GELFAND–ZETLIN POLYTOPES AND DEMAZURE CHARACTERS

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1. Introduction

An important feature of toric geometry is the interplay between a polarized projective toric variety and its convex polytope. For instance, the Hilbert polynomial can be computed by counting integer points in the dilations of the polytope. In a recent preprint [KST], we explore the interplay between algebraic and convex geometry in a non-toric case, namely, for Schubert varieties in a complete flag variety.

With a projective embedding of the flag variety, one can naturally associate a convex polytope, called the *Gelfand–Zetlin polytope*. In [Ko], Kogan assigned a collection of faces of the Gelfand–Zetlin polytope to each Schubert variety. Our main result is a formula for the Demazure character of a Schubert variety in terms of the exponential sums over the integer points in the union of these faces (Theorem 3.1). As a corollary, we get a formula for the Hilbert functions of Schubert varieties via the number of integer points (Corollary 3.2). This in turn implies a formula for the degrees of Schubert varieties via volumes (Corollary 3.3) similar to the Koushnirenko theorem in toric geometry. These results provide a generalization of [PS, Corollary 15.2] from *Kempf varieties* to all Schubert varieties.

Denote by X the variety of complete flags in $\mathbb{C}^n$, and by X^w the Schubert variety corresponding to a permutation $w \in S_n$ as in Section 2 (the codimension of X^w is equal to the length $l(w)$ of w). For every strictly dominant weight λ, denote by V_λ the highest weight irreducible GL_n-module with the highest weight λ. Recall that the Gelfand–Zetlin polytope P_λ is a convex integer polytope in $\mathbb{R}^d$, where $d = n(n-1)/2$, with the property that the integer points inside and at the boundary of P_λ parameterize a natural basis (*Gelfand–Zetlin basis*) in V_λ (see Section 2 for a precise definition of P_λ). In particular, with each integer

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point $z \in P_\lambda$ we can associate its weight $p(z)$ in the character lattice of GL_n.

Denote by $B^- \subset GL_n(\mathbb{C})$ the subgroup of lower-triangular matrices. Consider the projective embedding $X^w \subset X \hookrightarrow \mathbb{P}(V_\lambda)$. Denote by $\chi^w(\lambda)$ the Demazure character of the B^--module $V^-_{\lambda,w} := H^0(X^w, \mathcal{L}_\lambda)^*$ where $\mathcal{L}_\lambda$ is the restriction to $X^w \subset \mathbb{P}(V_\lambda)$ of the tautological line bundle on $\mathbb{P}(V_\lambda)$. For every λ and w, we prove that

$$\chi^w(\lambda) = \sum_{z \in A_{\lambda,w} \cap \mathbb{Z}^d} e^{p(z)}, \tag{1}$$

where $A_{\lambda,w} := \bigcup_{w(F_\lambda)=w} F_\lambda$ is the union of all *rc-faces, or reduced Kogan faces* F_λ (see Section 2) of P_λ with permutation w.

In the case $w = e$, that is, $X^w = X$ and $V^-_{\lambda,w} = V_\lambda$, formula (1) follows directly from the property of the Gelfand–Zetlin polytope mentioned above. For other w, it is usually not true that a subset of the Gelfand–Zetlin basis (in particular, the subset given by the integer points in $A_{\lambda,w}$) gives a basis in the Demazure module $V^-_{\lambda,w}$. Our proof of formula (1) uses the Demazure character formula and elementary arguments involving combinatorics and geometry of the Gelfand–Zetlin polytope. In particular, we use a combinatorial procedure for dealing with divided difference operators (called *mitosis*) introduced in [KnM]. Our proof yields a geometric realization of mitosis [KST, Remark 6.7]. As a byproduct, we construct a realization of a simplex as a cubic complex different from those previously known [KST, Proposition 6.6].

2. Definitions

Denote by G the group $GL_n(\mathbb{C})$. The Weyl group of G is identified with the symmetric group S_n: a permutation $w \in S_n$ corresponds to the element of G acting on the standard basis vectors e_i by the formula $e_i \mapsto e_{w(i)}$. For each $w \in S_n$, we define the *Schubert variety* X^w to be the closure of the B^-–orbit of w in the flag variety $X = G/B$. It is easy to check that the length $l(w)$ of w is equal to the codimension of X^w in X.

Let $V^-_{\lambda,w}$ be the Demazure B^-–module defined as the dual space to the space of global sections $H^0(X^w, \mathcal{L}_\lambda|_{X^w})$, where $\mathcal{L}_\lambda$ is the very ample line bundle on X corresponding to a strictly dominant weight λ. Note that by the Borel–Weil–Bott theorem $V^-_{\lambda,e}$ is isomorphic to the irreducible representation V_λ of G with the highest weight λ. Choose a basis of weight vectors in $V^-_{\lambda,w}$. Recall that the *Demazure character* $\chi^w(\lambda)$ of $V^-_{\lambda,w}$ is the sum over all weight vectors in the basis of the exponentials of the corresponding weights, or, equivalently,

$$\chi^w(\lambda) := \sum_{\mu \in \Lambda} m_{\lambda,w}(\mu) e^\mu,$$

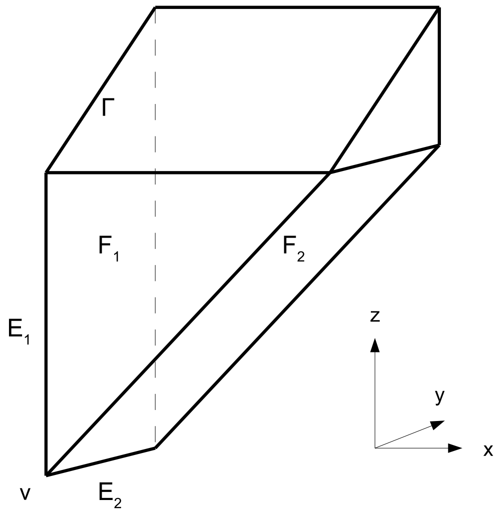

FIGURE 1. A Gelfand–Zetlin polytope for GL_3

where Λ is the weight lattice of GL_n and $m_{\lambda,w}(\mu)$ is the multiplicity of the weight μ in $V^-_{\lambda,w}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ be a strictly dominant weight of the group $GL_n(\mathbb{C})$, i.e. an n-tuple of integers λ_i such that $\lambda_i < \lambda_{i+1}$ for all $i = 1, \ldots, n-1$. The Gelfand–Zetlin polytope P_λ is a convex integer polytope in $\mathbb{R}^d$, where $d = n(n-1)/2$, defined by inequalities

$$\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \ldots & & \lambda_n \\
& \lambda_{1,1} & & \lambda_{1,2} & & \ldots & & \lambda_{1,n-1} & \\
& & \lambda_{2,1} & & \ldots & & \lambda_{2,n-2} & & \\
& & & \ddots & \ldots & & & & \\
& & & \lambda_{n-2,1} & & \lambda_{n-2,2} & & & \\
& & & & \lambda_{n-1,1} & & & &
\end{array} \qquad (GZ)$$

where $(\lambda_{1,1}, \ldots, \lambda_{1,n-1}; \lambda_{2,1}, \ldots, \lambda_{2,n-2}; \ldots; \lambda_{n-2,1}, \lambda_{n-2,2}; \lambda_{n-1,1})$ are coordinates in $\mathbb{R}^d$, and the notation

$$\begin{array}{ccc} a & & b \\ & c & \end{array}$$

means $a \le c \le b$. See Figure 1 for a picture of the Gelfand–Zetlin polytope for $G = GL_3$.

It will be convenient to represent faces of P by *face diagrams*. First, replace all λ_j and $\lambda_{i,j}$ in table (GZ) by dots. Every face of P is given by a system of equations of the form $a = b$, where a and b are coordinates represented by adjacent dots in two consecutive rows. To represent such an equation, we draw a line interval connecting the corresponding dots (these line intervals go from northeast to southwest or from northwest to southeast). Thus a system of equations defining a face of P gets represented by a collection of line intervals called the face diagram. *Rows* of a face diagram are defined as the collections of dots corresponding to the coordinates $\lambda_{i,j}$ with a fixed i, and *columns* are by definition collections of dots with a fixed j (columns look like diagonals in our pictures).

In what follows, we will consider faces of the Gelfand–Zetlin polytope given by the equations of the type $\lambda_{i,j} = \lambda_{i+1,j}$. We will call such faces *Kogan faces*. To each Kogan face F, we assign the permutation $w(F)$ as follows. First, assign to each equation $\lambda_{i,j} = \lambda_{i+1,j}$ the elementary transposition $s_{i+j} = (i+j, i+j+1)$. Now compose all elementary transpositions corresponding to the equations defining F by going from left to right in each row of the diagram for F and by going from the bottom row to the top one. We say that a Kogan face F is *reduced* if the decomposition for $w(F)$ obtained this way is reduced. Reduced Kogan faces of the Gelfand–Zetlin polytopes are in bijective correspondence with reduced *pipe-dreams* (see [Ko, 2.2.1] for more details).

Reduced Kogan faces for $n = 3$ (see Figure 1) are the vertex v, the edges E_1 and E_2, the front faces F_1, F_2, the back face Γ and P_λ itself. The corresponding permutations are $s_2s_1s_2$, s_2s_1, s_1s_2, s_2, s_2, s_1, id, respectively. Note that the faces F_1 and F_2 have the same permutation.

For each $\lambda = (\lambda_1, \dots, \lambda_n)$, consider the affine hyperplane $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ with coordinates $y_1, \dots, y_n$ given by the equation $y_1 + \dots + y_n + u_0 = 0$, where $u_0 = \lambda_1 + \dots + \lambda_n$. Choose coordinates $u_1, \dots, u_{n-1}$ in $\mathbb{R}^{n-1}$ such that $y_i = u_i - u_{i-1}$ for $i = 1, \dots, n-1$. Consider the following linear map $p : \mathbb{R}^d \to \mathbb{R}^{n-1}$ from the space $\mathbb{R}^d$ with coordinates $\lambda_{i,j}$ to the hyperplane $\mathbb{R}^{n-1} \subset \mathbb{R}^n$:

$$u_i = \sum_{j=1}^{n-i} \lambda_{i,j}.$$

In other terms, if we arrange the coordinates $\lambda_{i,j}$ into a triangular table as in (GZ), then u_i is the sum of all elements in the i-th row. In what follows, we identify $\mathbb{R}^n$ with the real span of the weight lattice Λ of G so that the i-th basis vector in $\mathbb{R}^n$ corresponds to the weight given by the i-th entry of the diagonal torus in G. Then the hyperplane $\mathbb{R}^{n-1}$ is the parallel translate of the hyperplane spanned by the roots of G. It is easy to check that the image of the Gelfand–Zetlin polytope $P_\lambda \subset \mathbb{R}^d$ under the map p is the weight polytope of the representation V_λ.

Let S be a subset of the Gelfand–Zetlin polytope P_λ (in what follows S will be a face or a union of faces). Define the *character* χ_S of S as the sum of formal exponentials $e^{p(z)}$ over all integer points $z \in S$, that is,

$$\chi(S) := \sum_{z \in S \cap \mathbb{Z}^d} e^{p(z)}.$$

The formal exponentials e^u, $u \in \mathbb{Z}^n$, generate the group algebra of Λ. Thus the character takes values in this group algebra.

3. Results

The main result of this section establishes a relation between the Demazure character of a Schubert variety and the character of the union of the corresponding faces.

Theorem 3.1. *For each permutation $w \in S_n$, the Demazure character $\chi^w(\lambda)$ is equal to the character of the union of all Kogan faces in the Gelfand–Zetlin polytope P_λ, whose permutation is w:*

$$\chi^w(\lambda) = \chi\left(\bigcup_{w(F_\lambda)=w} F_\lambda\right).$$

If w is a 132*–avoiding*, or *Kempf*, permutation (such permutations are also called *dominant*), then Theorem 3.1 reduces to [PS, Corollary 15.2]. Note that by [Ko, Proposition 2.3.2] a permutation w is Kempf if and only if there is a unique reduced Kogan face F such that $w(F) = w$, and this is exactly the face considered in [PS]. Hence, $\chi^w(\lambda) = \chi(F)$ in this case.

Let us now obtain several corollaries from this theorem. Firstly, we can easily describe the Hilbert function of the Schubert variety X^w embedded into $\mathbb{P}(H^0(X^w, \mathcal{L}_\lambda|_{X^w})^*) \subset \mathbb{P}(V_\lambda)$.

Corollary 3.2. *The dimension of the space $H^0(X^w, \mathcal{L}_\lambda|_{X^w})$ is equal to the number of integer points in the union of all reduced Kogan faces with permutation w:*

$$\dim H^0(X^w, \mathcal{L}_\lambda|_{X^w}) = \left|\bigcup_{w(F)=w} F_\lambda \cap \mathbb{Z}^d\right|.$$

In particular, the Hilbert function $H_{w,\lambda}(k) := \dim H^0(X^w, \mathcal{L}_\lambda^{\otimes k}|_{X^w})$ is equal to the Ehrhart polynomial of $\bigcup_{w(F_\lambda)=w} F_\lambda$, that is,

$$H_{w,\lambda}(k) = \left|\bigcup_{w(F_\lambda)=w} kF_\lambda \cap \mathbb{Z}^d\right| \tag{2}$$

for all positive integers k.

Secondly, we can compute the degree $\deg_\lambda(X^w)$ of the Schubert variety X^w in the embedding $X_w \hookrightarrow \mathbb{P}(V_\lambda)$. Denote by $\mathbb{R}F \subset \mathbb{R}^d$ the affine span of a face F. In the formulas displayed below, the volume form on $\mathbb{R}F$ is normalized so that the covolume of the lattice $\mathbb{Z}^d \cap \mathbb{R}F$ in $\mathbb{R}F$ is equal to 1. Then

Corollary 3.3. *We have*

$$\deg_\lambda(X^w) = (d - l(w))! \sum_{w(F_\lambda)=w} \mathrm{Volume}(F_\lambda) \tag{3}$$

Corollary 3.3 follows immediately from Corollary 3.2 by a standard argument from the theory of Newton polytopes [Kh], that is, by comparing the higher order terms in both sides of (2). Hence, (3) can be viewed as an asymptotic version of more precise identity (2). Note that in the general theory of Newton polytopes and Newton–Okounkov bodies developed recently by Kaveh and Khovanskii [KK] only asymptotic identities hold in most cases. So it is interesting that for Schubert varieties and corresponding unions of faces, we have an exact identity.

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