial properties of orbits of a Borel subgroup acting on a spherical flag varieties, and their closures. These orbits and their closures are direct analogues of Schubert cells and Schubert varieties in flag varieties G/P.

A combinatorial description of the set of these orbits for the case when G is of type A (i.e. $G = \mathrm{GL}(n)$) and a multiple flag variety is the direct product of two Grassmannians $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$, was obtained by Evgeny Smirnov in [52]. In this case the orbits are indexed by triples consisting of two Young diagrams and an involutive permutation of a special form. We also construct resolutions of singularities of B-orbit closures, analogous to the Bott–Samelson resolutions of Schubert varieties.

The geometry of B-orbit closures of double *cominuscule* flag varieties was studied by Piotr Achinger and Nicolas Perrin in [1]; generalizing the results of [52], they have shown that for G simply laced the B-orbit closures in these varieties are normal and have rational singularities (for the ground field of characteristic 0).

1.4. **Multiple flag varieties with finitely many G-orbits.** The property of X being spherical, i.e. the finiteness of the number of B-orbits on X, is equivalent to the finiteness of the number of G-orbits on the direct product $X \times G/B$ (there is a natural bijection between the B-orbits on the former variety and the G-orbits on the latter one). So each spherical double flag variety $G/P_1 \times G/P_2$ gives us a triple flag variety $G/P_1 \times G/P_2 \times G/B$ with finitely many G-orbits. We can state a natural question: what are the multiple flag varieties with finitely many G-orbits on them? This question is discussed in Section 5. The classification of such flag varieties was obtained by Peter Magyar, Jerzy Weyman, and Andrei Zelevinsky by methods of representation theory, for the case of G being equal to $\mathrm{GL}(V)$ or $\mathrm{Sp}(V)$; they also obtained a description of G-orbits and some results on the inclusion order on their closures.

1.5. **Multiple flag varieties with an open orbit.** The finiteness of the number of G-orbits on a multiple flag variety X implies that one of the orbits is open in X. The converse is, in general, not true. Thus we get a question of describing all multiple flag varieties with an open G-orbit. We are dealing with this question in Section 6. Vladimir Popov obtained a classification of all multiple flag varieties of the form $(G/P)^r$ (i.e. the products of several copies of the same flag variety), where P is a maximal parabolic subgroup. The existence of an open G-orbit on such a variety means that G acts on generic r-tuples of points in G/P transitively. These results were geenralized by Rostislav Devyatov for the case of P being not necessarily maximal, if G is not of the type A. Finally, in a recent paper by Izzet Coskun, Majid Hadian, and Dmitry Zakharov the authors provided a description of multiple flag varieties of the form $G/P_1 \times \cdots \times G/P_r$ with an open G-orbit, provided that G is of type A, and all the parabolic subgroups P_i are maximal (i.e., the variety is the product of Grassmannians).

Notation and conventions. The ground field is the field $\mathbb{C}$ of complex numbers.

Let G be a connected reductive algebraic subgroup over $\mathbb{C}$. We fix a *Borel subgroup* $B \subset G$ (i.e. a maximal connected solvable group) and a *maximal torus* $T \subset B$. The unipotent radical of B will be denoted by U; thus, $B \cong U \ltimes T$.

The root system associated to the triple $T \subset B \subset G$ will be denoted by R. We denote the positive and negative root systems by R^+ and R^- respectively. The Weyl group of R is denoted by $W \cong N(T)/T$. The simple root system corresponding to the triple (T, B, G) is called $\Delta \subset R^+$. The simple roots of simple algebraic groups are denoted by $\alpha_1, \dots, \alpha_r$, where $r = |\Delta|$. The fundamental weights dual to these roots are called $\omega_1, \dots, \omega_n$. The roots are numbered as in [7].

The weight lattice of the group G is denoted by Λ, the set of dominant weights is called Λ^+. For a dominant weight λ the irreducible representation of G with the highest weight λ is denoted by V_λ.

Structure of the paper. This text is organized as follows. In Section 2 we give preliminaries on flag varieties: we define the Schubert decomposition, the Bruhat order, we state the Borel–Weil theorem and give several equivalent definitions of a spherical variety. In Section 3 we give the classification of spherical double flag varieties, double flag varieties of complexity 1, and discuss the relation between these problems and the problem of decomposition of tensor products of irreducible G-modules. Section 4 is devoted to the study of combinatorial and geometric properties of orbits of a Borel subgroup in a spherical double flag variety and their closures. In Section 5 we discuss a generalization of the previous question: we study the situations when the number of G-orbits of a multiple flag variety is finite. Finally, in Section 6 we are dealing with results on multiple flag varieties with an open G-orbit.

2. Preliminaries

2.1. **Flag varieties.** Let G be a connected reductive algebraic group, let B be its Borel subgroup.

Definition 2.1. Let P be a connected algebraic subgroup in G containing B. Then P is called a *parabolic subgroup*, and the homogeneous space G/P is called a (generalized) *flag variety*.

Example 2.2. Let $G = \mathrm{GL}(n)$, and let B be the subgroup of nondegenerate upper-triangular matrices. Then each parabolic subgroup P is the stabilizer of a partial flag $V_\bullet = \langle e_1, \dots, e_{d_1}\rangle \subset \langle e_1, \dots, e_{d_2}\rangle \subset \dots \subset \langle e_1, \dots, e_{d_k}\rangle \subset \mathbb{C}^n$, where $1 \leq d_1 < \dots < d_k \leq n$ is a strictly increasing sequence of integers. Such a parabolic subgroup is called *standard*. In this case P is formed by block-triangular matrices with blocks of size $d_i - d_{i-1}$ on the diagonal, and the homogeneous space G/P is a variety of partial flags

$$Fl(d_1, \dots, d_k) = \{(U_1 \subset \dots \subset U_k \subset \mathbb{C}^n \mid \dim U_i = d_i\}.$$

Example 2.3. The previous example has two important particular cases, in a sense “opposite” to each other. If $k = 1$, subgroup P is a maximal parabolilc subgroup; in this case $G/P = \mathrm{Gr}(d, n)$ is the Grassmannian of d-dimensional subspaces in an n-dimensional space. On the contrary, if $P = B$, i.e. $(d_1, \dots, d_{n-1}) = (1, \dots, n-1)$, then $G/P = G/B$ is a *variety of complete flags*, or a *full flag variety*. Sometimes we will use the term “full flag variety” for G/B also in the case of the group G not equal to $\mathrm{GL}(n)$.

The parabolic subgroups $P \subset G$ containing B bijectively correspond to subsets in the system of simple roots Δ of the group G. We shall say that a parabolic subgroup corresponds to a set of simple roots $I \subseteq \Delta$, and denote it by P_I, if for its tangent algebra we have the following decomposition:

$$\mathfrak{p}_I = \mathfrak{t} \oplus \bigoplus_{\{\alpha \in R^+ \cup \mathbb{Z}(\Delta \setminus I)\}} \mathfrak{g}_\alpha,$$

where $\mathfrak{t}$ is the tangent algebra of the torus T and $\mathfrak{g}_\alpha$ are the root spaces in $\mathfrak{g}$ corresponding to the roots α.

In other words, the set of simple roots of the standard Levi subgroup in P_I equals $\Delta \setminus I$. So for "larger" subgroups we get "smaller" sets of simple roots: say, for the Borel subgroup $B = P_\Delta$. On the contrary, maximal parabolic subgroups correspond to subsets consisting of one simple root. In the case $G = \mathrm{GL}(n)$ for a flag variety $Fl(d_1, \ldots, d_k) = G/P_I$ we have $I = \{\alpha_{d_1}, \ldots, \alpha_{d_k}\}$. In particular, the Grassmannian $\mathrm{Gr}(d, n)$ corresponds to the set $I = \{\alpha_d\}$.

2.2. **Schubert decomposition.** For the group G we can consider its *Bruhat decomposition* (sometimes also called *Ehresmann–Bruhat decomposition*):

$$G = \bigsqcup_{w \in W} BwB.$$

The group G is presented as a disjoint union of double cosets of the Borel subgroup; these cosets are indexed by the elements of the Weyl group.

This decomposition gives us a decomposition of the full flag variety G/B into the union of orbits of B acting on G/B on the left:

$$G/B = \bigsqcup_{w \in W} BwB/B.$$

Let $w \in W$ be a Weyl group element, and let $\ell(w) \in \mathbb{Z}_+$ denote its *length*, i.e. the smallest number m such that w can be presented as the product of m simple reflections. (In the case $G = \mathrm{GL}(n)$ the Weyl group is the symmetric group: $W \cong S_n$, and the length of an element is just the length of a permutation, i.e. the number of its inversions: $\ell(w) = \#\{(i,j) \mid i < j, w(i) > w(j)\}$.) It is not hard to see that each B-orbit BwB/B is isomorphic to an affine space $\mathbb{A}^{\ell(w)}$ of dimension $\ell(w)$ equal to the length of w. Thus we get a cellular decomposition of the variety G/B.

Definition 2.4. The B-orbits on G/B are called *Schubert cells*. Their closures are called *Schubert varieties*; they will be denoted by $X_w = \overline{BwB/B}$. The cellular decomposition of G/B obtained in this way is called the *Schubert decomposition*.

The structure of the Schubert decomposition of G/B implies that the cohomology classes $[X_w] \in H^*(G/B, \mathbb{Q})$ which are Poincaré dual to the fundamental classes of Schubert varieties generate the cohomology ring $H^*(G/B,\mathbb{Q})$ as a $\mathbb{Q}$-vector space. This observation can be used for solving many problems of enumerative geometry: these problems can be restated in terms of computations in the ring $H^*(G/B,\mathbb{Q})$. This approach is called *Schubert calculus*; historically it appeared as the original motivation to introducing the Schubert decomposition of flag varieties. More on Schubert calculus can be found in the preface to the reprint of the original Schubert's book [48]; see also [25], [24], and [54].

2.3. **The Bruhat order.** The Schubert decomposition is a cellular decomposition, i.e. the closure of each cell is a union of cells. So the inclusion relation on the Schubert cell closures induces a partial order on the Weyl group. This order is called the *Bruhat order*.

Definition 2.5. We shall say that two Weyl group elements $v, w \in W$ are *comparable with respect to the Bruhat order*: $v \le w$, if $X_v \subset X_w$.

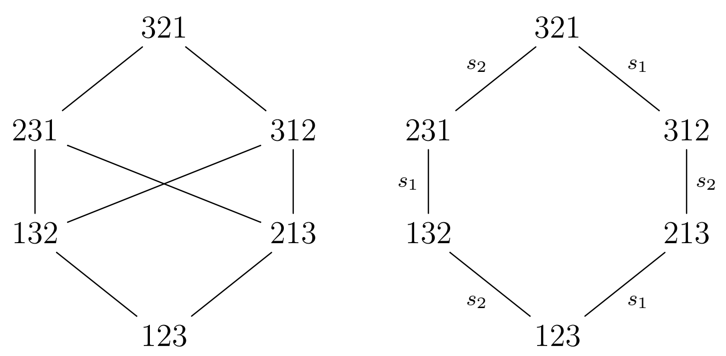
The Bruhat order admits the following combinatorial description (see, for instance, [23]):

Proposition 2.6. *Two elements v and w are comparable with respect to the Bruhat order: $v \le w$, if and only if there exists a sequence of reflections (not necessarily simple) $s_{i_1}, \dots, s_{i_r} \in W$ such that $w = s_{i_r} \dots s_{i_1} v$ and $\ell(s_{i_t} \dots s_{i_1} v) > \ell(s_{i_{t-1}} \dots s_{i_1} v)$ for each $t \le r$.*

We also can define the *weak Bruhat order*:

Definition 2.7. The elements v and w are *comparable with respect to the weak Bruhat order:* $v \le_w w$ if and only if there exists a sequence of *simple* reflections $s_{i_1}, \dots, s_{i_r} \in W$ such that $w = s_{i_r} \dots s_{i_1} v$, with $\ell(s_{i_t} \dots s_{i_1} v) > \ell(s_{i_{t-1}} \dots s_{i_1} v)$ for each $t \le r$.

It is clear that the relation $v \le_w w$ implies that $v \le w$; the converse is in general not true. Below we give the Hasse graph of the usual and weak Bruhat orders on the permutation group S_3 (for $G = \mathrm{GL}(3)$). We use the one-line notation for permutations: so, for instance, 321 is the permutation $1 \mapsto 3$, $2 \mapsto 2$, $3 \mapsto 1$.



The edges of the second graph are marked with simple transpositions: two vertices w and v are joined by an edge with s_i on it, if $w = s_i v$. We shall use the weak Bruhat order below for constructing Bott–Samelson resolutions of Schubert varieties, see Subsection 4.4.

2.4. **Parabolic subgroups in Weyl groups.** Let W be a Weyl group with generators $s_1, \dots, s_r$ and relations $s_i^2 = e$, $(s_i s_j)^{m_{ij}} = e$. Consider an arbitrary set of simple roots $I \subset \Delta$ and take the subgroup $W_I \subset W$ generated by all s_{α_i} such that $\alpha_i \in I$. (For example, if $I = \varnothing$, then $W_I = \{e\}$). This subgroup itself is a group generated by reflections; such subgroups are called *standard parabolic subgroups* in the Weyl group W.

Proposition 2.8 ([23, Proposition 1.10])**.** *Let R_I be the intersection of the root system R with the linear span $V_I = \langle \alpha_i \rangle$, $\alpha_i \in I$.*

- *R_I is a root system having I as its simple roots system. The group W_I is the reflection group associated with this root system.*
- *The length function on W_I (as on the reflection group) coincides with the restriction of the length function on W: namely, $\ell_I(w) = \ell(w)$ for each $w \in W_I$;*
- *Define $W^I = \{w \in W \mid \ell(ws_\alpha) > \ell(w) \text{ for each} \alpha \in \Delta_I\}$. Then each element w can be uniquely decomposed as the product $w = vu$, with $v \in W^I$, $u \in W_I$, such that $\ell(v) + \ell(u) = \ell(w)$. Moreover, v is a unique element of the minimal length in the left coset wW_I.*

The standard parabolic subgroups $W_I \subset W$ bijectively correspond to parabolic subgroups $P_I \subset G$ containing B. The set W^I parametrizes the B-orbits (Schubert cells) in the flag variety G/P_I:

$$G/P = \bigsqcup_{w \in W^I} BwP/P.$$

The embedding $W^I \hookrightarrow W$ induces on W^I the strong and the weak Bruhat orders; the strong order describes the inclusion order on the closures of Schubert cells, while the weak order is given by the action of minimal parabolic subgroups (see Subsection 4.3 below).

2.5. **Borel–Weil theorem.** This theorem states that all irreducible finite-dimensional representations of a reductive group G can be realized as the spaces of global sections of line bundles on G/B.

Let λ be an integer weight. It defines the character $\chi_\lambda \colon B \to \mathbb{C}^*$ of the Borel subgroup B, or, equivalently, a one-dimensional representation $\mathbb{C}_\lambda$ of the group B, with the action defined as follows: $b.z = \lambda(b)z$.

We can consider a homogeneous line bundle $G \times^B \mathbb{C}_\lambda = \mathcal{L}_\lambda$; this is a G-equivariant line bundle on G/B. All the G-equivariant line bundles on G/B can be obtained in such a way. Holomorphic global sections $\mathcal{L}_\lambda$ correspond to holomorphic maps

$$f \colon G \to \mathbb{C}_\lambda : f(gb) = \chi_\lambda(b) f(g) \qquad \forall b \in B, \quad g \in G.$$

They form a vector space $H^0(G/B, \mathcal{L}_\lambda)$. This space carries a natural structure of G-module:

$$g \cdot f(h) = f(g^{-1}h) \qquad \forall g, h \in G.$$

Theorem 2.9 (A. Borel, A. Weil)**.** *For a dominant weight λ, the space $H^0(G/B, \mathcal{L}_\lambda)$ is isomorphic (as a G-module) to the irreducible G-module V_{λ^*} with the highest weight λ^*. Otherwise $H^0(G/B, \mathcal{L}_\lambda) = 0$.*

(The proof can be found, for instance, in [50] or [19]).

Note that the module V_{λ^*} can be viewed as the G-module induced from the one-dimensional B-module $\mathbb{C}_{\lambda^*}$.

Similarly, the spaces of global sections of G-equivariant line bundles over partial flag varieties $H^0(G/P, \mathcal{L}_\lambda)$ can be viewed as irreducible G-modules $V_{\lambda^*} = \operatorname{Ind}_P^G \mathbb{C}_{\lambda^*}$. We will use this geometric realization of irreducible representations of G in the next section.

The generalization of this theorem, usually called the Borel–Weil–Bott theorem, gives a description of higher cohomology spaces $H^i(G/B, \mathcal{L}_\lambda)$ as G-modules; see, for instance, [16].

2.6. **Spherical varieties.** In the previous subsections we described the Schubert decomposition for flag varieties G/P. It can be viewed as a decomposition of the variety into the union of orbits of a Borel subgroup $B \subset G$. The number of these orbits is finite, so among them there is an open orbit. The property of existence of an open orbit of a Borel subgroup on a G-variety defines a class of G-varieties which are called *spherical varieties.*

In this subsection we only give several definitions of a spherical variety, which we shall need later. A detailed exposition of the theory of spherical varieties can be found in many sources, in particular, in Dmitry Timashev's book [56] or in the recent survey [40] by Nicolas Perrin.

Definition 2.10. Let X be a normal G-variety. The *complexity* $c_G(X) = c(X)$ of X is the minimal codimension of a B-orbit in X. The variety X is said to be *spherical* if $c(X) = 0$.

Let us give several equivalent definitions of a spherical variety.

Theorem 2.11 (see, for instance, [40, Thm 2.1.2])**.** *The following are equivalent:*

(1) *X is spherical;*
(2) $\mathbb{C}(X)^B = \mathbb{C}$*;*
(3) *X consists of finitely many B-orbits.*
For X quasiprojective these properties are equivalent to the following one:
(4) *If $\mathcal{L}$ is a G-equivariant line bundle, the G-module $H^0(X, \mathcal{L})$ is multiplicity-free (i.e. for any G-module W we have* $\dim \operatorname{Hom}_G(W, H^0(X, \mathcal{L})) \leq 1$*).*

Apart of flag varieties, there are other important classes of spherical varieties, in particular, toric varieties (in this case $G = B = T$) and symmetric spaces.

3. Multiple flag varieties and tensor product decompositions

3.1. **Double flag varieties of complexity 0 and 1.** In this section we give a classification of double flag varieties of complexity 0 and 1. First note that we can restrict ourselves to the case when G is a simple algebraic group. Indeed, each semisimple group can be decomposed into an almost direct product of simple groups $G = G_1 \dots G_s$, and parabolic subgroups $P, Q \subset G$ are decomposed into an almost direct product of parabolic subgroups $P_i, Q_i \subset G_i$. Then the complexity of a double flag variety $G/P \times G/Q$ equals

$$c_G(G/P \times G/Q) = c_{G_1}(G/P_1 \times G/Q_1) + \dots + c_{G_s}(G_s/P_s \times G_s/Q_s).$$

So our question is reduced to the case of simple groups.

If G is a classical group, let us take B to be the subgroup of upper-triangular matrices (in the orthogonal and symplectic cases we can suppose that G preserves a bilinear form with an antidiagonal matrix). Then the parabolic subgroups containing B have a block-diagonal form, so they can be described by the sizes of the blocks on the diagonal. The only exception is the group $\mathrm{SO}(n)$ for n even: not all of its parabolic subgroups have such a form. The remaining subgroups can be brought to this form by conjugating with a permutation of the two middle basis vectors; such parabolic subgroups will be denoted by the prime sign.

For the exceptional groups we describe parabolic subgroups $P_I \supseteq B$ by subsets of simple roots $I \subseteq \Delta$, as in 2.1.

In the cases of classical and exceptional groups, the following classification theorems hold.

Theorem 3.1 ([41, Theorem 1]). *Let G be a classical group (i.e. $\mathrm{SL}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(n)$). Then all double flag varieties of complexity 0 and 1 correspond to the pairs of parabolic subgroups (up to a permutation, for $\mathrm{SL}(n)$ also up to a simultaneous transposition with respect to the antidiagonal, and in the case of $\mathrm{SO}(2n)$ also up to the diagram automorphism of G), given in Tables 1, 2, 3.*

Theorem 3.2 ([41, Theorem 2]). (1) *For the groups of type E_8, F_4 and G_2 there are no double flag varieties of complexity 0 and 1.*
(2) *For the group of type E_6 double flag varieties of complexity 0 correspond to the following pairs of parabolic subgroups:*

$$(\{\alpha_1\},\{\alpha_1\}),(\{\alpha_1\},\{\alpha_2\}),(\{\alpha_1\},\{\alpha_4\}),(\{\alpha_1\},\{\alpha_5\}),(\{\alpha_1\},\{\alpha_6\}),(\{\alpha_2\},\{\alpha_5\}),$$
$$(\{\alpha_4\},\{\alpha_5\}),(\{\alpha_5\},\{\alpha_5\}),(\{\alpha_5\},\{\alpha_6\}),(\{\alpha_1\},\{\alpha_1,\alpha_5\}),(\{\alpha_5\},\{\alpha_1,\alpha_5\});$$

Table 1. Pairs of parabolic subgroups corresponding to double flag varieties in the groups $\mathrm{SL}(n)$

Number of blocks in P and Q	Complexity 0 P	Complexity 0 Q	Complexity 1 P	Complexity 1 Q
$(2,2)$	(p_1,p_2)	(q_1,q_2)		
$(2,3)$	(p_1,p_2) (p_1,p_2) $(2,p_2)$	$(1,q_1,q_2)$ $(q_1,1,q_3)$ (q_1,q_2,q_3)	$(3,p_2)$, $p_2\ge 3$ (p_1,p_2), $p_1,p_2\ge 3$ (p_1,p_2), $p_1,p_2\ge 3$	(q_1,q_2,q_3), $q_1,q_2,q_3\ge 2$ $(2,2,q_3)$, $q_3\ge 2$ $(2,q_2,2)$, $q_2\ge 2$
$(2,4)$			$(2,p_2)$ (p_1,p_2), $p_1,p_2\ge 2$ (p_1,p_2), $p_1,p_2\ge 2$	(q_1,q_2,q_3,q_4) $(1,1,1,q_4)$ $(1,1,q_3,1)$
$(2,s)$	$(1,p_2)$	$(q_1,\dots,q_s)$		
$(3,3)$			$(1,1,p_3)$ $(1,p_2,1)$	(q_1,q_2,q_3) (q_1,q_2,q_3)

Table 2. Pairs of parabolic subgroups corresponding to double flag varieties in the groups $\mathrm{SO}(n)$

Number of blocks in P and Q	Complexity 0 P	Complexity 0 Q	Complexity 1 P	Complexity 1 Q
$(2,2)$ $(2,2)$	(p,p) (p,p)	(p,p) $(p,p)'$		
$(2,3)$	(p,p) (p,p)	(q_1,q_2,q_1), $q_1\le 3$ $(q,2,q)$	$(6,6)$	$(4,4,4)$
$(2,4)$	(p,p) (p,p) $(4,4)$	$(1,q,q,1)$ $(1,q,q,1)$' $(2,2,2,2)'$	$(4,4)$ $(5,5)$ $(5,5)$ $(5,5)$ $(5,5)$	$(2,2,2,2)$ $(2,3,3,2)$ $(3,2,2,3)$ $(2,3,3,2)'$ $(3,2,2,3)'$
$(2,5)$			$(4,4)$	$(2,1,2,1,2)$
$(2,6)$ $(2,6)$			$(4,4)$ $(4,4)$	$(1,1,2,2,1,1)$ $(1,1,2,2,1,1)'$
$(3,3)$	$(1,p,1)$ $(p,1,p)$	(q_1,q_2,q_1) $(p,1,p)$	$(2,2,2)$ $(2,p,2)$, $p>1$	$(2,2,2)$ $(q,1,q)$
$(3,4)$	$(1,p,1)$	(q_1,q_2,q_2,q_1)	$(2,2,2)$	$(1,2,2,1)$
$(3,5)$			$(1,p,1)$ $(2,1,2)$	(q_1,q_2,q_3,q_2,q_1) $(1,1,1,1,1)$
$(3,6)$			$(1,p,1)$	$(q_1,q_2,q_3,q_3,q_2,q_1)$
$(4,4)$			$(1,2,2,1)$ $(1,2,2,1)$	$(1,2,2,1)$ $(1,2,2,1)'$

varieties of complexity 0 correspond to the following pairs of parabolic subgroups:

$$(\{\alpha_1\},\{\alpha_1,\alpha_2\}),(\{\alpha_1\},\{\alpha_1,\alpha_6\}),(\{\alpha_1\},\{\alpha_4,\alpha_5\}),(\{\alpha_1\},\{\alpha_5,\alpha_6\}),$$
$$(\{\alpha_5\},\{\alpha_1,\alpha_2\}),(\{\alpha_5\},\{\alpha_1,\alpha_6\}),(\{\alpha_5\},\{\alpha_4,\alpha_5\}),(\{\alpha_5\},\{\alpha_5,\alpha_6\});$$

TABLE 3. Pairs of parabolic subgroups corresponding to double flag varieties in the groups $\mathrm{Sp}(n)$

Number of blocks in P and Q	Complexity 0		Complexity 1	
	P	Q	P	Q
$(2,2)$	(p,p)	(p,p)		
$(2,3)$	(p,p)	$(1,q,1)$	(p,p)	$(2,q,2)$
$(2,4)$			$(2,2)$	$(1,1,1,1)$
$(3,3)$	$(1,p,1)$	(q_1,q_2,q_1)		
$(3,4)$			$(1,p,1)$	(q_1,q_2,q_2,q_1)
$(3,5)$			$(1,p,1)$	(q_1,q_2,q_3,q_2,q_1)

(3) *For the group of type E_7 double flag varieties of complexity 0 correspond to the following pairs of parabolic subgroups:*

$$(\{\alpha_1\},\{\alpha_1\}),(\{\alpha_1\},\{\alpha_6\}),(\{\alpha_1\},\{\alpha_7\});$$

variety of complexity 1 corresponds to the following pairs of parabolic subgroups: $(\{\alpha_1\},\{\alpha_2\})$.

3.2. **Cox rings of double flag varieties.** Let us give the definition of the Cox ring for a projective variety X in the case when the Picard group $\mathrm{Pic}(X)$ is a free abelian group of finite rank. Let $\mathrm{Pic}(X)$ be freely generated by the classes of line bundles $\mathcal{L}_1,\dots,\mathcal{L}_s$. Then each line bundle over X is isomorphic to $\mathcal{L}_1^{k_1}\otimes\dots\otimes\mathcal{L}_s^{k_s}$, where $k_1,\dots,k_s\in\mathbb{Z}$.

Definition 3.3. The *Cox ring* of X is the space

$$R(X)=\bigoplus_{k_i\in\mathbb{Z}} H^0(X,\mathcal{L}_1^{k_1}\otimes\dots\otimes\mathcal{L}_s^{k_s}).$$

The multiplication on $R(X)$ is given by the tensor product of sections.

Remark 3.4. The ring $R(X)$ is multigraded by the group $\mathrm{Pic}(X)$. The sections of line bundles are exactly the multihomogeneous elements of $R(X)$.

A more general definition of Cox rings can be found, for example, in [2, Sec. 1.4]

Further we shall need the structure of Cox rings of flag varieties. It will be convenient to present flag varieties as the quotient spaces G/P^-, where P^- is a parabolic subgroup containing the Borel subgroub B^- which is *opposite* to B; in other words, the tangent algebra $\mathfrak{p}^-$ contains all the root spaces corresponding to the negative roots.

Let $I=\{\alpha_{i_1},\dots,\alpha_{i_r}\}\subset\Delta$ be a nonempty subset of the simple root system, let $P=P_I$ be the corresponding parabolic subgroup, and let $P^-=P_I^-$ be the opposite to P_I parabolic subgroup. It is well known (see, for example, [11]) that the Picard group $\mathrm{Pic}(G/P^-)\cong\mathbb{Z}^r$ is freely generated by the classes of Schubert divisors, i.e. by the classes of Schubert varieties of codimension one. These divisors form the set of all B-invariant prime divisors on G/P^-; they have the form $D_{i_k} = \overline{Bs_{i_k}P^-/P^-}$, where $\alpha_{i_k} \in I$ and s_{i_k} is the simple reflection corresponding to root α_{i_k}.

Let $D = \sum m_{i_k} D_{i_k} \in \mathrm{Pic}(G/P_I^-)$, let $\lambda = \sum m_{i_k}\omega_{i_k}$ be the weight of the canonical section of the bundle $\mathcal{O}(D)$. Then $H^0(G/P_I^-, \mathcal{O}(D)) \cong V_\lambda$, if $m_{i_1}, \dots, m_{i_r} \geq 0$, and zero otherwise. Hence,

$$R(G/P^-) \simeq \bigoplus_{\substack{\lambda = m_{i_1}\omega_{i_1} + \ldots + m_{i_r}\omega_{i_r} \\ m_{i_1}, \ldots, m_{i_r} \geqslant 0}} V_\lambda.$$

Now consider the double flag variety $X = G/P^- \times G/Q^-$, where $P = P_I$, $Q = P_J$, $I = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, $J = \{\alpha_{j_1}, \dots, \alpha_{j_t}\}$. Its Picard group is freely generated by the preimages of the Schubert divisors on G/P^- and G/Q^- under the canonical projections $X \to G/P^-$ and $X \to G/Q^-$ respectively. The Cox ring of the double flag variety X can be presented as follows:

$$R(X) = R(G/P^-) \otimes R(G/Q^-) \simeq \bigoplus_{\substack{\lambda = m_{i_1}\omega_{i_1} + \ldots + m_{i_r}\omega_{i_r},\, m_{i_1}, \ldots, m_{i_r} \geqslant 0 \\ \mu = n_{j_1}\omega_{j_1} + \ldots + n_{j_t}\omega_{j_t},\, n_{j_1}, \ldots, n_{j_t} \geqslant 0}} V_\lambda \otimes V_\mu.$$

It is multigraded with the multidegree given by an integer $(r+t)$-vector.

3.3. **Decomposition of tensor products of irreducible representations.** In this section we show how the algebras of U-invariants in the Cox rings of double flag varieties allow us to decompose tensor products of irreducible G-modules. We follow the exposition from the papers [42] and [43].

Let a ring A be graded by an abelian group E; then by A_ρ we denote the homogeneous component of A corresponding to an element $\rho \in E$.

Let $X = G/P^- \times G/Q^-$, with $P = P_I$, $Q = P_J$, $I = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, $J = \{\alpha_{j_1}, \dots, \alpha_{j_t}\}$. Let $\lambda = \sum m_{i_k}\omega_{i_k}$, $\mu = \sum n_{j_l}\omega_{j_l}$. Then the submodule $V_\lambda \otimes V_\mu \subset R(X)$ consists of the multihomogeneous elements of multidegree $(m_{i_1}, \dots, m_{i_r}, n_{j_1}, \dots, n_{j_t}) =: (\bar{m}, \bar{n})$, so it coicides with $R(X)_{(\bar{m},\bar{n})}$.

The multiplicity of occurence of V_ν in $V_\lambda \otimes V_\mu$ is equal to the dimension of the space $(V_\lambda \otimes V_\mu)_\nu^U$ of U-invariants that have the weight ν with respect to the action of the torus T. This space can be identified with the subspace. So,

$$V_\lambda \otimes V_\mu \simeq R(X)_{(\bar{m},\bar{n})} \simeq \bigoplus_\nu V_\nu^{\oplus d(\bar{m},\bar{n},\nu)},$$

where $d(\bar{m},\bar{n},\nu) = \dim R(X)^U_{(\bar{m},\bar{n}),\nu}$.

In the following cases the dimensions $d(\bar{m},\bar{n},\nu)$ and $d(\bar{m},\nu)$ occuring in the tensor product decomposition rules can be easily computed. Theorem 3.7 below implies that these cases include the cases of complexity 0 and 1.

Theorem 3.5. *Let $X = G/P^- \times G/Q^-$, where $P = P_I$, $Q = P_J$, $I = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, $J = \{\alpha_{j_1}, \dots, \alpha_{j_t}\}$. Let $\lambda = \sum m_{i_k}\omega_{i_k}$, $\mu = \sum n_{j_l}\omega_{j_l}$ and $(\bar m, \bar n) := (m_{i_1}, \dots, m_{i_r}, n_{j_1}, \dots, n_{j_t})$.*

Suppose that the algebra $R(X)^U$ is free, with elements of its minimal system of generators of weights $\nu_1, \dots, \nu_d$ and multidegrees $(\bar m_1, \bar n_1), \dots, (\bar m_d, \bar n_d)$. Then the following decomposition holds:

$$V_\lambda \otimes V_\mu \simeq \bigoplus_{k_1(\bar m_1, \bar n_1)+\ldots+k_d(\bar m_d, \bar n_d)=(\bar m, \bar n)} V_{k_1\nu_1+\ldots+k_d\nu_d}$$

Proof. This is obviously implied by the statements above. □

Theorem 3.6. *Let X, λ, μ, $(\bar m, \bar n)$ be as in Theorem 3.5, and the algebra $R(X)^U$ is a hypersurface (i.e. its generators satisfy a unique relation), with the weights and the multidegrees of its minimal system of homogeneous generators equal to $\nu_1, \dots, \nu_d$ and $(\bar m_1, \bar n_1), \dots, (\bar m_d, \bar n_d)$ respectively, and let the defining relation be of the weight ν_0 and the multidegree $(\bar m_0, \bar n_0)$. Then the following decomposition holds:*

$$V_\lambda \otimes V_\mu \simeq \bigoplus_{k_1(\bar m_1, \bar n_1)+\ldots+k_d(\bar m_d, \bar n_d)=(\bar m, \bar n)} V_{k_1\nu_1+\ldots+k_d\nu_d} - \\ - \bigoplus_{l_1(\bar m_1, \bar n_1)+\ldots+l_d(\bar m_d, \bar n_d)=(\bar m, \bar n)-(\bar m_0, \bar n_0)} V_{\nu_0+l_1\nu_1+\ldots+l_d\nu_d} \tag{3.1}$$

By the "difference" of two representations we mean a representation such that the multiplicity of the occurence of each irreducible V_ν is equal to the difference of the multiplicities of occurence of V_ν in these two representations.

Proof. Consider the following exact sequence:

$$0 \to (F_1) \xrightarrow{\varphi_1} \mathbb{C}[t_1, \dots, t_d] \xrightarrow{\varphi_2} R(X)^U \to 0,$$

where F_1 is the defining relation, φ_1 is a natural embedding, $\varphi_2(t_i) = f_i$. We introduce the following $\mathbb{Z}^{r+t+l}$-grading on the polynomial algebra $\mathbb{C}[t_1, \dots, t_d]$, where $l = \operatorname{rk} G$: assign to each variable t_i multidegree $(\bar m_i, \bar n_i)$ and weight ν_i. Then the maps φ_1 and φ_2 preserve the grading. The multiplicity of occurence of V_ν in $V_\lambda \otimes V_\mu$ equals $\dim R(X)^U_{(\bar m, \bar n), \nu}$, the multiplicity of its occurence in the first factor of the right-hand side of the isomorphism (3.1) equals $\dim \mathbb{C}[t_1, \dots, t_d]_{(\bar m, \bar n), \nu}$, while its multiplicity of occurence in its second factor (3.1) equals $\dim (F_1)_{(\bar m, \bar n), \nu}$. Since the sequence is exact, we have $\dim R(X)^U_{(\bar m, \bar n), \nu} = \dim \mathbb{C}[t_1, \dots, t_d]_{(\bar m, \bar n), \nu} - \dim (F_1)_{(\bar m, \bar n), \nu}$. The theorem is proven. □

3.4. U-invariants in the Cox rings of double flag varieties.

Theorem 3.7 ([42], [43])**.** *Let $X = G/P^- \times G/Q^-$ be of complexity 0 or 1, with P^- and Q^- parabolic subgroups of a simple group G, containing the Borel subgroup B^- and opposite to P and Q. Let P*

and Q correspond to subsets $I = \{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, $J = \{\alpha_{j_1}, \dots, \alpha_{j_t}\}$ of simple roots. Then the algebra $R(X)^U$ is generated by the elements of weights and multidegrees listed in Tables 4 and 5 and by elements of the corresponding fundamental weights $\omega_{i_1}, \dots, \omega_{i_r}, \omega_{j_1}, \dots, \omega_{j_t}$ of multidegrees $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, 1)$ respectively. If X is of complexity 0, these elements freely generate $R(X)^U$. If X is of complexity 1, in Table 5 we list the number of relations on these elements (there is either one or none), the weight and the multidegree of the relation. If the relation occurs, it is of the following form: the sum of all monomials oft the given weight and multidegree in the generators equals 0.

This classification theorem implies the following corollary

Corollary 3.8. (1) *Let X be a spherical double flag variety. Then the algebra $R(X)^U$ is free.*
(2) *Let X be a double flag variety of complexity 1. Then $R(X)^U$ is either free or a hypersurface.*

Remark 3.9. Some particular cases of this corollary were known earlier: part (1) was obtained by Littelmann in [31]). The algebra of U-invariants of $R(X)$ for X of complexity one, corresponding to the product of two flag varieties for maximal parabolic subgroups, was computed by Panyushev [38].

Table 4: Weights and multidegrees of generators for the U-invariant subalgebra in the Cox ring for complexity 0

I	**J**	**degree**	**weight**
		$\mathbf{SL_n}$	
α_i	α_j $i \leqslant j$	$(1,1)$	$\omega_{i-k} + \omega_{j+k}$, $k = 1, \dots, \min\{i, n-j\}$
α_i	α_1, α_j $i \leqslant j$	$(1,1,0)$ $(1,0,1)$ $(1,1,1)$	ω_{i+1} $\omega_{i-k} + \omega_{j+k}$, $k = 1, \dots, \min\{i, n-j\}$ $\omega_{i-k+1} + \omega_{j+k}$ $k = \max\{1, 2-(j-i)\}, \dots, \min\{i-1, n-j\}$
α_i	α_1, α_j $i > j$	$(1,1,0)$ $(1,0,1)$ $(1,1,1)$	ω_{i+1} $\omega_{i+k} + \omega_{j-k}$, $k = 1, \dots, \min\{j, n-i\}$ $\omega_{i+k} + \omega_{j-k+1}$, $k = 2, \dots, \min\{j-1, n-i\}$
α_i	α_j, α_{j+1} $i \geqslant j+1$	$(1,1,0)$ $(1,0,1)$	$\omega_{i+k} + \omega_{j-k}$, $k = 1, \dots, \min\{j, n-i\}$ $\omega_{i+k} + \omega_{j-k+1}$, $k = 1, \dots, \min\{j+1, n-i\}$
α_2	α_i, α_j $i > j$ $i, j-i, n-j \geqslant 2$	$(1,1,0)$ $(1,0,1)$ $(1,1,1)$	$\omega_1 + \omega_{i+1}$, ω_{i+2} $\omega_1 + \omega_{j+1}$, ω_{j+2} $\omega_{i+1} + \omega_{j+1}$

α_1	$\alpha_{i_1},\ \ldots,\ \alpha_{i_s}$	$(1,1,0,0,\ldots,0)$ $(1,0,1,0,\ldots,0)$ $\ldots$ $(1,0,0,0,\ldots,1)$	ω_{i_1+1} ω_{i_2+1} $\ldots$ ω_{i_s+1}
		$\mathbf{Sp_{2l}}$, $(l \geqslant 2)$	
α_1	α_i $i \leqslant l-1$	$(1,1)$ $(2,1)$	$\omega_{i-1},\ \omega_{i+1}$ ω_i for $i>1$
α_1	α_l	$(1,1)$ $(2,1)$	ω_{l-1} ω_l
α_l	α_l	$(1,1)$	$2\omega_k,\ k=0,\ldots,l-1$
		$\mathbf{SO_{2l}}$, $(l \geqslant 4)$	
α_1	α_i $i \leqslant l-3$	$(1,1)$ $(2,1)$	$\omega_{i-1},\ \omega_{i+1}$ ω_i for $i>1$
α_1	α_{l-2}	$(1,1)$ $(2,1)$	$\omega_{l-3},\ \omega_{l-1}+\omega_l$ ω_{l-2}
α_1	α_{l-1}	$(1,1)$	ω_l
α_2	α_l	$(1,1)$ $(1,2)$	$\omega_1+\omega_{l-1},\ \omega_l$ ω_{l-2}
α_3	α_l $l \geqslant 6$	$(1,1)$ $(1,2)$ $(2,2)$	$\omega_1+\omega_l,\ \omega_2+\omega_{l-1},\ \omega_{l-1}$ $\omega_1+\omega_{l-2},\ \omega_{l-3}$ $\omega_2+\omega_{l-2}$
α_{l-1}	α_l	$(1,1)$	$\omega_{l-2k-1},\ k=1,\ldots,[\frac{l-1}{2}]$
α_l	α_l	$(1,1)$	$\omega_{l-2k},\ k=1,\ldots,[\frac{l}{2}]$
α_1	$\alpha_i,\ \alpha_l$ $i \leqslant l-3$	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$	$\omega_{i-1},\ \omega_{i+1}$ ω_i for $i>1$ ω_{l-1}
α_1	$\alpha_{l-2},\ \alpha_l$	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$	$\omega_{l-3},\ \omega_{l-1}+\omega_l$ ω_{l-2} ω_{l-1}
α_1	$\alpha_{l-1},\ \alpha_l$	$(1,1,0)$ $(1,0,1)$ $(1,1,1)$	ω_l ω_{l-1} ω_{l-2}
α_l	$\alpha_1,\ \alpha_2$	$(1,1,0)$ $(1,0,1)$ $(2,0,1)$	ω_{l-1} $\omega_1+\omega_{l-1},\ \omega_l$ ω_{l-2}
α_l	α_1,α_{l-1}	$(1,1,0)$ $(1,0,1)$ $(1,1,1)$	ω_{l-1} $\omega_{l-2k-1},\ k=1,\ldots,[\frac{l-1}{2}]$ $\omega_{l-2k},\ k=1,\ldots,[\frac{l-2}{2}]$
α_l	α_1,α_l	$(1,1,0)$ $(1,0,1)$ $(1,1,1)$	ω_{l-1} $\omega_{l-2k},\ k=1,\ldots,[\frac{l}{2}]$ $\omega_{l-2k+1},\ k=2,\ldots,[\frac{l-1}{2}]$
α_l	α_{l-1},α_l	$(1,1,0)$ $(1,0,1)$	$\omega_{l-2k-1},\ k=1,\ldots,[\frac{l-1}{2}]$ $\omega_{l-2k},\ k=1,\ldots,[\frac{l}{2}]$

$\mathbf{SO_8}$			
α_4	α_2,α_3	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$	$\omega_1+\omega_3$, ω_4 ω_2 ω_1
$\mathbf{SO_{10}}$			
α_3	α_5	$(1,1)$ $(1,2)$	$\omega_1+\omega_5$, $\omega_2+\omega_4$, ω_4 $\omega_1+\omega_3$, ω_2
$\mathbf{SO_{2l+1}}$, $(l\geqslant 3)$			
α_1	α_i $i\leqslant l-2$	$(1,1)$ $(2,1)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$
α_1	α_{l-1}	$(1,1)$ $(2,1)$	ω_{l-2}, $2\omega_l$ ω_{l-1}
α_1	α_l	$(1,1)$ $(1,2)$	ω_l ω_{l-1}
α_l	α_l	$(1,1)$	ω_k, $k=0,\dots,l-1$
$\mathbf{E_6}$			
α_1	α_1	$(1,1)$	ω_2, ω_5
α_1	α_2	$(1,1)$ $(2,1)$	$\omega_1+\omega_5$, ω_3, ω_6 $\omega_2+\omega_5$, ω_4
α_1	α_4	$(1,1)$ $(2,1)$	ω_2, ω_5, $\omega_5+\omega_6$ ω_3, ω_6
α_1	α_5	$(1,1)$	0, ω_6
α_1	α_6	$(1,1)$ $(2,1)$	ω_1, ω_4 ω_2
α_1	α_1, α_5	$(1,1,0)$ $(1,0,1)$ $(1,1,1)$	ω_2, ω_5 0, ω_6 ω_4
$\mathbf{E_7}$			
α_1	α_1	$(1,1)$	0, ω_2, ω_6
α_1	α_6	$(1,1)$ $(2,1)$	ω_1, ω_7 ω_2
α_1	α_7	$(1,1)$ $(2,1)$ $(2,2)$	ω_2, ω_5, ω_6 ω_3, ω_7 ω_4

Table 5: Weights and multidegrees of generators for the U-invariant subalgebra in the Cox ring for complexity 1

I	**J**	**degree**	**weight**	**relations**
$\mathbf{SL_n}$				

α_2 $\alpha_i,\ \alpha_j,\ \alpha_m$ $i<j<m$	$(1,1,0,0)$ $(1,0,1,0)$ $(1,0,0,1)$ $(1,1,1,0)$ $(1,1,0,1)$ $(1,0,1,1)$	$\omega_{2-k}+\omega_{i+k},\ k=\max\{1,3-i\},\dots,2$ $\omega_1+\omega_{j+1},\ \omega_{j+2}$ $\omega_{2-k}+\omega_{m+k},\ k=1,\dots,\min\{2,n-m\}$ $\omega_{i+1}+\omega_{j+1}$ for $j-i>1$ $\omega_{i+1}+\omega_{m+1}$ $\omega_{j+1}+\omega_{m+1}$ for $m-j>1$	$(2,1,1,1)$ $\omega_1+\omega_{i+1}+\omega_{j+1}+$ $+\omega_{m+1}$ 1 relation
α_i $\alpha_1,\alpha_2,\alpha_3$ $i,n-i\geqslant 3$	$(1,1,0,0)$ $(1,0,1,0)$ $(1,0,0,1)$ $(1,1,0,1)$	ω_{i+1} $\omega_1+\omega_{i+1},\ \omega_{i+2}$ $\omega_1+\omega_{i+2},\ \omega_2+\omega_{i+1},\ \omega_{i+3}$ $\omega_2+\omega_{i+2}$	$(2,1,1,1)$ $\omega_1+\omega_2+\omega_{i+1}+$ $+\omega_{i+2}$ 1 relation
α_i $\alpha_1,\alpha_2,\alpha_{n-1}$ $i,n-i\geqslant 3$	$(1,1,0,0)$ $(1,0,1,0)$ $(1,0,0,1)$ $(1,1,0,1)$ $(1,0,1,1)$	ω_{i+1} $\omega_1+\omega_{i+1},\ \omega_{i+2}$ ω_{i-1} ω_i $\omega_1+\omega_i,\ \omega_{i+1}$	$(2,1,1,1)$ $\omega_1+\omega_i+\omega_{i+1}$ 1 relation
α_3 $\alpha_i,\ \alpha_j$ $i,j-i,n-j\geqslant 2$	$(1,1,0)$ $(1,0,1)$ $(1,1,1)$ $(2,1,1)$	$\omega_{3-k}+\omega_{i+k},\ k=\max\{1,4-i\},\dots,3$ $\omega_{3-k}+\omega_{j+k}$ $k=1,\dots,\min\{3,j-i\}$ $\omega_1+\omega_{i+1}+\omega_{j+1},\ \omega_{i+k}+\omega_{j+3-k}$ $k=1,\dots,\min\{2,j-i-1\}$ $\omega_2+\omega_{i+2}+\omega_{j+2}$	$(3,2,2)$ $\omega_1+\omega_2+\omega_{i+1}+$ $+\omega_{i+2}+\omega_{j+1}+\omega_{j+2}$ 1 relation
α_i α_2,α_4 $i,n-i\geqslant 4$	$(1,1,0)$ $(1,0,1)$ $(1,1,1)$ $(2,1,1)$	$\omega_1+\omega_{i+1},\ \omega_{i+2}$ $\omega_1+\omega_{i+3},\ \omega_2+\omega_{i+2},\ \omega_3+\omega_{i+1},\ \omega_{i+4}$ $\omega_1+\omega_3+\omega_{i+2},\ \omega_3+\omega_{i+3}$ $\omega_2+\omega_{i+1}+\omega_{i+3}$	$(3,2,2)$ $\omega_1+\omega_2+\omega_3+$ $+\omega_{i+1}+\omega_{i+2}+\omega_{i+3}$ 1 relation
α_i α_2,α_{n-2} $i,n-i\geqslant 4$	$(1,1,0)$ $(1,0,1)$ $(1,1,1)$ $(2,1,1)$	$\omega_1+\omega_{i+1},\ \omega_{i+2}$ $\omega_{i-1}+\omega_{n-1},\ \omega_{i-2}$ $\omega_1+\omega_{i-1},\ \omega_1+\omega_i+\omega_{n-1}$ $\omega_{i+1}+\omega_{n-1},\ \omega_i$ $\omega_{i-1}+\omega_{i+1}$	$(3,2,2)$ $\omega_1+\omega_{i-1}+\omega_i+$ $+\omega_{i+1}+\omega_{n-1}$ 1 relation
α_i,α_j α_1,α_2	$(1,0,1,0)$ $(1,0,0,1)$ $(0,1,1,0)$ $(0,1,0,1)$ $(1,1,0,1)$	ω_{i+1} $\omega_{2-k}+\omega_{i+k},\ k=\max\{1,3-i\},\dots,2$ ω_{j+1} $\omega_{2-k}+\omega_{j+k}$ $k=1,\dots,\min\{2,n-j\}$ $\omega_{i+1}+\omega_{j+1}$ for $j-i>1$	$(1,1,1,1)$ $\omega_1+\omega_{i+1}+\omega_{j+1}$ 1 relation
α_i,α_j α_1,α_{n-1}	$(1,0,1,0)$ $(1,0,0,1)$ $(0,1,1,0)$ $(0,1,0,1)$ $(1,0,1,1)$ $(0,1,1,1)$	ω_{i+1} ω_{i-1} ω_{j+1} ω_{j-1} ω_i for $i>1$ ω_j for $n-j>1$	$(1,1,1,1)$ $\omega_i+\omega_j$ for $j-i>1$ no relations; for $j-i=1$ 1 relation
$\mathbf{Sp_{2l}}$, $(l\geqslant 2)$			

α_l α_2 $l \geqslant 4$	$(1,1)$ $(1,2)$ $(2,2)$	$\omega_1+\omega_{l-1}$, ω_{l-2} $\omega_1+\omega_{l-1}$, $2\omega_1+\omega_l$, ω_l $2\omega_{l-1}$	$(3,4)$ $2\omega_1+2\omega_{l-1}+\omega_l$ 1 relation
α_1 α_i, α_l	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$ $(2,0,1)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$ ω_{l-1} ω_l	$(2,1,1)$ $\omega_i+\omega_l$ for $l-i>1$ no relations; for $l-i=1$ 1 relation
α_1 α_i, α_j $i<j<l$	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$ $(2,0,1)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$ ω_{j-1}, ω_{j+1} ω_j	$(2,1,1)$ $\omega_i+\omega_j$ for $j-i>1$ no relations; for $j-i=1$ 1 relation
		$\mathbf{Sp_4}$	
α_2 α_1	$(1,1,0)$ $(1,2,0)$ $(1,0,1)$	ω_1 ω_2 0, $2\omega_1$	$(2,1,1)$ $2\omega_1+\omega_2$ 1 relation
		$\mathbf{Sp_6}$	
α_3 α_2	$(1,1)$ $(1,2)$ $(2,2)$	ω_1, $\omega_1+\omega_2$ $2\omega_1+\omega_3$, ω_3 $2\omega_2$	$(3,4)$ $2\omega_1+2\omega_2+\omega_3$ 1 relation
		$\mathbf{SO_{2l}}$, $(l \geqslant 4)$	
α_1 α_i,α_j $i<j<l-2$	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$ $(2,0,1)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$ ω_{j-1}, ω_{j+1} ω_j	$(2,1,1)$ $\omega_i+\omega_j$ for $j-i>1$ no relations; for $j-i=1$ 1 relation
α_1 α_i,α_{l-2} $i<l-2$	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$ $(2,0,1)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$ ω_{l-3}, $\omega_{l-1}+\omega_l$ ω_{l-2}	$(2,1,1)$ $\omega_i+\omega_{l-2}$ for $l-2-i>1$ no relations; for $l-2-i=1$ 1 relation
α_1 α_i,α_j,α_l $i<j<l-2$	$(1,1,0,0)$ $(2,1,0,0)$ $(1,0,1,0)$ $(2,0,1,0)$ $(1,0,0,1)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$ ω_{j-1}, ω_{j+1} ω_j ω_{l-1}	$(2,1,1,0)$ $\omega_i+\omega_j$ for $j-i>1$ no relations; for $j-i=1$ 1 relation

α_1	$\alpha_i,\alpha_{l-2},\alpha_l$ $i<l-2$	$(1,1,0,0)$ $(2,1,0,0)$ $(1,0,1,0)$ $(2,0,1,0)$ $(1,0,0,1)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$ ω_{l-3}, $\omega_{l-1}+\omega_l$ ω_{l-2} ω_{l-1}	$(2,1,1,0)$ $\omega_i+\omega_{l-2}$ for $l-2-i>1$ no relations; for $l-2-i=1$ 1 relation
α_1	$\alpha_i,\alpha_{l-1},\alpha_l$ $i<l-1$	$(1,1,0,0)$ $(2,1,0,0)$ $(1,0,1,0)$ $(1,0,0,1)$ $(1,0,1,1)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$ ω_l ω_{l-1} ω_{l-2}	$(2,1,1,1)$ $\omega_i+\omega_{l-1}+\omega_l$ for $l-1-i>1$ no relations; for $l-1-i=1$ 1 relation
			SO_8	
α_4	α_2,α_4	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$	$\omega_1+\omega_3$, ω_4 ω_2 0, ω_2	$(2,1,1)$ $\omega_2+\omega_4$ 1 relation
α_4	$\alpha_1,\alpha_3,\alpha_4$	$(1,1,0,0)$ $(1,0,1,0)$ $(1,0,0,1)$ $(1,1,1,0)$	ω_3 ω_1 0, ω_2 ω_2	$(2,1,1,1)$ $\omega_1+\omega_3+\omega_4$ no relations
α_4	$\alpha_2,\alpha_3,\alpha_4$	$(1,1,0,0)$ $(2,1,0,0)$ $(1,0,1,0)$ $(1,0,0,1)$	$\omega_1+\omega_3$, ω_4 ω_2 ω_1 0, ω_2	$(2,1,0,1)$ $\omega_2+\omega_4$ 1 relation
α_4	$\alpha_1,\alpha_2,\alpha_4$	$(1,1,0,0)$ $(1,0,1,0)$ $(2,0,1,0)$ $(1,0,0,1)$	ω_3 $\omega_1+\omega_3$, ω_4 ω_2 0, ω_2	$(2,0,1,1)$ $\omega_2+\omega_4$ 1 relation
α_4	$\alpha_1,\alpha_2,\alpha_3$	$(1,1,0,0)$ $(1,0,1,0)$ $(2,0,1,0)$ $(1,0,0,1)$ $(1,1,0,1)$	ω_3 $\omega_1+\omega_3$, ω_4 ω_2 ω_1 ω_2	$(2,1,1,1)$ $\omega_1+\omega_2+\omega_3$ 1 relation
			SO_{10}	
α_5	α_2,α_5	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$ $(1,1,1)$ $(2,1,1)$	$\omega_1+\omega_4$, ω_5 ω_3 ω_1, ω_3 ω_3 $\omega_2+\omega_4$	$(2,1,1)$ $\omega_3+\omega_5$ 1 relation
α_5	α_2,α_4	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$ $(1,1,1)$	$\omega_1+\omega_4$, ω_5 ω_3 0, ω_2 $\omega_1+\omega_3$	$(2,1,1)$ $\omega_2+\omega_5$ no relations

α_5	α_3,α_5	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$	$\omega_1+\omega_5$, $\omega_2+\omega_4$, ω_4 $\omega_1+\omega_3$, ω_2 ω_1, ω_3	$(2,1,1)$ $\omega_1+\omega_3+\omega_5$ 1 relation
α_5	α_3,α_4	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$	$\omega_1+\omega_5$, $\omega_2+\omega_4$, ω_4 $\omega_1+\omega_3$, ω_2 0, ω_2	$(2,1,1)$ $\omega_2+\omega_4$ 1 relation
$\mathbf{SO_{12}}$				
α_4	α_6	$(1,1)$ $(1,2)$	$\omega_1+\omega_5$, $\omega_2+\omega_6$, $\omega_3+\omega_5$, ω_6 $\omega_1+\omega_3$, ω_2, $\omega_2+\omega_4$, ω_4	$(2,3)$ $\omega_2+\omega_4+\omega_6$ 1 relation
$\mathbf{SO_{2l+1}}$, $(l \geqslant 3)$				
α_2	α_l $l \geqslant 4$	$(1,1)$ $(1,2)$ $(2,2)$	ω_1+ω_l, ω_l ω_1+ω_{l-1}, ω_{l-2}, ω_{l-1} ω_1+ω_{l-1}	$(2,3)$ $\omega_1+\omega_{l-1}+\omega_l$ 1 relation
α_1	α_i,α_j $i<j<l-1$	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$ $(2,0,1)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$ ω_{j-1}, ω_{j+1} ω_j	$(2,1,1)$ $\omega_i+\omega_j$ for $j-i>1$ no relations; for $j-i=1$ 1 relation
α_1	α_i,α_{l-1} $i<l-1$	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$ $(2,0,1)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$ ω_{l-2}, $2\omega_l$ ω_{l-1}	$(2,1,1)$ $\omega_i+\omega_{l-1}$ for $l-1-i>1$ no relations; for $l-1-i=1$ 1 relation
α_1	α_i,α_l	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$ $(1,0,2)$	ω_{i-1}, ω_{i+1} ω_i for $i>1$ ω_l ω_{l-1}	$(2,1,2)$ $\omega_i+2\omega_l$ for $l-i>1$ no relations; for $l-i=1$ 1 relation
$\mathbf{SO_7}$				
α_2	α_3	$(1,1)$ $(1,2)$	ω_1+ω_3, ω_3 ω_1+ω_2, ω_1, ω_2	$(2,3)$ $\omega_1+\omega_2+\omega_3$ 1 relation
$\mathbf{E_6}$				
α_1	α_1, α_2	$(1,1,0)$ $(1,0,1)$ $(2,0,1)$	ω_2, ω_5 $\omega_1+\omega_5$, ω_3, ω_6 $\omega_2+\omega_5$, ω_4	$(2,1,1)$ $\omega_1+\omega_2+\omega_5$ 1 relation
α_1	α_1, α_6	$(1,1,0)$ $(1,0,1)$ $(2,0,1)$	ω_2, ω_5 ω_1, ω_4 ω_2	$(2,1,1)$ $\omega_1+\omega_2$ no relations

		$(2,1,1)$	ω_3	
α_1	$\alpha_4,\ \alpha_5$	$(1,1,0)$ $(2,1,0)$ $(1,0,1)$	$\omega_2,\ \omega_5,\ \omega_5+\omega_6$ $\omega_3,\ \omega_6$ $0,\ \omega_6$	$(2,1,1)$ $\omega_5+\omega_6$ 1 relation
α_1	$\alpha_5,\ \alpha_6$	$(1,1,0)$ $(1,0,1)$ $(2,0,1)$ $(1,1,1)$	$0,\ \omega_6$ $\omega_1,\ \omega_4$ ω_2 ω_3	$(2,1,1)$ $\omega_1+\omega_6$ no relations
			$\mathbf{E_7}$	
α_1	α_2	$(1,1)$ $(2,1)$	$\omega_1,\ \omega_1+\omega_6,\ \omega_3,\ \omega_7$ $\omega_2,\ \omega_2+\omega_6,\ \omega_5,\ \omega_6$	$(3,2)$ $\omega_1+\omega_2+\omega_6$ 1 relation

4. Orbits of a Borel subgroup on a multiple flag variety

As we discussed earlier in Subsection 2.1, the orbits of a Borel subgroup B on a flag variety G/P are nothing but Schubert cells. They are indexed by the cosets in the Weyl group W/W^I; one Schubert cell lies in the closure of another one if and only if the corresponding Weyl group elements are comparable with respect to the Bruhat order. Moreover, the closures of these orbits (they are called Schubert varieties) are normal, Cohen–Macaulay and have rational singularities.

Now consider the set of B-orbits on an arbitrary spherical multiple flag variety. One can ask similar questions: how to describe this set combinatorially? When does one orbit belong to the closure of another orbit? What can be said about the geometry of these closures, in particular, on their singular loci?

The answers for these questions are known only for several spherical multiple flag varieties. We present these results in this section.

4.1. **The direct product of two Grassmannians.** Let the group $G = \mathrm{GL}(n)$ act on the direct product of two Grassmannians $X = \mathrm{Gr}(k,n) \times \mathrm{Gr}(l,n)$. We already know that the variety X is spherical (see Theorem 3.1). Our next goal is to get a combinatorial description of B-orbits on X.

The group G is diagonally embedded into the direct product $G\times G$, with each copy of G acting on the corresponding Grassmannian. The orbits of a Borel subgroup $B\times B \subset G\times G$ are easy to describe: each such orbit is the direct product of two Schubert cells $C_\alpha \times C_\beta \subset \mathrm{Gr}(k,n)\times \mathrm{Gr}(l,n)$, where (α,β) is a pair of Young diagrams dominated by the rectangles of size $k\times n-k$ and $l \times n-l$, respectively.

Further we will describe how is a $(B\times B)$-orbit $C_\alpha\times C_\beta$ decomposed into B-orbits.

Let $\alpha = (\alpha_1,\dots,\alpha_k)$ be a Young diagram, with $n-k\geq \alpha_1 \geq \dots \geq \alpha_k \geq 0$. It corresponds to a "bit string": a sequence of zeroes and ones

$\mathbf{a} = (a_1, \dots, a_n)$ in the following way:

$$a_i = \begin{cases} 1, & i = \alpha_k + 1, \alpha_{k-1} + \alpha_k + 2, \dots, \alpha_1 + \dots + \alpha_k + k, \\ 0 & \text{otherwise.} \end{cases}$$

This sequence can be interpreted in the following way: the (lower) boundary of a Young diagram located inside a rectangle of size $k \times n-k$ is a broken line going from the lower-left into the upper-right corner of the rectangle. The i-th segment of this broken line is vertical if $a_i = 1$ and horizontal otherwise. So the bit string consists of k ones and $n-k$ zeroes.

Similarly we can consider the bit string $\mathbf{b} = (b_1, \dots, b_n)$ corresponding to the diagram $\beta \subset l \times n - l$. It consists of l ones and $m - l$ zeroes.

Definition 4.1. Let $\mathbf{a}, \mathbf{b}$ be two bit strings of length n. We shall say that an involutive permutation $w \in S_n$, $w^2 = \mathrm{Id}$, is *consistent* with the pair $(\mathbf{a}, \mathbf{b})$ if for each $i < n$ such that $w(i) > i$ we have $a_i = b_i = 0$ and $a_{w(i)} = b_{w(i)} = 1$.

Theorem 4.2 ([52], cf. also [34]). *There is a bijection between the B-orbits $\mathcal{O} \subset C_\alpha \times C_\beta$ and involutive permutations $w \in S_n$, consistent with the pair $(\mathbf{a}, \mathbf{b})$.*

Let us introduce the notion of the *common part* of the pair $(\mathbf{a}, \mathbf{b})$. It is a bit string $\mathbf{c}(\mathbf{a}, \mathbf{b}) = (a_{i_1}, \dots, a_{i_r})$, such that $i_1, \dots, i_r$ are the indices corresponding to the equal entries of the sequences $\mathbf{a}$ and $\mathbf{b}$: $a_{i_1} = b_{i_1}, \dots, a_{i_r} = b_{i_r}$. The length of c can be arbitrary, not exceeding n (in particular, $\mathbf{c}$ can be empty).

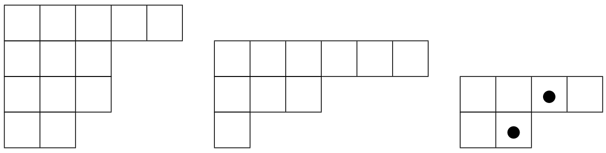
Definition 4.3. The *common diagram* for the Young diagrams α and β corresponding to bit strings $\mathbf{a}$ and $\mathbf{b}$ is the diagram $c(\lambda, \mu)$ which corresponds to the bit string $c(\mathbf{a}, \mathbf{b})$.

Example 4.4. Let $n = 9$, $k = 4$, $l = 3$. Consider $\alpha = (5, 3, 3, 2)$, $\beta = (6, 3, 1)$. Then $\mathbf{a} = (0, 0, 1, 0, 1, 1, 0, 0, 1)$, $\mathbf{b} = (0, 1, 0, 0, 1, 0, 0, 0, 1)$. The bit strings $\mathbf{a}$ and $\mathbf{b}$ coincide in the positions $1, 2, 4, 5, 7, 8, 9$; hence $c(\mathbf{a}, \mathbf{b}) = (0, 0, 1, 0, 0, 1)$ and $c(\alpha, \beta) = (4, 2)$.

Let $w = (i_1, j_1) \dots (i_s, j_s)$ be an involutive permutation presented as the product of independent transpositions, with $i_t < j_t$, and consistent with the pair $(\mathbf{a}, \mathbf{b})$. The latter condition means that $a_{i_1} = b_{i_1} = \dots = a_{i_s} = b_{i_s} = 0$, $a_{j_1} = b_{j_1} = \dots = a_{j_s} = b_{j_s} = 1$.

Involutive permutations that are consistent with a given pair $(\mathbf{a}, \mathbf{b})$ can be represented in the following way. Let us put marks (say, dots) into the boxes of the common Young diagram $c(\alpha, \beta)$ which correspond to the pairs $(i_1, j_1), \dots, (i_s, j_s)$. It turns out that no two marked boxes appear in the same row or in the same column. So we get a *rook placement*: each marked box of the Young diagram can be interpreted as a chess field occupied by a rook in such a way that no two rooks attack each other.

Example 4.5. The figure shows the Young diagrams α, β from Example 4.4 and their common diagram $c(\alpha, \beta)$ with marked boxes corresponding to the permutation $w = (2,4)(7,9)$.



4.2. **Order on B-orbits in a given $(B \times B)$-orbit.** Let $(\mathbf{a}, \mathbf{b})$ be a pair of bit strings. It defines a $(B \times B)$-orbit in $\mathrm{Gr}(k, V) \times \mathrm{Gr}(l, V)$, i.e. the product of two Schubert cells $C_\alpha \times C_\beta$. This $(B \times B)$-orbit is decomposed into B-orbits; these are indexed by the involutive permutations consistent with the pair $(\mathbf{a}, \mathbf{b})$. Let $w, v \in S_n$ be two such permutations, and let $\mathcal{O}_w, \mathcal{O}_v \subset C_\alpha \times C_\beta$ be the corresponding B-orbits. We give a criterion of inclusion of their closures: $\mathcal{O}_w \subset \overline{\mathcal{O}}_v$.

Consider the set of involutive permutations from S_n (not necessarily consistent with $(\mathbf{a}, \mathbf{b})$. To each such permutation $w \in S_n$ we can assign its *rank matrix* $R(w) = (r_{ij}(w))$. This is a strictly upper-triangular $(n \times n)$-matrix with nonnegative integer entries defined by the following rule:

$$r_{ij}(w) = \begin{cases} \#\{k \leq n \mid i \leq w(k) < k \leq j\}, & i < j; \\ 0 & \text{otherwise.} \end{cases}$$

The identity permutation corresponds to the zero rank matrix.

Example 4.6. Let $w = (13)(26)(47) \in S_7$; then

$$R(w) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let us introduce a partial order $\preceq$ on the involutive permutation (this order has nothing to do with the Bruhat order) by the following rule: $w \preceq v$ if and only if $r_{ij}(w) \leq r_{ij}(v)$ for each $1 \leq i < j \leq n$. This order has a unique minimal element, namely, the identity permutation; for $n \leq 3$ there is more than one maximal element.

Theorem 4.7. [53] *Let $\mathcal{O}_w$, $\mathcal{O}_v$ be two B-orbits belonging to the same $(B \times B)$-orbit in the product of two Grassmannians. Then $\mathcal{O}_w \subset \overline{\mathcal{O}}_v$ if and only if $w \preceq v$.*

The same order on involutive permutations appears in the works by Anna Melnikov on nilpotent upper-triangular matrices with the zero square. Namely, let $X = \{x \in \mathfrak{n} \subset \mathfrak{gl}_n(\mathbb{C}) \mid x^2 = 0\}$ be the variety of such matrices. Then the group of upper-triangular matrices $B_n \subset \mathrm{GL}_n(\mathbb{C})$ acts on X by conjugations. This action has finitely many orbits. As it is shown in [36], [37], these orbits are indexed by the involutive permutations in S_n. The inclusion order on these orbits is the same: one orbit lies in the closure of another one if and only if the corresponding permutations are comparable with respect to the order $\preceq$. An equivalent description of this order was obtained independently by Allen Knutson and Paul Zinn-Justin, see [27, Sec. 2].

4.3. **Weak order on B-orbit closures.** Our next goal is to construct resolutions of singularities for B-orbit closures in double Grassmannians. For this we shall use our explicit combinatorial description of these orbits.

First let us state some general facts on spherical varieties. Let X be a spherical G-variety, let Y be a B-orbit closure. Take a simple root $\alpha \in \Delta$ and consider the corresponding minimal parabolic group $P_\alpha = B \cup Bs_\alpha B$. The codimension of B in P_α is equal to 1, and $P_\alpha/B \cong \mathbb{P}^1$. We distinguish between the two cases: either $P_\alpha Y = Y$ or $P_\alpha Y = Y'$, where $\dim Y' = \dim Y + 1$. Suppose the second alternative holds. In this case we shall say that the simple root α *raises* the orbit closure Y to the orbit closure Y'. This relation can be extended to a partial order relation on the set of B-orbit closures (or, equivalently, B-orbits) on X. We will refer to this order as to the *weak order*.

Definition 4.8. We shall say that an orbit closure Y *is less than or equal to Y' in the sense of the weak order* (notation: $Y \preceq Y'$), if there exists a sequence of minimal parabolic subgroups $P_{\alpha_1}, \dots, P_{\alpha_r}$ such that $Y' = P_{\alpha_r} \dots P_{\alpha_1} Y$.

Remark 4.9. If $X = G/B$ is a full flag variety, the weak order on X coincides with the weak order on the Weyl group W, defined in Subsection 2.3: if $w = s_i v$, then for the corresponding Schubert varieties $X_w = P_{\alpha_i} X_v$.

It is clear that $Y \preceq Y'$ implies $Y \subseteq Y'$. The converse is false: for example, if two B-orbit closures are compatible with respect to the weak order, the corresponding B-orbits must belong to the same G-orbit, while for the usual inclusion (or degeneration) order this is not necessarily true. This explains the term "weak order".

4.4. **Parabolic induction and Bott–Samelson resolutions.** Take two B-orbit closures Y and Y' such that $Y' = P_\alpha Y$, i.e. Y raises to Y' by minimal parabolic subgroup P_α. Consider a B-equivariant fiber bundle

$$P_\alpha \times^B Y = \{(p, y) \mid p \in P_\alpha, y \in Y\}/(p, y) \sim (pb^{-1}, by), \quad b \in B.$$

This is a fiber bundle over $\mathbb{P}^1$ with the fiber Y; the projection to the base map is just the projection onto the first factor: $(p, x) \mapsto pB$.

Further, there is a map from $P_\alpha \times^B Y$ to Y':

$$\pi_{\alpha,Y} \colon P_\alpha \times^B Y \to Y', \qquad (p, y) \mapsto py.$$

Obviously,this map is a B-equivariant morphism of algebraic varieties.

The following statement is a standard fact from the theory of spherical varieties (see [46], [26, Lemma 3.2], [9]).

Theorem 4.10. *The map $\pi_{\alpha,Y} \colon P_\alpha \times^B Y \to Y'$ is either birational or generically two-to-one (the preimage of a general point consists of two points).*

It turns out that for certain classes of spherical varieties the map $\pi_{\alpha,Y}$ is always birational. Sometimes it allows us to construct resolutions of singularities for the orbit closures. First this approach was applied to Schubert varieties in flag varieties in the paper [6] by Raoul Bott and Hans Samelson; an algebraic reformulation is due to Michel Demazure [15] and H. C. Hansen [22]. Namely, the following statement holds.

Proposition 4.11 (see, for example, [12][Sec. 2.1])**.** *Let $X = G/P$ be a flag variety. Then*

(1) *$Y_{\min} = eP \subset G/P$ is a unique minimal orbit for the weak order on X;*
(2) *For each two Schubert varieties Y, Y' satisfying $Y' = P_\alpha Y$ the map $\pi_{\alpha,Y}$ is birational.*

This proposition allows us to construct resolutions of singularities for all orbit closures. Namely, let $Y \subset X$ be a B-orbit closure. Since $Y_{\min} \preceq Y$ (the minimal orbit is unique, so it is less than any other orbit), there exists a sequence of minimal parabolic subgroups such that $Y = P_{\alpha_r} \dots P_{\alpha_1} Y_{\min}$. According to the second part of the previous proposition the following map from the iterated $\mathbb{P}^1$-bundle, denoted by Z, into Y:

$$\pi_{\alpha_1,\dots,\alpha_r} \colon Z = P_{\alpha_r} \times^B P_{\alpha_{r-1}} \times^B \dots \times^B P_{\alpha_1} \times^B Y_{\min} \to Y,$$

is birational. But $Y_{\min}$ consists of one point, so, in particular, it is a smooth variety. So the iterated $\mathbb{P}^1$-bundle Z is smooth as well. We get the following result.

Theorem 4.12. *The map $\pi_{\alpha_1,\dots,\alpha_r} \colon Z \to Y$ is a resolution of singularities.*

Remark 4.13. Generally speaking, the sequence of parabolic subgroups raising $Y_{\min}$ to Y can be chosen in more than one way; so in such a way we can get different resolutions of singularities. Moreover, we can proceed in a more economic way by constructing fiber bundles having a Grassmannian instead of $\mathbb{P}^1$ as their base. As it was shown by Andrei

Zelevinsky, if $X = \mathrm{Gr}(k, n)$, among these resolutions we can always find so-called *small resolutions of singularities.* This allowed him to give a geometric description of the Kazhdan–Lusztig polynomials, in particular, to show their positivity. Details can be found in [58].

4.5. **Double cominuscule flag varieties.**

Definition 4.14. The minimal parabolic subgroup P_α associated to the root $\alpha \in \Delta$ is said to be *cominuscule* if the root α appears in the decomposition of the highest root with multiplicity 1. The fundamental weight ω dual to α is called a *cominuscule weight.* In this case the flag variety G/P_α is also said to be a *cominuscule flag variety.*

Here is the list of cominuscule weights and the corresponding flag varieties for the simple algebraic groups:

A_n**:** All fundamental weights are cominuscule; $G/P_k \cong \mathrm{Gr}(k, n+1)$.
B_n**:** ω_1; the variety $G/P_1 \cong Q^{2n-1}$ is an odd-dimensional quadric;
C_n**:** ω_n; the variety $G/P_n \cong \mathrm{LGr}(n)$ is a Lagrangian Grassmannian;
D_n**:** $\omega_1, \omega_{n-1}, \omega_n$; the corresponding varieties are an even-dimensional quadric $G/P_1 = Q^{2n-2}$ and orthogonal Grassmannians $G/P_{n-1} \cong G/P_n \cong \mathrm{OGr}(n)$.
E_6**:** ω_1, ω_6; the corresponding variety is the Cayley (octonionic) projective plane $\mathbb{OP}^2$;
E_7**:** ω_7; the variety is $G/P_7 \cong G_\omega(\mathbb{O}^3, \mathbb{O}^6)$ (see [45], [29]).

For the groups of type E_8, F_4, and G_2 there are no cominuscule weights.

Cominuscule flag varieties are remarkable due to several algebraic and geometric properties. In particular:

- The unipotent radical of the subgroup P_α is abelian;
- The Bruhat order on the B-orbits on G/P_α is a distributive lattice;
- The Levi subgroup in P_α acts in the tangent space to G/P_α at the point eP_α with finitely many orbits.

The classification theorem due to Peter Littelmann [31] (see also Section 3 above) implies the following theorem.

Theorem 4.15. *Each double cominuscule flag variety (i.e., the direct product of two cominuscule flag varieties) is spherical with respect to the diagonal action of G.*

The paper [1] by Piotr Achinger and Nicolas Perrin is devoted to the study of the geometry of B-orbit closures on double cominuscule flag varieties. The main result of this paper is the following theorem.

Theorem 4.16 ([1, Theorem 1])**.** *Let G be a simply laced reductive algebraic group (i.e., with the prime factors of types A, D, or E). Let $P, Q \subset G$ be two cominuscule parabolic subgroups containing a fixed*

Borel subgroup $B \subset G$, and $X = G/P \times G/Q$. Then the closures of B-orbits in X are normal Cohen–Macaulay varieties with rational singularities.

If the group G has type A, a double cominuscule flag variety is the direct product of two Grassmannians. In this case normality and rationality of singularities for its B-orbit closures was proven by Grzegorz Bobiński and Grzegorz Zwara [3] by methods of the representation theory of quivers.

To prove this result in the general case the authors study in detail the structure of B-orbits on double cominuscule flag varieties and observe the following two facts, which are of independent interest. For double Grassmannians (of type A) they were observed in [52].

Proposition 4.17. *The B-orbits on X that are minimal with respect to the weak order are $(B\times B)$-invariant, i.e., they are products of Schubert varieties in G/P and G/Q.*

Proposition 4.18. *Let G be simply laced, i.e., let it have only the prime factors of type A, D, E. Let Y and Y' be two B-orbit closures in $X = G/P \times G/Q$, comparable with respect to the weak order, with $\dim Y' = \dim Y + 1$ (i.e. $Y' = P_\alpha Y$ for a minimal parabolic subgroup P_α). Then the map $P \times^B Y \to Y'$ is birational.*

This proposition allows us to construct a resolution of singularities for the B-orbit closures on X that is similar to the Bott–Samelson resolution (Theorem 4.12).

Namely, let Y be the closure of some B-orbit. Consider the orbit closure $Y_{\min}$ that is minimal with respect to the weak order, such that $Y_{\min} \preceq Y$. In other words, there exists a sequence of minimal parabolic subgroups such that $Y = P_{\alpha_r} \dots P_{\alpha_1} Y_{\min}$. According to the previous proposition, the map

$$\pi_{\alpha_1,\dots,\alpha_r} \colon P_{\alpha_r} \times^B P_{\alpha_{r-1}} \times^B \dots \times^B P_{\alpha_1} \times^B Y_{\min} \to Y$$

is birational.

The difference with the case of flag varieties is as follows. First, a minimal orbit that precedes a given one with respect to the weak order, is not necessarily unique; second, the orbit closure $Y_{\min}$ may be singular. However, according to Proposition 4.17, all the minimal B-orbit closures $Y_{\min}$ are $(B \times B)$-invariant, i.e., they are products of Schubert varieties in G/P and G/Q. Taking the direct product of Bott–Samelson resolutions for these varieties, we get a birational isomorphism $\pi_{\min} \colon Z_{\min} \to Y_{\min}$ with $Z_{\min}$ smooth. So the composition map

$$P_{\alpha_r} \times^B P_{\alpha_{r-1}} \times^B \dots \times^B P_{\alpha_1} \times^B Z_{\min} \overset{\mathrm{Id}\times\pi_{\min}}{\to}$$
$$\overset{\mathrm{Id}\times\pi_{\min}}{\to} P_{\alpha_r} \times^B P_{\alpha_{r-1}} \times^B \dots \times^B P_{\alpha_1} \times^B Y_{\min} \overset{\pi_{\alpha_1,\dots,\alpha_r}}{\to} Y$$

is a birational isomorphism of a smooth variety and Y, i.e., a resolution of singularities.

The construction of this resolution of singularities is a key step for the proof of Theorem 4.16. The proof of normality, Cohen–Macaulayness and rationality of singularities for the B-orbit closures is based on the facts on spherical varieties given in the papers [9] and [10] and in general is similar to the original proof of normality of the Schubert varieties in G/P due to Seshadri (see [51] or, for instance, [12]).

The requirement for G to be simply laced is essential for Theorem 4.16. Let $G = \mathrm{Sp}(3)$ act in the standard way on the space $\mathbb{C}^6$ with a nondegenerate skew-symmetric bilinear form, and let P be the stabilizer of a maximal (three-dimensional) isotropic subspace in $\mathbb{C}^6$. Then the variety G/P is a Lagrangian Grassmannian $LGr(3,6)$. In [1, Prop. 5.1] the authors give an example of a B-orbit in $G/P \times G/P$ with a nonnormal closure. For this variety Proposition 4.18 is also violated: some of the maps $P \times^B Y \to Y'$ have a two-point generic fibes, hence they are not birational.

5. G-orbits on multiple flag varieties

5.1. **Multiple flag varieties with finitely many G-orbits.** In the previous sections we were considering orbits of a Borel subgroup $B \subset G$ acting diagonally on the direct product of two flag varieties $X = G/P_1 \times G/P_2$. These orbits bijectively correspond to the orbits of the group G acting diagonally on the direct product of three flag varieties $G/P_1 \times G/P_2 \times G/B$; this correspondence preserves the inclusion relation between the orbit closures. In other words, the variety X is spherical if and only if the group G acts on $X \times G/B$ with finitely many orbits.

This situation can be generalized: we can consider an arbitrary set of flag varieties G/P_i instead of $X \times G/B$. We get the following question:

Problem 5.1. *For which tuples of parabolic subgroups $(P_1, \dots, P_r)$ does the group G act on $G/P_1 \times \cdots \times G/P_r$ with finitely many orbits? How to describe the orbits of this action in combinatorial terms?*

The answer to this question was given for the groups $\mathrm{GL}(n)$ and $\mathrm{Sp}(2n)$ by P. Magyar, J. Weyman, and A. Zelevinsky in [34], [35]. For an arbitrary reductive group, in particular for orthogonal groups, classification of such tuples of parabolic subgroups is yet unknown.

The criterion of finiteness for the number of orbits on a multiple flag variety of type A provided by Magyar, Weyman, and Zelevinsky uses ideas and results from the quiver theory; this result is very similar to the description of finite-type quivers due to P. Gabriel [21]. Let us state it.

Definition 5.2. Denote by a *composition* of a nonnegative integer n an ordered collection of nonnegative integers $\mathbf{a} = (a_1, \dots, a_p)$ such that their sum is equal to n. These numbers are called the *parts* of the composition $\mathbf{a}$. The smallest number of $a_1, \dots, a_p$ is called the *minimum* of $\mathbf{a}$ and denoted by $\min(\mathbf{a})$.

To each composition $\mathbf{a}$ corresponds a partial flag variety $\mathrm{Fl}_{\mathbf{a}}$ of $\mathrm{GL}(n)$, consisting of flags $V_1 \subset \dots \subset V_p \cong \mathbb{C}^n$ such that $\dim V_i/V_{i-1} = a_i$. We shall say that a tuple of compositions $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ of a given number n is of *finite type* if the group $\mathrm{GL}(n)$ acts diagonally on $\mathrm{Fl}_{\mathbf{a}_1} \times \dots \times \mathrm{Fl}_{\mathbf{a}_k}$ with finitely many orbits. Moreover, we shall refer to a one-component composition as to a *trivial* one; it corresponds to a one-point flag variety. So we can suppose that all the compositions in a given tuple are nontrivial.

Theorem 5.3. *If a tuple of nontrivial compositions* $(\mathbf{a}_1, \dots, \mathbf{a}_k)$ *is of finite type, then* $k \leq 3$.

Sketch of the proof. Let us show that a quadruple of nontrivial compositions cannot have finite type. We shall do this for "the smallest" quadruple, i.e., for four compositions $(1, 1)$ of the number 2. In this case the corresponding multiple flag variety is the product of four projective lines $\mathbb{P}^1$. Quadruples of points on $\mathbb{P}^1$ up to the action of $\mathrm{GL}(2)$ are indexed by their cross-ratio, and it takes infinitely many values. $\square$

So each finite-type tuple of compositions consists of at most three components. Adding, if necessary, trivial ones, we can suppose that there are exactly three of them. Denote them by $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Let p, q, r denote the number of parts in these compositions; without loss of generality suppose that $p \leq q \leq r$.

Theorem 5.4. *A triple of compositions* $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *is of finite type if and only if it belongs to one of the following classes:*

- $A_{p,q}$: $(p, q, r) = (1, q, r)$, $1 \leq q \leq r$;
- D_{r+2}: $(p, q, r) = (2, 2, r)$, $2 \leq r$;
- E_6: $(p, q, r) = (2, 3, 3)$;
- E_7: $(p, q, r) = (2, 3, 4)$;
- E_8: $(p, q, r) = (2, 3, 5)$;
- $E_{r+3}^{(a)}$: $(p, q, r) = (2, 3, r)$, $3 \leq r$, $\min(\mathbf{a}) = 2$;
- $E_{r+3}^{(b)}$: $(p, q, r) = (2, 3, r)$, $3 \leq r$, $\min(\mathbf{b}) = 1$;
- $S_{p,q}$: $(p, q, r) = (2, q, r)$, $2 \leq q \leq r$, $\min(\mathbf{a}) = 1$.

5.2. **Description of orbits.** In this subsection we give a combinatorial description of the set of $\mathrm{GL}(V)$-orbits on $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ for each finite-type triple of compositions. For a compoition $\mathbf{a} = (a_1, \dots, a_p)$ (possibly with zero parts) let us write

$$|\mathbf{a}| = a_1 + \dots + a_p, \qquad \|\mathbf{a}\|^2 = a_1^2 + \dots + a_p^2.$$

The number of parts p will be called the *length* of a composition and denoted by $\ell(\mathbf{a})$.

For a given triple (p,q,r) denote by $\Lambda_{p,q,r}$ an additive subgroup formed by all triples of compositions $(\mathbf{a},\mathbf{b},\mathbf{c})$, such that $|\mathbf{a}|=|\mathbf{b}|=|\mathbf{c}|=n$, $(\ell(\mathbf{a}),\ell(\mathbf{b}),\ell(\mathbf{c}))=(p,q,r)$. Denote the *Tits quadratic form* by

$$Q(\mathbf{a},\mathbf{b},\mathbf{c})=\dim\mathrm{GL}(V)-\dim\mathrm{Fl}_{\mathbf{a}}(V)-\dim\mathrm{Fl}_{\mathbf{b}}(V)-\dim\mathrm{Fl}_{\mathbf{c}}(V),$$

where $\dim V=|\mathbf{a}|=|\mathbf{b}|=|\mathbf{c}|=n$. A simple computation shows that

$$Q(\mathbf{a},\mathbf{b},\mathbf{c})=\frac{1}{2}(\|\mathbf{a}\|^2+\|\mathbf{b}\|^2+\|\mathbf{c}\|^2-n^2).$$

Denote the set of "simple roots" $\Pi_{p,q,r}$ as the set of $\mathbf{d}=(\mathbf{a},\mathbf{b},\mathbf{c})$ satisfying $Q(\mathbf{d})=1$.

The following theorem allows us to reduce the description of orbits to a purely combinatorial problem.

Theorem 5.5. *Let $(\mathbf{a},\mathbf{b},\mathbf{c})\in\Lambda_{p,q,r}$ be a finite-type triple of compositions. Then $\mathrm{GL}(V)$-orbits in $\mathrm{Fl}_{\mathbf{a}}(V)\times\mathrm{Fl}_{\mathbf{b}}(V)\times\mathrm{Fl}_{\mathbf{c}}(V)$ bijectively correspond to tuples of nonnegative integers $(m_{\mathbf{d}})$, such that $\mathbf{d}\in\Pi_{p,q,r}$, satisfying the following identity in the semigroup $\Lambda_{p,q,r}$:*

$$(\mathbf{a},\mathbf{b},\mathbf{c})=\sum m_{\mathbf{d}}\mathbf{d}.$$

The set $\Pi_{p,q,r}$ also has an explicit description. Let $\mathbf{a}$ be a composition. Denote by $\mathbf{a}^+$ the composition obtained from $\mathbf{a}$ by removing its zero parts and rearranging the remaining parts in the decreasing order. Denote $\underbrace{(a,\dots,a)}_{p\text{ times}}=(a^p)$.

Theorem 5.6. *A triple $(\mathbf{a},\mathbf{b},\mathbf{c})\in\Lambda_{p,q,r}$ belongs to $\Pi_{p,q,r}$ if and only if the triple $(\mathbf{a}^+,\mathbf{b}^+,\mathbf{c}^+)$, considered up to a reordering, belongs to the following list:*

- $\{(1,1,1)\}$;
- $\{(3^2),(2^3),(2,1,1,1,1)\}$;
- $\{(4,2),(2^3),(1^6)\}$;
- $\{(m+1,m),(m,m,1),(1^{2m+1}\}$, $m\geq 2$;
- $\{(m,m),(m-1,m,1),(1^{2m}\}$, $m\geq 2$;
- $\{(m-1,1),(1^m),(1^m)\}$, $m\geq 2$.

Bijection in Theorem 5.5 has the following combinatorial interpretation. Condider the following additive category $\mathcal{F}_{p,q,r}$. Its *objects* are families $(V;A,B,C)$, where V is a vector space, (A,B,C) is a triple of flags in V belonging to the multiple flag variety $\mathrm{Fl}_{\mathbf{a}}(V)\times\mathrm{Fl}_{\mathbf{b}}(V)\times\mathrm{Fl}_{\mathbf{c}}(V)$ for a certain $(\mathbf{a},\mathbf{b},\mathbf{c})\in\Lambda_{p,q,r}$. The collection $(\mathbf{a},\mathbf{b},\mathbf{c})$ is called the *dimension vector* of the corresponding object. A *morphism* from $(V;A,B,C)$ to $(V';A',B',C')$ is a linear map $f\colon V\to V'$ such that $f(A_i)\subset A'_i$, $f(B_i)\subset B'_i$, $f(C_i)\subset C'_i$ for each i; direct sums are defined componentwise.

The category $\mathcal{F}_{p,q,r}$ can be considered as a subcategory in the category of representations of a quiver $Q_{p,q,r}$. This is a quiver with $p+$

$q+r-2$ vertices formed into three branches of length p, q and r, with all arrows oriented towards the central vertex. This category is defined by the following condition: all the “arrows”, i.e., linear maps between spaces in the vertices of the quiver, are embeddings. This is an additive subcategory, closed under taking extensions, but not quotients (so it is not abelian): if $I, J \in \mathcal{F}_{p,q,r}$ are two objects with $I \subset J$, then the quotient J/I does not necessarily belong to $\mathcal{F}_{p,q,r}$.

Isomorphism classes of objects from $\mathcal{F}_{p,q,r}$ with a given dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ coincide with the orbits of the $\mathrm{GL}(V)$-action on $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$. Hence the finiteness of the number of orbits is equivalent to the finiteness of the isomorphism classes of objects with a given dimension vector from $\mathcal{F}_{p,q,r}$. Each object in this category can be presented as a direct sum of indecomposable objects; according to Krull–Remak–Schmidt theorem, such a decomposition is unique up to an isomorphism of $(V; A, B, C)$. Thus each isomorphism class of an object is given by the multiplicities of indecomposable objects occuring as its direct summands. Moreover, it turns out that an indecomposable object is uniquely determined by its dimension vector. So if we know the list of indecomposable objects for a given category, then the problem of clsasification of $\mathrm{GL}(V)$-orbits on $\mathrm{Fl}_{\mathbf{a}}(V) \times \mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$ is reduced to a purely combinatorial problem of presentation of the dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ as a nonnegative integer linear combination of the dimension vectors of indecomposable objects. Also note that if the quiver $Q_{p,q,r}$ is of finite type (i.e., its graph is a Dynkin diagram of type A, D, or E), this automatically implies the finiteness of the number of orbits; these are the first five cases in Theorem 5.4.

Theorem 5.7. *For each $\mathbf{d} \in \Pi_{p,q,r}$ there exists a unique isomorphism class $I_{\mathbf{d}} \in \mathcal{F}_{p,q,r}$ with the dimension vector $\mathbf{d}$. For each finite-type triple $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \Lambda_{p,q,r}$ each object with the dimension vector $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ can be uniquely decomposed into the direct sum of objects of the form $I_{\mathbf{d}}$.*

Moreover, this method allows us not only to classify the $\mathrm{GL}(V)$-orbits on a multiple flag variety, but also to indicate an explicit representative in each orbit; see [34, Theorem 2.9].

Example 5.8. Consider a variety of type $A_{q,r}$, i.e. the direct product of two flag varieties $\mathrm{Fl}_{\mathbf{b}}(V) \times \mathrm{Fl}_{\mathbf{c}}(V)$. Let $\mathbf{b} = (b_1, \dots, b_q)$, $\mathbf{c} = c_1, \dots, c_r$. A pair of flags (B, C) corresponds to an object $(V; A, B, C)$ in the category $\mathcal{F}_{1,q,r}$, where A is a trivial flag: $0 = A_0 \subset A_1 = V$. According to Theorem 5.5, the indecomposable objects in this category have the reduced dimension vector $(1, 1, 1)$, i.e. they are of the form $I_{ij} = (V'; A', B', C')$ with $i \leq q$, $j \leq r$; this means that $\dim V' = 1$, $B' = (0 = B'_0 = \dots = B'_{i-1} \subset B_i = \dots = B_q = V'$, $C' = (0 = C'_0 = \dots = C'_{j-1} \subset C_j = \dots = C_r = V'$ (i.e. the dimension jumps in each flag are unique and occur on i-th and j-th places respectively).

This means that the $\mathrm{GL}(V)$-orbits in $\mathrm{Fl}_{\mathbf{b}}(V)\times\mathrm{Fl}_{\mathbf{c}}(V)$ are parametrized by matrices (m_{ij}) of size $q\times r$ with integer nonnegative entries. The sum of elements in each row of such a matrix is equal to $b_1,\dots,b_q$, and the column sums equal $c_1,\dots,c_r$. Such an orbit corresponds to the direct sum of indecomposable objects $\bigoplus_{i,j} m_{ij}I_{ij}$. In particular, if $\mathbf{b}=\mathbf{c}=(1^n)$, we are dealing with the direct product of two complete flags, with $\mathrm{GL}(V)$-orbits given by permutations (they bijectively correspond to Schubert cells in G/B).

5.3. **Orbit closures on multiple flag varieties.** Another natural question is to give the description of the generalized Bruhat order on a multiple flag variety: what are the conditions for one G-orbit on $X=\mathrm{Fl}_{\mathbf{a}}(V)\times\mathrm{Fl}_{\mathbf{b}}(V)\times\mathrm{Fl}_{\mathbf{c}}(V)$ to be inside the closure of another one? This question also can be answered in terms of the category $\mathcal{F}_{p,q,r}$.

Let Ω_F and $\Omega_{F'}$ be two G-orbits on X corresponding to the isomorphism classes of objects M and M' in the category $\mathcal{F}_{p,q,r}$ (in terms of Theorem 5.5). We shall say that $F\overset{\deg}{<}F'$ if $\Omega_F\subset\overline{\Omega}_{F'}$. This partial order will be called *degeneration order*.

The following result is due to Christine Riedtmann [47].

Proposition 5.9. *If $F\overset{\deg}{<}F'$, then for each indecomposable object $I_{\mathbf{d}}$, where $\mathbf{d}\in\Pi_{p,q,r}$, we have the following inequality:*

$$\dim\mathrm{Hom}(I_{\mathbf{d}},F)\geq\dim\mathrm{Hom}(I_{\mathbf{d}},F').$$

One can ask whether this necessary condition is also sufficient, i.e., do these inequalities (for each indecomposable object) imply the inclusion of the orbit closures. The results of Klaus Bongartz [4, §2], [5, §4] imply that this is true if the graph of the quiver $Q_{p,q,r}$ is a simply-laced Dynkin diagram, i.e. has the type A_n, D_n, E_6, E_7, E_8. In the cases of A_n and D_n this can also be checked directly (see [53]). Moreover, Peter Magyar [33] showed that these inequalities imply the degeneration order on orbits for the quiver of type $S_{p,q}$.

5.4. **The case $S_{p,q}$.** This is an interesting "non-Dynkin" case corresponds to $\mathrm{GL}(V)$-varieties of the form $G/P_1\times G/P_2\times\mathbb{P}(V)$. In other words, the group $\mathrm{GL}(V)$ acts with finitely many orbits on triples consisting of a two flags of a given type and a line in V. In particular, setting $P_1=B$, $P_2=P$, we get that the variety $G/P\times\mathbb{P}(V)$ is spherical. To the best of our knowledge, this was first observed by Michel Brion in [8].

In [33] Magyar gives a description of G-orbits on $G/B\times G/B\times\mathbb{P}(V)$ (they are given by "decorated" permutations, i.e. permutations from S_n, with $n=\dim V$, with a certain distinguished subset in $\{1,\dots,n\}$) and proves a simple criterion, stated in linear-algebraic terms. It allows to find out whether one G-orbit is inside the closure of another one.

He also describes the covering relations, i.e. pairs of orbits such that the first one is contained inside the second one and has codimension 1.

This spherical variety also plays an important role in the description of the *mirabolic Robinson–Schensted–Knuth correspondence*, see [57], [18].

5.5. **Symplectic multiple flag varieties of finite type.** In the paper [35] Problem 5.1 was solved for the group $G = \mathrm{Sp}_{2n}$. Moreover, similarly to the case of $G = \mathrm{GL}(V)$ the authors reduced the problem of combinatorial description of orbits on a multiple flag variety to a purely combinatorial one and indicated a representative in each orbit.

The main tool for this is the following (rather unexpected) observation: two multiple flags in a symplectic $2n$-dimensional space V belong to the same Sp_{2n}-orbit if and only if they belong to the same GL_{2n}-orbit. Thus the problem is essentially reduced to the case of $\mathrm{GL}(V)$.

Let us give a classification of multiple flag varieties of finite type for the group Sp_{2n}. Let V be a $2n$-dimensional symplectic vector space with a nondegenerate skew-symmetric bilinear form $\langle\, ,\, \rangle$. The group of automorphisms of V preserving this form is $\mathrm{Sp}(V) = \mathrm{Sp}_{2n}$. A subspace $U \subset V$ is called *isotropic* if $\langle U, U\rangle = 0$.

Let $\mathbf{a} = (a_1, \ldots, a_p)$ be a *symmetric* composition of the number $2n$, i.e. let $a_i = a_{p-i+1}$ for each i. Consider the space of flags $0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_p = V$ satisfying the condition $\dim A_i/A_{i-1} = a_i$. Such a flag is called *isotropic* if it is formed by isotropic subspaces and their orthogonals; the space of isotropic flags will be denoted by $\mathrm{Sp}\,\mathrm{Fl}_{\mathbf{a}}$:

$$\mathrm{Sp}\,\mathrm{Fl}_{\mathbf{a}} = \{A \in \mathrm{Fl}_{\mathbf{a}}(V) \mid \langle A_i, A_{p-i}\rangle = 0 \quad \text{for each } i\}.$$

We have obtained a realization of a partial flag manifold Sp_{2n}/P. The variety of complete symplectic flags Sp_{2n}/B corresponds to the dimension vector (1^{2n}).

A tuple of symmetric compositions $(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ is called a Sp_{2n}-*finite type tuple* if the group Sp_{2n} acts on $\mathrm{Sp}\,\mathrm{Fl}_{\mathbf{a}_1} \times \cdots \times \mathrm{Sp}\,\mathrm{Fl}_{\mathbf{a}_k}$ with finitely many orbits.

Similarly to the case $\mathrm{GL}(V)$ we get an analogue of Theorem 5.3: a k-tuple is of finite type only if $k \leq 3$. Here we list all the triples of compositions $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of Sp_{2n}-finite type. We suppose that the compositions do not contain zeroes and the number of nonzero parts is equal to (p, q, r) respectively, with $p \leq q \leq r$.

Theorem 5.10. *A triple of compositions* $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *is of* Sp_{2n}*-finite type if and only if it belongs to one of the following classes:*

- $\mathrm{Sp}\,A_{p,q}$**:** $(p, q, r) = (1, q, r)$, $1 \leq q \leq r$;
- $\mathrm{Sp}\,D_{r+2}$**:** $(p, q, r) = (2, 2, r)$, $2 \leq r$;
- $\mathrm{Sp}\,E_6$**:** $(p, q, r) = (2, 3, 3)$;
- $\mathrm{Sp}\,E_7$**:** $(p, q, r) = (2, 3, 4)$;
- $\mathrm{Sp}\,E_8$**:** $(p, q, r) = (2, 3, 5)$;

$\operatorname{Sp} E^{(b)}_{r+3}$**:** *$(p,q,r) = (2,3,r)$, $3 \le r$, the nonzero parts of* $\mathbf{b}$ *equal* $(1, 2n-2, 1)$*;*

$\operatorname{Sp} Y_{r+4}$**:** *$(p,q,r) = (3,3,r)$, $3 \le r$, the nonzero parts of one of the compositions equal* $(1, 2n-2, 1)$*.*

6. Multiple flag varieties with an open G-orbit

As we mentioned in Subsection 2.6, finiteness of the number of B-orbits on a G-variety X is equivalent to the existence of an open B-orbit on X. This, in turn, is equivalent to the existence of an open G-orbit on $X \times G/B$. Note that for an arbitrary parabolic subgroup P this is not true in general: the existence of an open P-orbit on X does not imply the finiteness of the number of P-orbits.

So we can ask the following question:

Problem 6.1. *For which multiple flag varieties $X = G/P_1 \times \cdots \times G/P_d$ there is an* open *G-orbit for the diagonal action of G on X?*

6.1. **Locally n-transitive actions on flag varieties.** Vladimir Popov [44] obtained an answer for this question in the following important particular case:

Question 6.2. *Let G be a connected simple linear algebraic group, let P be its maximal parabolic subgroup. For which pairs (G, P) does the group G act on $G/P \times G/P \times G/P$ with an open orbit?*

This question motivates the following definition.

Definition 6.3. Let n be a positive integer, and let G be an algebraic group acting algebraically on an irreducible variety X. Denote this action by $\alpha\colon G \times X \to X$. We shall say that α is *locally n-transitive* if the diagonal action $\alpha^n\colon G \curvearrowright X^n$ is locally transitive, i.e. has an open orbit. (If the initial action itself is not locally transitive, we shall say that it is *locally 0-transitive*).

Informally, local n-transitivity means that "almost any" n-tuple of points of X can be taken by the group action into "almost any other" n-tuple. It is clear that local n-transitivity of an action implies its local m-transitivity for each $0 < m \le n$. An upper estimate is also obvious: α cannot be locally n-transitive for $n \dim X > \dim G$.

Definition 6.4. The *local transitivity degree* of an action α is the number
$$\operatorname{gtd}(\alpha) := \sup n,$$
where the supremum is being taken over all n such that α is locally n-transitive. *Maximal transitivity degree* of a connected algebraic group is the number
$$\operatorname{gtd}(G) := \sup \operatorname{gtd}(\alpha),$$
where the supremum is being taken over all nontrivial actions α of G on all possible irreducible varieties.

In the paper [44] the following results on the maximal transitivity degree of connected algebraic groups were obtained.

Theorem 6.5. *Let G be a nontrivial connected algebraic group. Then,*

1. *if G is solvable,* $\mathrm{gtd}(G) \leq 2$*;*
2. *if G is nilpotent,* $\mathrm{gtd}(G) = 1$*;*
3. *if G is a reductive group and $\widetilde{G} \to G$ is an isogeny, then* $\mathrm{gtd}(\widetilde{G}) = \mathrm{gtd}(G)$*;*
4. *if $G = Z \times S_1 \times \cdots \times S_d$, where Z is an algebraic torus and $S_1, \ldots, S_d$ are connected simple algebraic groups, then*
$$\mathrm{gtd}(G) = \max_i \mathrm{gtd}(S_i);$$
5. *maximal transitivity degree of simple groups G is given in the following table:*

type of G	A_l	B_l	C_l	D_l	E_6	E_7	E_8	F_4	G_2
$\mathrm{gtd}(G)$	$l+2$	3	3	3	4	3	2	2	2

We get a natural question: on which varieties does a simple algebraic group (or, more generally, a reductive group) act in "the most transitive way", i.e. for which G-varieties the given maximum is attained? It turns out that among these varieties there always is a flag manifold corresponding to a certain maximal parabolic subgroup.

Theorem 6.6. *Let G be a connected nonabelian reductive group. Then there exists a maximal parabolic subgroup $P \subset G$ such that the local transitivity degree of the standard action of G on G/P equals the maximal transitivity degree* $\mathrm{gtd}(G)$ *of the group G.*

The following theorem lists the local transitivity degrees of G acting on all G/P with P maximal parabolic. It is clear that the G-action on G/P is always 2-transitive. For certain parabolic groups listed in the table below this transitivity degree is larger.

Theorem 6.7. *Let G be an arbitrary group, $d \geq 3$, let P_i be a maximal parabolic subgroup in G corresponding to the root α_i. Then the diagonal action of G on the multiple flag variety $(G/P_i)^n$ has an open orbit if and only if $n \leq 2$ or the triple (G, n, i) appears in the following table:*

Type of G	(n, i)
A_l	$n < \frac{(l+1)^2}{i(l+1-i)}$
B_l, $l \geq 3$	$n = 3$, $i = 1, l$
C_l, $l \geq 2$	$n = 3$, $i = 1, l$
D_l, $l \geq 4$	$n = 3$, $i = 1, l-1, l$
E_6	$n = 3, 4$, $i = 1, 6$
E_7	$n = 3$, $i = 7$

It is interesting that the action of the group SL_{l+1} on Grassmannians is "the most transitive": its local transitivity degree can be greater than or equal to 5, while for all the remaining groups it can only be equal to two, three, and (only in the case E_6) four. Namely:

Corollary 6.8. *Let G be a connected simple algebraic group of type A_l. Then*

$$\operatorname{gtd}(G : G/P_i) \begin{cases} = 3, & \text{if } 2i = l+1; \\ \geq 4 & \text{otherwise.} \end{cases}$$

This provides the answer for Question 6.2:

Corollary 6.9. *Let G be a connected simple linear algebraic group, P its maximal parabolic subgroup. G acts on $(G/P)^3$ with an open orbit if and only if P is conjugate to a standard minuscule or cominuscule parabolic subgroup.*

6.2. **The case of non-maximal parabolic subgroups.** Popov's results were generalized by Rostislav Devyatov [17] for the case of varieties G/P where G is a simple algebraic group not locally isomorphic to SL_l and P is an arbitrary (not necessarily maximal) parabolic subgroup.

Denote the intersection of several standard maximal parabolic subgroups by $P_{i_1,\ldots,i_s} = P_{i_1} \cap \cdots \cap P_{i_s}$. Then the following result holds.

Theorem 6.10 ([17])**.** *Let G be a simple algebraic group of type different from A_l, and let $P = P_{i_1,\ldots,i_s}$ be a non-maximal standard parabolic subgroup in G. The local transitivity degree of the standard action of G on G/P is equal to 3 for the cases listed in the following table and to 2 otherwise.*

Type of G	P
D_l, $l \geq 5$ *odd*	$P_{1,l-1}$, $P_{1,l}$
D_l, $l \geq 4$ *even*	$P_{1,l-1}$, $P_{1,l}$, $P_{l-1,l}$

Moreover, Devyatov shows directly that for these varieties the number of G-orbits on $(G/P)^3$ is infinite. Thus we get the following result.

Theorem 6.11. *Let G be a simple algebraic group, let $P \subset G$ be a parabolic subgroup, $n \geq 3$. The following are equivalent:*

- *G acts on $(G/P)^n$ with finitely many orbits;*
- *$n = 3$, P is maximal, and $(G/P)^n$ has an open G-orbit;*
- *$n = 3$ and $G/P \times G/P$ is spherical.*

6.3. **Products of Grassmannians with an open $\mathrm{GL}(n)$-orbit.** Popov and Devyatov consider the action of a group G on the product of several copies of the same flag manifold G/P. Izzet Coskun, Majid Hadian, and Dmitry Zakharov in [13] consider the group $\mathrm{GL}(n)$ acting on the product of several not necessarily isomorphic Grassmannians $X = \mathrm{Gr}(d_1, n) \times \cdots \times \mathrm{Gr}(d_k, n)$ and give a partial answer to the following question, which is a particular case of Problem 6.1.

Question 6.12. *For which tuples of dimensions* $(d_1, \dots, d_k; n)$ *does the group* $\mathrm{GL}(n)$ *act on* $\mathrm{Gr}(d_1, n) \times \dots \times \mathrm{Gr}(d_k, n)$ *with an open orbit? (Such a dimension vector is said to be* dense*).*

Dimension reasons allow us to formulate the following immediate necessary condition on the existence of such an orbit:

$$\sum_{i=1}^{k} d_i(n - d_i) \leq n^2 - 1. \tag{$*$}$$

The summand -1 in the right-hand side appears due to the fact that the center of $\mathrm{GL}(n)$ acts on X trivially. So we will further speak about the action of $\mathrm{PGL}(n)$ rather than $\mathrm{GL}(n)$.

The following example shows that this necessary condition is not sufficient.

Example 6.13. Consider the dimension vector $(1, 1, 2, 2; 3)$. For this vector $X = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{2*} \times \mathbb{P}^{2*}$, $\dim X = 8 = \dim \mathrm{PGL}(3)$, so the necessary condition holds.

An element of X can be treated as a configuration $(p_1, p_2, \ell_1, \ell_2)$ of two points and two lines in $\mathbb{P}^2$. Let us show that the action of $\mathrm{PGL}(3)$ does not have an open orbit. Let ℓ be a line containing both p_1 and p_2, and let q_1 and q_2 be the intersection points of ℓ with ℓ_1 and ℓ_2, respectively. Then the cross-ratio of p_1, p_2, q_1, q_2 on ℓ is $\mathrm{PGL}(3)$-invariant; moreover, if we fix p_1 and p_2 and vary ℓ_1 and ℓ_2, we can get an arbitrary value of this cross-ratio. This means that the codimension of $\mathrm{PGL}(3)$-orbits on X is at least 1.

The same example can be generalized for the case of arbitrary dimension. Consider the following dimension vector: $(1, 1, n-1, n-1; n)$, where $n \geq 3$. In this case $\dim X = 4(n-1) \leq n^2 - 1$; the elements of X correspond to quadruples (p_1, p_2, H_1, H_2) consisting of two points and two hyperplanes in $\mathbb{P}^{n-1}$. Such quadruples also have a continuous invariant: let $\ell = \langle p_1, p_2 \rangle$, $q_i = \ell \cap H_i$, $i = 1, 2$. Then the cross-ratio of the four points p_1, p_2, q_1, q_2 on ℓ is preserved by the action of $\mathrm{PGL}(n)$.

In both examples we were able to find a smaller configuration of subspaces, obtained from the initial one by taking sums and intersections, such that for this configuration the inequality $(*)$ does not hold anymore. Indeed, four points on a line (p_1, p_2, q_1, q_2) define an element of the 4-dimensional variety $(\mathbb{P}^1)^4$, so that the three-dimensional group $\mathrm{PGL}(2)$ acting on this variety cannot have an open orbit. Conjecturally, for each product of Grassmannians such that the action of $\mathrm{PGL}(n)$ on it does not have an open orbit, we can point out an "obstruction" to its existence: a configuration of subspaces obtained from the initial one by sums and intersections of subspaces, such that for this configuration the inequality $(*)$ is not satisfied.

The question of density or nondensity of certain dimension vectors often can be reduced to the question of density or nondensity of vectors in a smaller space. The following obvious statement is often used as the “induction base”:

Lemma 6.14. *A dimension vector* $(d_1, \ldots, d_k; n)$ *is dense if*

$$\sum_{i=1}^{k} d_i \leq n.$$

Further Coskun, Hadian, and Zakharov classify all the dimension vectors with a small number of components. As we have already seen, the group $\mathrm{GL}(n)$ always acts on the direct product of at most three Grassmannians with an open orbit (in fact, even with finitely many of them, see Subsection 5). It turns out that for the four components “almost all” dimension vectors are dense.

Theorem 6.15. *Let* $\mathbf{d}$ *be a dimension vector of length* $k \leq 4$*. It is not dense if and only if* $k = 4$ *and* $\mathbf{d} = (a, b, c, d; n)$ *with* $a+b+c+d = 2n$*.*

Proof. Let us give the proof of the “if” statement. First, if $a = b = c = d = n/2$, there is no open orbit because of the dimension reasons: $4(n/2)(n - n/2) = n^2 > n^2 - 1 = \dim \mathrm{PGL}(n)$.

If not all a, b, c, d are equal, let us prove the statement by induction over $a + b + c + d = 2n$. We will need the following lemma.

Lemma 6.16 ([13, Lemma 4.2])**.** *Let* $\mathbf{d} = (a_1, \ldots, a_r, b_1, \ldots, b_s; n)$ *be a dimension vector such that* $\sum_{i=1}^{r} a_i = n - k < n$ *and* $\sum_{j=1}^{s}(n - b_j) \leq n - k$*. Then* $\mathbf{d}$ *is dense if and only if the vector* $\mathbf{d}' = (a_1, \ldots, a_r, b_1 - k, \ldots, b_s - k; n - k)$ *is dense.*

Without loss of generality (by changing the order of subspaces and, if necessary, taking their duals) we can suppose that $a + b < n$. Now apply the previous lemma. Let $(V_1, V_2, V_3, V_4; V)$ be a configuration of vector spaces corresponding to the vector $(a, b, c, d; n)$. Consider the subspace $W := V_1 + V_2$; it has dimension $a + b$. Then the configuration of subspaces $(V_1, V_2, W \cap V_3, W \cap V_4; W)$ has the dimension vector $\mathbf{d}' = (a, b, a + b + c - n, a + b + d - n; a + b)$. The sum of its components is equal to $2(a + b) < 2n$. Hence $\mathbf{d}'$ is nondense by induction hypothesis. So $\mathbf{d}$ is also nondense. □

Reasoning in a similar way, we can give an algorithm that allows to get a full list of dense dimension vectors such that their maximal component $\max d_i$ does not exceed a given number l. Let us give the answer for $l \leq 3$. Introduce the following notation: if a component a occurs d times, denote it by a^d.

Let $l = 1$. This case is trivial: the vector $(1^r; n)$ is dense for $r \leq n+1$ and nondense otherwise (the group $\mathrm{PGL}(n)$ acts transitively on tuples of at most $n + 1$ points in general position).

Further, consider the case $l = 2$, $\mathbf{d} = (1^a, 2^b; n)$ (i.e. configurations of a points and b lines in $\mathbb{P}^{n-1}$).

Theorem 6.17. *All the dense vectors with the maximal component not exceeding* 2 *are listed below.*

- $(1^a, 2^b; n)$, *where* $a + 2b \le n + 1$*;*
- $(1^a, 2^b; n)$, *where* $a + 2b = n + 2$ *and* $a \le 3$*;*
- *Finitely many "exceptional" vectors with* $a+2b \ge n+3$*:* $(2^3; 3)$, $(1, 2^3; 3)$, $(2^4; 3)$, $(1, 2^3; 4)$, *and* $(2^4; 5)$.

Theorem 6.18. *All the dense vectors with the maximal component not exceeding* 3 *are listed below.*

- $(1^a, 2^b, 3^c; n)$, *where* $a + 2b + 3c \le n + 1$*;*
- $(1^a, 2^b, 3^c; n)$, *where* $a + 2b + 3c = n + 2$ *and* $a \le 3$*;*
- $(1^a, 2^b, 3^c; n)$, *where* $a + 2b + 3c = n + 3$, $a + b \le 4$ *and* $(a, b) \ne (2, 2)$*;*
- *Finitely many "exceptional" vectors with* $a + 2b + 3c \ge n + 4$*:* $(2, 3^2; 4)$, $(2^3, 3, 4)$, $(1, 2, 3^2; 4)$, $(3^3; 4)$, $(1, 3^3; 4)$, $(2, 3^3; 4)$, $(3^4; 4)$, $(1, 3^4; 4)$, $(2^3, 3; 5)$, $(1, 2, 3^2; 5)$, $(3^3; 5)$, $(1, 3^3; 5)$, $(2, 3^3; 5)$, $(3^4; 5)$, $(1, 3^3; 6)$, $(2^2, 3^2; 6)$, $(2, 3^3; 6)$, $(2, 3^3; 7)$, $(3^4; 8)$, $(2, 3^4; 9)$, *and* $(3^5; 11)$.

Further the situation is similar: for the vector $\mathbf{d} = (1^{e_1}, 2^{e_2}, \dots, k^{e_k}; n)$ with $e_1 + 2e_2 + \dots + ke_k = n + l + 1$, $l < k$ the criterion of its density can be written down explicitly: it is reduced to the density of a vector in a smaller ambient space and with a smaller maximal component. Namely, the following theorem holds.

Theorem 6.19.
1. *Let* $\mathbf{d} = (1^{e_1}, 2^{e_2}, \dots, k^{e_k}; n)$ *with* $e_1 + 2e_2 + \dots + ke_k = n + l + 1$ *and* $l < k$*. Then* $\mathbf{d}$ *is dense if and only if the vector* $(1^{e_1}, 2^{e_2}, \dots, l^{e_l}; l + 1)$ *is dense.*
2. *For a given* k *the number of dense vectors with* $e_1 + 2e_2 + \dots + ke_k \ge n + k + 1$ *is finite.*

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