Frobenius endomorphisms of linear spaces

RESEARCH STATEMENT

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During the last century many efforts were devoted to classifying Frobenius endomorphisms of matrices, i.e., transformations preserving certain matrix invariants. The first result in this area was obtained in 1896 by Frobenius [20], who has characterized all bijective linear transformations on the space of complex matrices $M_n(\mathbb{C})$ that preserve the determinant function and proved that such transformations are necessary of the form

$$T(X) = PXQ \text{ for all } X \in M_n(\mathbb{C}) \quad \text{or} \quad T(X) = PX^tQ \text{ for all } X \in M_n(\mathbb{C}).$$

In 1925 Schur [48] generalized Frobenius theorem for subdeterminants of a fixed order r in some specific way. The aforesaid Frobenius theorem was generalized in 1949 by Dieudonné [17] to arbitrary fields and for the transformations preserving the set of singular matrices. The detailed and self-contained information on Frobenius endomorphisms can be found in the special volumes of the journal Linear and Multilinear Algebra, volumes 33 and 48, completely devoted to the survey of results in this area (see [40, 42]).

The general setting of this problem can be formulated as follows. Let $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ be a linear (additive, semilinear, non-linear) transformation of matrices of a fixed order n over a certain field $\mathbf{F}$. Let us consider a subset $S \subseteq M_n(\mathbf{F})$, or a matrix functional $\rho : M_n(\mathbf{F}) \to Q$, where Q is a given set (ρ can be a determinant, trace, rank, permanent, etc.), or a matrix property $\mathcal{P}$ (nilpotence, idempotence, singularity, etc.), or a matrix relation $\mathcal{R}$ (similarity, commutativity, order, etc.). We assume that the transformation T preserves one of the pointed properties: in the first case the condition $X \in S$ implies the condition $T(X) \in S$. In the second case $\rho(X) = \rho(T(X))$ for all matrices $X \in M_n(\mathbf{F})$, etc. The question is to characterize transformations preserving one of the S, ρ, $\mathcal{P}$, or $\mathcal{R}$.

The aim of this project is to develop some general methods to classify Frobenius endomorphisms and to solve some important open problems with the help of developed methods.

Past research

1. Transformations preserving zeros of matrix polynomials.

The following problem is due to Kaplansky, see [36, 37], and Watkins, see [49], 1976:

Let $\mathfrak{p}(x_1, \ldots, x_k)$ be an arbitrary element of a free associative algebra (a polynomial in pairwise non-commuting variables $x_1, \ldots, x_k$ of degree $\deg \mathfrak{p} > 1$) over an algebraically closed field $\mathbb{F}$ of zero characteristic. Let $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a bijective linear transformation of the linear space of $n \times n$ matrices with the entries in $\mathbb{F}$. We assume that for any sequence $(A_1, \ldots, A_k)$ of matrices for which $\mathfrak{p}(A_1, \ldots, A_k) = 0$ it holds that $\mathfrak{p}(\Phi(A_1), \ldots, \Phi(A_k)) = 0$. The problem is to characterize such transformations.

Using the methods and results from algebraic geometry Howard in [34] solved this problem for bijective linear transformations of matrices over an algebraically closed fields which preserve the set of zeros of a given polynomial in one variable. It is pointed out in [47] that inspite of the active investigations in this field, see [9, 11, 15, 19, 34, 42, 47, 49], some success was achieved only for several concrete polynomials and general question remained open even for homogeneous multilinear polynomials.

In the works [30, 31] we solve this problem in generic case. i.e., for arbitrary homogeneous multilinear polynomials with non-zero sum of coefficients and homogeneous multilinear polynomials of some special structure with zero sum of coefficients. In order to do this we put forward and developed a new method *of elementary operators* which gives a possibility to answer the question in even more general setting, namely, without assumptions of bijectivity or linearity. In particular, we proved the following theorem:

Theorem 1. *[30] Let $\mathbb{F}$ be an arbitrary algebraically closed field,* $\operatorname{char}\mathbb{F} \neq 2$*. Assume $n \geq 4$ and $k \geq 3$. If a surjection $\Phi : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ strongly preserves the zeros of $\mathfrak{p}(x_1, \ldots, x_k)$ then there exist a field isomorphism $\varphi : \mathbb{F} \to \mathbb{F}$, a functions $\gamma : M_n(\mathbf{F})\backslash\{0\} \to \mathbb{F}\backslash\{0\}$, $\mu : M_n(\mathbf{F}) \to \mathbb{F}$, and an invertible matrix S such that*

(i) $T(A) = \gamma(A)\, SA^{\varphi}S^{-1} + \mu(A)\, I$ *for all* $A \in M_n(\mathbf{F})$*, or*

(ii) $T(A) = \gamma(A)\, S\,(A^{\varphi})^t\, S^{-1} + \mu(A)\, I$ *for all* $A \in M_n(\mathbf{F})$*.*

Similar results are obtained for polynomials with zero sum of coefficients of some special structure.

2. **Monotone transformations**.

Various partial order relations on matrix algebras are widely investigated due to their applications in algebra and statistics, see [3, 16]. In the series of papers [2, 24, 25, 27, 26] we develop a new technique to characterize monotone transformations which allows us to work not only with a certain specified order relation but to deal with additive transformations that are monotone with respect to any abstract order relations which satisfy some natural restrictions, these orders are called *regular*. In particular we show that over the field of real numbers all such monotone transformations are linear and either bijective or equal to zero. As a corollary we obtain the complete characterization of additive transformations over the field of real and complex numbers which preserve one of the following order relations: Drazin order, see [18], left and right $*$-orders, see [4], diamond order, see [3]. Our main result is the following:

Theorem 2. *[27] Let $\prec$ be an regular order. Assume that $T : M_{m,n}(\mathbf{F}) \to M_{m,n}(\mathbf{F})$ is an additive transformation which is monotone with respect to $\prec$. Then T has one of the following forms:*

1) $T([x_{i,j}]) = P[\phi(x_{i,j})]Q$ for all $X = [x_{i,j}] \in M_{m,n}(\mathbf{F})$, where $\phi : \mathbf{F} \to \mathbf{F}$ is a field endomorphism , P, Q are invertible matrices of appropriate sizes,

2) if $m = n$, $T([x_{i,j}]) = P[\phi(x_{i,j})]^tQ$ for all $X = [x_{i,j}] \in M_n(\mathbf{F})$, where $\phi : \mathbf{F} \to \mathbf{F}$ is a field endomorphism, $P, Q \in GL_n(\mathbf{F})$, Y^t denotes the transpose matrix to the matrix Y.

3) $T([x_{i,j}]) = 0$ for all $X = [x_{i,j}] \in M_{m,n}(\mathbf{F})$.

In [27] it is shown that for some partial orderings the matrices P and Q are of some special type, for example, if $\prec$ is a Drazin order, then the matrices P and Q are unitary.

In [12] we also obtained a characterization of linear transformations which are monotone with respect to the non-regular orders generated by the group inverse matrix, see [41, 33].

3. **Frobenius endomorphisms over tropical algebras and semirings**.

Briefly, a semiring differs from a ring by the fact that not every element requires to have the additive inverse. The most common examples of semirings which are not rings are non-negative integers $\mathbb{Z}^+$, non-negative rationals $\mathbb{Q}^+$ and non-negative reals $\mathbb{R}^+$ with the usual addition and multiplication, Boolean algebras, and tropical algebras.

In the papers [1, 8, 28] we developed some general technique to work with Frobenius endomorphisms over semirings and obtained the following results. In particular, the following theorem is an analog of Dieudonné theorem [17]

Theorem 3. *[8] Let $T : \mathcal{M}_{m,n}(\mathcal{S}) \to \mathcal{M}_{m,n}(\mathcal{S})$ be a surjective linear operator. Then the following statements are equivalent*

1. *T preserves the set of $\mathcal{S}$-singular matrices,*
2. *T preserves the set of $\mathcal{S}$-nonsingular matrices,*
3. *there exist permutation matrices $P, Q \in \mathcal{M}_{m,n}(\mathcal{S})$ and a matrix $B = [b_{i,j}] \in \mathcal{M}_{m,n}(\mathcal{S})$ with invertible entries $b_{i,j}$ for all (i, j), such that $T(X) = P(X \circ B)Q$ for all $X \in \mathcal{M}_{m,n}(\mathcal{S})$ or $T(X) = P(X \circ B)^t Q$ for all $X \in \mathcal{M}_{m,n}(\mathcal{S})$. Here $X \circ B$ denotes the Hadamard or Schur product.*

Since there is no subtractivity in antinegative semirings, it is impossible to define the determinant in a usual way. It is common to consider the bi-determinant which is a pair of elements $(\sum\limits_{\sigma\in A_n} a_{1,\sigma(1)}\cdots a_{n,\sigma(n)}, \sum\limits_{\sigma\in S_n\setminus A_n} a_{1,\sigma(1)}\cdots a_{n,\sigma(n)})$, where S_n denotes the symmetric group on the set $\{1, \ldots, n\}$, A_n denotes its subgroup of even permutations, see [22, 23]. The following theorem is a semiring analog of Frobenius theorem [20].

Theorem 4. *[8] Let $T : \mathcal{M}_{m,n}(\mathcal{S}) \to \mathcal{M}_{m,n}(\mathcal{S})$ be a surjective linear transformation. Then* bideterminant $T(X)$ = bideterminant X *for all $X \in \mathcal{M}_{m,n}(\mathcal{S})$ if and only if there exist permutation matrices $P, Q \in \mathcal{M}_{m,n}(\mathcal{S})$ of the same parity and diagonal matrices D, E with* bideterminant $(DE) = (1, 0)$ *such that $T(X) = PDXEQ$ for all $X \in \mathcal{M}_{m,n}(\mathcal{S})$ or $T(X) = PDX^tEQ$ for all $X \in \mathcal{M}_{m,n}(\mathcal{S})$.*

Some generalizations of these results are obtained for antinegative semirings which can be embedded to rings, see [8]. In [5, 6] we applied the developed technique to Frobenius endomorphisms for some combinatorial matrix properties. In [9, 10] Frobenius endomorphisms of zeros of matrix polynomials over semirings are characterized.

4. **Frobenius endomorphisms of polynomial spaces**.

Let $\mathbb{R}[x]$ denote the ring of univariate polynomials with real coefficients and denote by $\mathbb{R}_n[x]$ its linear subspace consisting of all polynomials of degree less than or equal to n. Following the classical approach of Pólya-Schur theory [44] we studied linear operators acting on $\mathbb{R}[x]$ and preserving either the set of positive univariate polynomials or similar sets of non-negative and elliptic polynomials. The following results have been obtained, see [32]:

Theorem 5. *Let $U_Q : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear ordinary differential operator of order $k \geq 1$ with polynomial coefficients $Q = (q_0(x), q_1(x), \ldots, q_k(x))$, $q_i(x) \in \mathbb{R}[x]$, $i = 0, \ldots, k$, $q_k(x) \not\equiv 0$, i.e.,*

$$U_Q = q_0(x) + q_1(x)\frac{d}{dx} + q_2(x)\frac{d^2}{dx^2} + \ldots + q_k(x)\frac{d^k}{dx^k}. \qquad (1)$$

Then for any coefficient sequence Q the operator U_Q does not preserve the set of non-negative (resp., positive or elliptic) polynomials of degree $2k$.

Corollary 6. *There are no linear ordinary differential operators of positive finite order which preserve the set of non-negative (resp., positive or elliptic) polynomials in $\mathbb{R}[x]$.*

Slightly generalizing a one hundred years old result of Remak [46] and Hurwitz [35] (see also Problem 38 in [45, Ch. 7]) we obtained the characterization of infinite order linear ordinary differential operators

$$U_\alpha = \alpha_0 + \alpha_1 \frac{d}{dx} + \alpha_2 \frac{d^2}{dx^2} + \ldots + \alpha_k \frac{d^k}{dx^k} + \ldots$$

with constant coefficients which preserves positivity.

Working plan

My plan is to continue the work in this area.

In particular, to extend the method of elementary operators in order to classify Frobenius endomorphisms for zeros of non-necessary homogeneous multilinear polynomials, and thus to obtain the complete solution of Kaplansky-Watkins problem in singular case and in the case of non-homogeneous polynomials. The conjecture is that they will be of the same form as in Theorem 1 in the case of non-zero sum of the coefficients. And in the case of zero sum, the result will be the composition of the aforesaid transformations with taking a polynomial in the preimage matrix.

I plan to consider non-additive transformations which are monotone with respect to diamond order and apply the method of elementary operators to their classification and prove the following theorem on partial orders generated by the generalized group inverse matrix:

Theorem 7. *Let $<$ be either a sharp order or a cn-order. Assume that $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ is a non-zero additive transformation which is monotone with respect to $<$. Then T has one of the following forms:*

1) $T([x_{i,j}]) = P[\phi(x_{i,j})]P^{-1}$ for all $X = [x_{i,j}] \in M_n(\mathbf{F})$, where $\phi : \mathbf{F} \to \mathbf{F}$ is a field endomorphism, P is an invertible matrix,

2) $T([x_{i,j}]) = P[\phi(x_{i,j})]^t P^{-1}$ for all $X = [x_{i,j}] \in M_n(\mathbf{F})$.

I am going to apply the results from [1] and [29] to the investigations of mean-payoff games, in particular, to prove the following:

Theorem 8. *Under natural non-singularity assumptions the system of linear tropical inequalities $Ax \leq Bx$ has a solution $x \in \mathbb{R}_+^n$ non-identically $-\infty$ if and only Player Max has a winning state in the mean payoff game with dynamic programming operator $f(x) = A^\sharp Bx$.*

Theorem 9. *Suppose that the system $Ax \leq Bx$ is non-singular and consists of finitely many inequalities. Consider the polyhedral cone $P := \{x \in \mathbb{R}_+^n;\ Ax \leq Bx\}$, and define the support S of P to be the union of the supports of the elements of P:*

$$S := \{i \in [n];\ \exists u \in P, u_i \neq -\infty\} \ .$$

Then S coincides with the set of initial states with a nonnegative value for the associated mean payoff game, that is:

$$S = \{i \in [n];\ \chi_i(f) \geq 0\} \ , \tag{2}$$

where $f : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is such that $f(x) = A^\sharp Bx$.

Also the results from [5, 6] will be applied for the investigations of strongly connected directed graphs.

On the base of the developed technique I plan to investigate the Polya problem of computing the permanent of a matrix by means of its determinant. Namely, it is planned to prove the following

Theorem 10. *Suppose $n \geq 3$, and let $\mathbb{F}$ be a finite field with $\operatorname{char}\mathbb{F} \neq 2$ and of sufficiently large cardinality (which depends only on n). Then, no bijective map $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ satisfies*

$$\operatorname{per} \mathrm{A} = \det \Phi(\mathrm{A}). \tag{3}$$

When $n = 3$ the conclusion holds for any finite field with $\operatorname{char}\mathbb{F} \neq 2$.

Then we plan to remove the asymptotic nature of Theorem 10 by some combinatorial arguments. It is planned to obtain analogous result also for the symmetric and Hermitian (with respect to the involution $x \to x^{p^{\frac{k}{2}}}$ where $|\mathbb{F}| = p^k$ and k is even) matrices.

It is planned also to solve the following converse problem, posed by Gibson in [21]: what are the necessary and sufficient conditions on a given (0,1)-matrix for the possibility of conversion of permanent into the determinant by adding $\pm$ to the elements. We plan to consider corresponding problems for symmetric and Hermitian matrices as well.

Teaching experience and plans

Since 1999 I work at the Department of Mathematics and Mechanics of MSU, since 2001 I work there on the full time position. Since 2007 my affiliation is Associate professor. I give seminars in all obligatory disciplines delivering by the Faculty of Algebra, namely Introduction to Algebra, Linear Algebra, General Algebra, Linear Algebra for economists. Also since 2000 I constantly deliver different one year lecture courses for M.Sc. students and Ph.D. students in my research area, namely, in the matrix theory and the ring theory. I am a co-organizer of several scientific seminars at MSU. In January, 2003, I was a visiting professor in the Sung Kyun Kwan University and Pohang Institute of Science and Technology, Korea and delivered lectures on Frobenius endomorphisms; in fall, 2004, and during the summer, 2005, I was a visiting professor in the University of Dortmund and delivered lectures in Linear Algebra; in October, 2005, I was a visiting professor in Birmingham University, where I delivered lectures on my research area were delivered under the grant of London Mathematical Society; in 2006 and 2008 I was a visiting professor in Paris at INRIA and Ecole Polytechnique, and delivered lecture courses on Frobenius endomorphisms. Currently 8 students, including a Ph.D. student, are working under my supervision in MSU. One of my Ph.D. students has prepared the dissertation which was already predefended and submitted in October for the defense. Also I am currently working on the preparation of a textbook "Frobenius Endomorphisms of Matrix Spaces" based on the lecture courses delivered by myself in Moscow, Dortmund, and Paris, and on the preparation of the computer course in Linear Algebra and its Applications, which will be delivered by our faculty.

References

[1] Akian M., Gaubert S., Guterman A. Rank functions for matrices over max-algebras // American Mathematical Society, Contemporary Mathematics, **495** (2009) 1-33.

[2] Alieva A., Guterman A. Monotone Linear Transformations on Matrices are Invertible // Comm. in Algebra. 2005. V. 33. P. 3335-3352.

[3] Baksalary J. K., Hauke J. A further algebraic version of Cochran's theorem and matrix partial orderings // Linear Algebra Appl. **127** (1990) 157-169.

[4] Baksalary J. K., Mitra S. K. Left-star and right-star partial orderings // Linear Algebra Appl. **149** (1991) 73-89.

[5] Beasley L. B., Guterman A. E. Operators preserving primitivity for matrix pairs // Matrix Methods: Theory, Algorithms, Applications, Word Scientific Publishing (2008) 2-20.

[6] Beasley L. B., Guterman A. E. The characterization of operators preserving primitivity for matrix k-tuples // Linear Algebra and its Applications, **430** (2009) 1762-1777.

[7] Beasley L. B., Guterman A. E. Rank inequalities over semirings // Journal of Korean Math. Soc., **42**, 2, (2005) 223-241.

[8] Beasley L.B., Guterman A.E., Lee S.-G., Song S.-Z. Frobenius and Dieudonné theorems over semirings // Linear and Multilinear Algebra, **55**, no. 1, (2007) 19-34.

[9] Beasley L.B., Guterman A.E., Lee S.-G., Song S.-Z. Linear preservers of zeros of matrix polynomials // Linear Algebra and its Appl., **401** (2005) 325-340.

[10] Beasley L.B., Guterman A.E., Lee S.-G., Song S.-Z. Linear transformations preserving the Grassmannian over $M_n(\mathbb{Z}+)$ // Linear Algebra and its Appl., **393** (2004) 39-46.

[11] Beidar K.I., Fong Y. On additive isomorphisms on prime rings preserving polynomials // J. Algebra, **217** (1999) 650-667.

[12] Bogdanov I. I., Guterman A. E. Monotone matrix mappings defined by the group inverse and simultaneous diagonalizability // Mat. Sbornik **198**, no. 1, (2007) 3-20.

[13] Botta P., Pierce S., Watkins W. Linear transformations that preserve the nilpotent matrices // Pacific J. Math. **104** (1983) 39-46.

[14] Bresar M., Šemrl P. Linear transformations preserving potent matrices // Proc. Amer. Math. Soc. **119**, no. 1, (1993) 81-86.

[15] Chebotar M.A., Fong Y., Lee P.-H. On maps preserving zeros of the polynomial $xy-yx^*$ // Linear Algebra Appl. **48** (2005) 230-243.

[16] Cochran W. G. The distribution of quadratic forms in a normal system, with applications to the analysis of covariance // Proc. Cambridge Philos. Soc. **30** (1934) 178-191.

[17] Dieudonné J. Sur une généralisation du groupe orthogonal à quatre variables // Arch. Math. **1** (1949), 282-287.

[18] Drazin M. P. Natural structures on semigroups with involution // Bull. Amer. Math. Soc. **84**, no. 1, (1978) 139-141.

[19] Fošner A. Nonlinear commutativity preserving maps on $\mathcal{M}_n(\mathbb{R})$ // Linear and Multilinear Algebra. **53**, no. 5, (2005) 323-344.

[20] Frobenius G. Über die Darstellung der endlichen Gruppen durch lineare Substitutionen // Sitzungsber. Berlin: Preuss. Akad. Wiss. Berlin, 1897. 994-1015.

[21] Gibson P. M. Conversion of the permanent into the determinant // Proc. Amer. Math. Soc. **27** (1971) 471-476.

[22] J.S. Golan, Semirings and their applications, updated and expanded version of the theory of semirings, with applications to mathematics and theoretical computer science, Kluwer Academic Publishers, Dordrecht, 1999.

[23] J.S. Golan, Semirings and affine equations over them: Theory and applications, Mathematics and its Applications, Vol. 556, Kluwer Academic Publishers, Dordrecht, 2003.

[24] Guterman A. Linear Preservers for Drazin Star Partial Order // Comm. in Algebra, **29**, no. 9, (2001) 3905-3917.

[25] Guterman A. Linear Preservers for Matrix Inequalities and Partial Orderings // Linear Algebra Appl.**331** (2001) 75-87.

[26] Guterman A. Monotone matrix maps preserve non-maximal rank // Lecture Notes in Pure and Applied Mathematics. Marcel Dekker. 2003, 311-328.

[27] Guterman A. Monotone additive transformations on matrices // Mat. Zametki, **81**, no. 5, (2007) 681-692.

[28] Guterman A. Transformations of non-negative integer matrices preserving the determinant // Uspehi Mat. Nauk, **58**, no. 6, (2003) 147-148.

[29] Guterman A. Rank and determinant functions for matrices over semirings, London Mathematical Society Lecture Notes, **347** (2007) 1-33.

[30] Guterman A., Kuzma B. Preserving zeros of a polynomial // Communications in Algebra, **37**, no. 11, (2009) 3801-3827.

[31] Guterman A., Kuzma B. Maps preserving zeros of matrix polynomials // Doklady of Russian Academy of Sciences, Mathematics, **427**, no. 3, (2009) 300-302.

[32] Guterman A., Shapiro B. On linear operators preserving the set of positive polynomials // Journal of Fixed Point Theory and Applications, **3**, no. 2, (2008), 411-429.

[33] Hartwig R. E., Mitra S. K. Partial orders based on outer inverses // Linear Algebra Appl. **176** (1982) 3-20.

[34] Howard R. Linear maps that preserve matrices annihilated by a polynomial // Linear Algebra Appl. 1980. V. 30. P. 167-176.

[35] Hurwitz A. Über definite Polynome // Math. Ann. **73** (1913) 173-176.

[36] Kaplansky I. Algebraic and analytic aspects of operator algebras // American Mathematical Society, Providence 1970.

[37] Kaplansky I. // Mappings preserving various properties. 1976.

[38] Linear and Multilinear Algebra, **48** (2001) 1-178.

[39] Marcus M., Filippenko I. Invariance of the nonvanishing specializations of polynomials // Linear and Multilinear Algebra. 1977. V. 5. P. 99-105.

[40] Li C.-K., Pierce S. Linear preserver problems // Amer. Math. Monthly **108**, no. 7, (2001) 591-605.

[41] Mitra S. K. On group inverses and the sharp order // Linear Algebra Appl. **92** (1987) 17-37.

[42] Pierce S. and others. A Survey of Linear Preserver Problems // Linear and Multilinear Algebra. **33** (1992) 1-119.

[43] Petek T. A note on additive commutativity-preserving mappings // Publ. Math. Debrecen. **56** (2000) 53-61.

[44] Pólya G., Schur I. Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen, J. Reine Angew. Math. **144** (1914), 89-113.

[45] Pólya P., Szegö G. Problems and theorems in analysis. II. Theory of functions, zeros, polynomials, determinants, number theory, geometry. Translated from the German by C. E. Billigheimer. Reprint of the 1976 English translation. Classics in Mathematics. Springer-Verlag, Berlin, 1998. xii+392 pp.

[46] Remak R. Bemerkung zu Herrn Stridsbergs Beweis des Waringschen Theorems, Math. Ann. **72** (1912) 153-156.

[47] Šemrl P. Non-linear commutativity preserving maps // Acta Sci. Math. (Szeged). **71** (2005) 781-819.

[48] Schur I. Einige Bemerkungen zur Determinantentheorie // Akad. Wiss. Berlin, S.-Ber. Preuß. (1925), 454-463.

[49] Watkins W. Linear maps that preserve commuting pairs of matrices // Linear Alg. Appl. **14** (1976) 29-35.