Vladlen Timorin: Research project

I am doing Geometry in a broad sense: my interests range from dynamical systems to algebraic and differential geometry. I have publications in several mathematical fields, and continue to pursue several different lines of research. However, what follows is my most recent and active research project focused on topological dynamics of rational functions.

Rational functions and regluing. A rational function of one complex variable (i.e. a ratio of two polynomials) is among the simplest and most basic objects in algebra. However, an extremely rich and complicated structure is revealed when one starts to iterate a rational function, i.e. consider it from the point of view of dynamical systems. The Riemann sphere $\mathbb{C}P^1$, on which a rational function acts, gets divided into two fully invariant sets: an open set, called the *Fatou set*, on which the dynamics is stable (e.g. in the sense of Lyapunov) and simple, and a closed set, called the *Julia set*, on which the dynamics is unstable and chaotic. Julia sets tend to have fractal shapes and very intricate geometric properties.

The following commutative diagram appears in a great variety of contexts:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ f\big\downarrow & & \big\downarrow g \\ X & \xrightarrow[\Phi]{} & Y \end{array} \qquad (*)$$

Suppose e.g. that $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a rational function, $g : S^2 \to S^2$ is a continuous map, and Φ a homeomorphism. Then g is *topologically conjugate* to f. Even if f is given by an explicit formula (say, $f(z) = z^2 - 1.5$), it may be very hard to understand its dynamical properties, i.e. what the f-orbits of points are doing. On the other hand, we may have a good model g for f, which is not given by explicit formula but has in a sense explicit dynamics. The meaning of "explicit dynamics" is hard to formalize but, for a good model g, it should be clear what the "model Fatou set" and the "model Julia set" are. Moreover, the topology of the Julia set should be made explicit (e.g. we may know that the Julia set is a Cantor set, or a Sierpinski carpet), and it should also be clear which parts of the Julia set map to which parts (e.g. we may have a Markov partition of the Julia set).

Topological models for quadratic polynomials with locally connected Julia sets and all periodic points repelling were constructed by Douady and Hubbard [DH] and, using a different language, by Thurston. Thurston's construction represents Julia sets as quotients of the unit circle by explicitly defined equivalence relations. On the other hand, there are quadratic polynomials of the form $f(z) = z^2 + c$, for which no explicit topological models are known. There are also general ways to build new topological models out of known topological models. A well-known *mating* construction by Douady and Hubbard defines a model of a rational function by gluing the models of two polynomials together, in a certain explicit way. Another construction is *capture* (it has been defined in a thesis of B. Wittner [W], and extensively studied and generalized by M. Rees [R92]): it allows to define a model for a rational function with a specific periodic or pre-periodic behavior of one critical point. There are few more such constructions, however, much more general principles of making topological models are needed.

The main objective of this project is to *develop systematic ways of building topological models for rational functions.* The procedure I suggest can be described by the same diagram $(*)$; however, it must be interpreted differently. It would be too restrictive to regard Φ as a map. Instead, I propose to regard it as a *topological correspondence*, i.e. a closed subset of $X \times Y$, which is not necessarily a map, but is explicit enough to understand topological dynamics on Y in terms of that on X. It will be instructive to think of topological correspondences as multivalued continuous maps.

I plan to use the following class of topological correspondences. Let Z be a topological sphere, $D \subset Z$ a closed topological disk, and $\pi_X : Z \to X$, $\pi_Y : Z \to Y$ quotient maps that are injective outside of D. The correspondence $E = \pi_X \times \pi_Y(Z) \subset X \times Y$ is called a *regluing correspondence* (we also say that E is a regluing of $\pi_X(D)$ into $\pi_Y(D)$). A simple example of regluing is the following. Set $X = Y = \mathbb{C}P^1$, and consider the multivalued function $\sqrt{z^2-1}$. For every simple curve $\alpha : [-1,1] \to \mathbb{C}$ such that $\alpha(-t) = -\alpha(t)$ and $\alpha(1) = 1$, there is a branch of $\sqrt{z^2-1}$ defined over the complement to $\alpha[-1,1]$. The closure of the graph of this branch defines a regluing of $\alpha[-1,1]$ into a simple curve connecting i with $-i$.

Consider diagram $(*)$, in which Φ is understood as a regluing correspondence. Given a rational function $f : \mathbb{C}P^1 \to \mathbb{C}P^1$, the diagram defines g; however, this will also be a topological correspondence rather than a function: it will have multiple values at some points. To have a precise setting, assume that f is a function of z^2 (e.g. any rational function is Möbius conjugate to a function of this form), and $\alpha : [-1,1] \to \mathbb{C}$ is a simple curve such that $\alpha(-t) = -\alpha(t)$. Let Φ be a regluing of α given by the multivalued function $\sqrt{z^2 - \alpha(1)^2}$. Then $f \circ \Phi^{-1}$ extends to a rational function. However, the correspondence $g = \Phi \circ f \circ \Phi^{-1}$ is not a well-defined map, because it has multiple values over f-pullbacks of $\alpha[-1,1]$. We can consider a regluing of these two curves etc. As a result, we obtain a sequence of regluing correspondences. A *limit* of this sequence can be understood in the following sense.

Consider a sequence (X_n, E_n) consisting of topological spaces X_n and topological correspondences $E_n \subset X_n \times X_{n+1}$. Form new spaces $\hat{X}_n$ consisting of sequences $(x_n, x_{n+1}, \dots)$ such that $(x_m, x_{m+1}) \in E_m$ for all $m \geq n$ (this goes like in the inverse limit construction). We define the topology on $\hat{X}_n$ as that induced from the embedding of $\hat{X}_n$ into $\prod_{m \geq n} X_m$. Then we have well-defined maps $\sigma_n : \hat{X}_n \to \hat{X}_{n+1}$ (forgetting the first term). The direct limit X_∞ of this sequence of maps is called the *limit* of (X_n, E_n).

We now have the following tasks:

(1) Under some general assumptions, prove that the limit of regluing correspondences (or a certain quotient of it) is homeomorphic to the 2-sphere.

(2) Suppose $X_1 = \mathbb{C}P^1$, and $f_1 : X_1 \to X_1$ a rational function. Form a sequence (X_n, E_n) of regluing correspondences as above. There are topological correspondences $f_n : X_n \to X_n$ that commute with E_n. They define a continuous map $f_\infty : X_\infty \to X_\infty$. Give criteria for f_∞ being topologically conjugate to a rational function.

(3) For some interesting classes of rational functions on $\mathbb{C}P^1$, give topological models in terms of the maps $f_\infty : X_\infty \to X_\infty$.

(4) Find some conditions, under which the limit X_∞ carries a canonical conformal structure, and f_∞ is holomorphic with respect to this structure.

Progress already made. I have made initial progress in parts (1), (3) and (4). The results described below deal with the following particular case of the regluing construction. Let E_1 be the regluing correspondence that "cuts along a simple curve $C = \alpha[-1, 1]$ and glues it back in a different way", as explained above. Then E_1 defines a sequence E_n of regluing correspondences (by a version of Thurston's algorithm applied to f_1). Suppose that all iterated pullbacks of C under f_1 are disjoint and do not contain critical points. In this case, we say that (X_∞, f_∞) is obtained from (X_1, f_1) by *regluing of disjoint simple curves* (namely, all pullbacks of C under f_1). Since the curves are disjoint, all regluings can be made simultaneously: there is a (multivalued) map Φ from X_1 to X_∞ that is single-valued and continuous on the complement to pullbacks of C, and that "reglues" all these pullbacks.

Part (1) (the space X_∞ is homeomorphic to the 2-sphere) is proved [2] in this case. It follows from a purely topological fact:

Theorem [2]. *Consider a countable set $\mathcal{Z}$ of disjoint compact connected locally connected non-separating sets in S^2. Suppose that $\mathcal{Z}$ forms a null-sequence. For every $A \in \mathcal{Z}$, fix a continuous map $\Pi_A : S^2 \to S^2$ such that Π_A restricts to an orientation-preserving homeomorphism between the complement to a closed disk Δ and $S^2 - A$, and $\Pi_A(\Delta) = A$. The equalizer $X_{\mathcal{Z}}$ of all maps Π_A is homeomorphic to S^2.*

Intuitively, the space $X_{\mathcal{Z}}$ is obtained from S^2 by blowing up all elements of $A \in \mathcal{Z}$ according to the maps Π_A. This result uses Moore's axiomatic characterization of topological 2-spheres. More general results are possible (an indication and some preparation is given in [4]).

For part (3), I considered slices $Per_k(0)$ in the space of Möbius conjugacy classes of quadratic rational functions (following the guidelines of M. Rees [R92] and J. Milnor [M93]). The slice $Per_k(0)$ is defined by the property that one critical point is periodic of a given period k (the zero in the notation stands for the multiplier of a periodic cycle). The first slice $Per_1(0)$ identifies with the space of quadratic polynomials $z \mapsto z^2 + c$, the most studied parameter family. *Hyperbolicity* is the simplest dynamical behavior: a hyperbolic rational function combines a strong contraction on the Fatou set with a strong expansion on the Julia set. The set of hyperbolic maps is open (in every reasonable parameter space), the components of this set are called *hyperbolic components*. Hyperbolic components in $Per_k(0)$ are classified into four types [R90, M93] A, B, C and D, according to the types of mutual behavior of the two critical points. E.g. a hyperbolic function of type B in $Per_k(0)$ is defined by the property that the non-periodic critical point lies in the immediate basin of the critical periodic cycle. A hyperbolic rational function of type C in $Per_k(0)$ has the property that the non-periodic critical point is attracted by the critical periodic cycle but is not in the immediate basin. I have proved the following

Theorem [2]. *All rational functions on the boundaries of type C hyperbolic components in $Per_k(0)$ but not on the boundaries of type B hyperbolic components are obtained from hyperbolic quadratic rational functions (for which explicit topological models are known) by regluing of disjoint simple curves.*

For most type C components, this gives topological models for all boundary maps — the situation is better than that in the family $z^2 + c$, where all hyperbolic components have many complicated maps on the boundary, whose models are not currently known (there are no type C or B components in $Per_1(0)$). In $Per_2(0)$, there is only one type B component, and all maps on its boundary (except for parabolics) can be obtained from $1/z^2$ by regluing of disjoint simple curves [1]. Moreover,

Theorem [1]. *All maps on the boundary of the only type B component in $Per_2(0)$ are simultaneously matings and anti-matings.*

Part (4) is done [3] in the case, where a simple regluing of a quadratic polynomial leads to a quadratic polynomial with totally disconnected Julia set. Since we know which quadratic polynomial we obtain as f_∞, the issue is to prove that the map Φ is holomorphic in a certain generalized sense. Let Z be a countable union of disjoint simple curves. Assume that Z has zero Lebesgue measure. We say that a map $\Phi : \mathbb{C} - Z \to \mathbb{C}$ is *holomorphic modulo Z* if there is a function $\Psi : Z \to \mathbb{C}$ such that

$$\int_{\mathbb{C}-Z} \Phi\, \overline{\partial}\omega = \int_Z \Psi\, \omega$$

for every smooth (1,0)-form ω on $\mathbb{C}$ with compact support. Intuitively, this definition says that the distributional differential $\overline{\partial}\Phi$ must be a sum of countably many δ-like (0,1)-currents supported in Z.

Theorem [3]. *Consider a quadratic polynomial $f : z \mapsto z^2 + c$ with connected Julia set such that the critical value c is accessible from the basin of infinity. There exists a countable union Z of disjoint simple curves of zero area, and a quadratic polynomial g with totally disconnected Julia set such that $\Phi \circ f = g \circ \Phi$ on $\mathbb{C} - Z$, where $\Phi : \mathbb{C} - Z \to \mathbb{C}$ is a holomorphic map modulo Z.*

Future plans. First, the requirement that all pullbacks of C under f_1 be disjoint should be relaxed. Assume only that 1) the forward orbits of critical points and of the endpoints of C are disjoint from C, and 2) no point of C returns to C infinitely many times (under the dynamics of f_1).

Conjecture (Part (1)). *A suitable topological quotient Y of X_∞ is homeomorphic to the sphere, and f_∞ descends to a branched covering $g : Y \to Y$.*

To prove this conjecture, I plan to use a relative version of Moore's theorem introduced in [4].

Conjecture (Part (2)). *Suppose additionally that the map g is critically finite, i.e. the forward orbits of all critical points are finite. Then g has no Thurston obstructions (therefore, it is Thurston equivalent to a rational function h). The function g is even topologically conjugate to h.*

Checking that a critically finite branched covering has no Thurston obstructions is a technical task, which is sometimes very complicated. However, in this case, I do not expect principal difficulties. This result can be used as follows: consider a rational function f_1 that is not critically finite (and whose combinatorics may be complicated). Do a regluing surgery to obtain a critically finite branched covering g topologically conjugate to a critically finite rational function h. In general, h is much simpler than f_1. Assume that the topological dynamics of h is known. Then we can often describe the topological dynamics of f_1 by "undoing" the surgery.

It is clear that all intermediate spaces X_n have canonical conformal structures.

Task (Part (4)). *Define a "degenerate limit conformal structure" on X_∞ in terms of conformal structures on X_n. Prove that f_∞ is holomorphic with respect to this "degenerate" conformal* *structure. Finally, the degenerate conformal structure on X_∞ induces a non-degenerate conformal structure on Y, and g is holomorphic with respect to this structure.*

A part of the problem is to define the notion of "degenerate conformal structure" on a topological space (that is not necessarily nice).

A possible progress in part (3) includes the description of the boundaries of type B hyperbolic components in $Per_k(0)$ in terms of regluing. A rather straightforward transfer can be made from $Per_k(0)$ to $Per_k(\lambda)$ (parameter slice of quadratic rational functions having a k-cycle with multiplier λ) with $|\lambda| < 1$. The case $|\lambda| = 1$ is much subtler. However, it is very interesting, and I believe the regluing surgery can be done in these parameter slices in the same way as in $Per_k(0)$ (the Siegel case is of course easier than the Cremer case). Regluing methods can also be tested on parameter spaces of rational functions with a preperiodic critical point (say, of given period and preperiod). I believe that all (or, at least, all sufficiently nice) such functions can be obtained from polynomials by regluing (which makes the fixed critical point at infinity into a preperiodic critical point). There is a combinatorial aspect in this. Namely, starting with a critically finite polynomial and a path connecting infinity to a preperiodic point, we can define a Thurston equivalence class of branched coverings (e.g. by applying a path homeomorphism in the sense of M. Rees). Regluing surgery associated with this path (if it leads to a rational function) must belong to the same class. The question is to find whether the given Thurston equivalence class represents a rational function, and, if yes, which rational function (these questions are similar to those addressed in [BN, T, R95, R] etc.).

Related joint projects. Finally, let me mention several joint projects that are related to this project of mine. In a joint project with A. Blokh, we study "combinatorial models" for cubic polynomials. Our goal is to obtain a description of the "combinatorial cubic Mandelbrot set" similar to well-known descriptions of "combinatorial quadratic Mandelbrot set" (which is homeomorphic to the actual Mandelbrot set provided that the latter is locally connected — local connectivity of the Mandelbrot set is a major open problem). Since cubic Mandelbrot set lives in 4-dimensional space and is hard to visualize, we approach the problem by considering certain generic 2-dimensional slices of it. A possible topological picture of these slices includes a repeated regluing surgery in the parameter picture. In a joint project with M. Rees, we try to find persistent Markov partitions for rational functions in $Per_k(0)$ — the first case to consider is the "airplane region" in $Per_3(0)$. In a joint project with M. Lyubich, we are trying to prove a simple criterion of hyperbolicity for 2D Henon maps.

Teaching experience and plans. I have been teaching various mathematical courses at all levels in Russia, Canada, USA and Germany. My current employment at the State University Higher School of Economics comes with teaching and supervising students. I also plan to give a special topics course in holomorphic dynamics at the Independent University of Moscow.

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