The report of Mikhail Bondarko for 2007–2009

1 My plans (from the research proposal)

In my research proposal I listed the following goals of the project: "The main goal of the project is the study of different triangulated categories of motives, their t-structures and realizations. In particular, a new method of attaching weights to cohomology functors will be studied; this includes the study of so-called 'weight structures', related t-structures, and Gersten-type resoultions. These results would be applied to the study of motives, cohomological functors on schemes and to K-theory. In particular, I plan to study (existing and new) theories of relative motives. I also plan to define certain non-reduced theories of motives. The infinitesmall part of such a theory could be related with my previous results on formal groups and group schemes."

2 My successes and failures

I was not able to define any 'reasonable' local motives (those that would be compatible with 'reduced' ones).

I have developed the theory of weight structures quite successfully. Now this is a theory that has several application to motives (including weights for arbitrary cohomology of motives), and also to (topological) spectra and other triangulated categories. The (very interesting!) connection of weight structures with t-structures was also studied in detail.

In particular, weight structures were successfully used to study Gersten resolutions and coniveau spectral sequences (though to this end I had to introduce a new motivic category not mentioned in the original project).

I have also invented a program to describe weights for relative motives. It is described in §8.2 of [2]; the details have to be written down still.

3 Publications

In 2007–2009 three of my papers were published: [6], [5], and [1]. The first two are not related to motives (though [5] has some relation with the 'local' part of the project that was not really successfull).

[1] is my first 'motivic' paper. In it the following results were established (actually, for the proof of part VI of Theorem 3.1 in its current form weight structures are essential; so in loc.cit. a weaker statement was proved, and the present version was proved in [2]).

Theorem 3.1. *I A full description of Voevodsky's DM_{gm}^{eff} in terms of 'twisted' Suslin cubical complexes (in the sense of Kapranov and Bondal) was given. In particular, for any motivic complex M (for instance, the Suslin complex of an arbitrary variety) there exists a quasi-isomorphic complex M' 'constructed from'* *the Suslin complexes of smooth projective varieties; M' is unique up to homotopy equivalence.*

II Voeovodsky's $DM_{gm} \otimes \mathbb{Q}$ is anti-equivalent to the Hanamura's motivic category.

III There exist a conservative exact weight complex functor $t : DM_{gm}^{eff} \subset DM_{gm} \to K^b(Chow^{eff}) \subset K^b(Chow)$.

IV t induces isomorphisms $K_0(DM_{gm}^{eff}) \to K_0(Chow)$ and $K_0(DM_{gm}) \to K_0(Chow)$; they are isomorphisms of rings.

V For any cohomological functor $H : DM_{gm} \to \underline{A}$ (here $\underline{A}$ is an abelian category) and $X \in Obj DM_{gm}$ there exists a weight spectral sequence $T : H^i(P^{-j}) \implies H^{i+j}(X)$ where (P^i) is a representative of $t(X)$. T is canonical and motivically functorial starting from E_2. This yields Deligne's weight spectral sequences and weight filtrations for mixed Hodge and étale cohomology of varieties (and also a certain 'weight spectral sequence' for motivic cohomology).

VI A motif (an object of Voevodsky's DM_{gm}) is a mixed Tate one whenever its weight complex is.

In 2009 [2] was accepted by the Journal of K-theory; [3] was submitted to Documenta Math.

4 Basic properties of weight structures

Weight structures (defined in [2] and studied further in [3]; see [4] for a survey) were central in my research project.

I showed that parts III–VI of Theorem 3.1 follow from a very general relevant formalism for triangulated categories; this setting was not previously described in literature. One considers a set of axioms that are (in a certain sense) 'dual' to the axioms of t-structures. Several properties of weight structures are similar to those of t-structures; yet other ones are quite distinct.

Below $\underline{C}$ and $\underline{D}$ will be triangulated categories; $\underline{A}$ will be abelian.

A category $\underline{C}$ with a weight structure w has an additive heart $\underline{Hw}$ with the property that there are no morphisms of positive degrees between objects of the heart in $\underline{C}$. Any *bounded* weight structure yields a conservative *weight complex* functor to the *weak homotopy category of complexes* over the heart. Moreover, w gives a Postnikov tower of any object of $\underline{C}$ whose 'factors' belong to $\underline{Hw}$; such a *weight Postnikov tower* is canonical and functorial 'up to cohomology zero maps'. Applying any (co)homological functor $\underline{C} \to \underline{A}$ ($\underline{A}$ is an abelian category) to this tower one obtains a 'weight spectral sequence' whose E_1-terms are (co)homology of the corresponding objects of the heart. This spectral sequence is canonical and functorial starting from E_2.

Now I describe this theory in more detail.

Definition 4.1 (Weight structures and their hearts). I A pair of subclasses $\underline{C}^{w\geq 0}, \underline{C}^{w\leq 0} \subset Obj\underline{C}$ for a triangulated category C will be said to define a weight structure w if $\underline{C}^{w\geq 0}, \underline{C}^{w\leq 0}$ satisfy the following conditions:

(i) $\underline{C}^{w\geq 0}, \underline{C}^{w\leq 0}$ contain all direct summands of their objects.

(ii) $\underline{C}^{w\le 0}[1]\subset \underline{C}^{w\le 0}$, $\underline{C}^{w\ge 0}\subset \underline{C}^{w\ge 0}[1]$.
(iii) For any $X\in \underline{C}^{w\ge 0}$, $Y\in \underline{C}^{w\le 0}[1]$ we have $\underline{C}(X,Y)=0$.
(iv) For any $X\in Obj\underline{C}$ there exists a distinguished triangle

$$B[-1]\to X\to A\stackrel{f}{\to} B \tag{1}$$

such that $A\in \underline{C}^{\le 0}, B\in \underline{C}^{w\ge 0}$.

II A category $\underline{Hw}$ whose objects are $\underline{C}^{w=0}=\underline{C}^{w\ge 0}\cap \underline{C}^{w\le 0}$, $\underline{Hw}(X,Y)=\underline{C}(X,Y)$ for $X,Y\in \underline{C}^{w=0}$, will be called the *heart* of the weight structure w.

$\underline{Hw}$ is additive. In contrast to the situation with t-structures, there cannot be any non-trivial '$\underline{C}$-extensions' of objects of $\underline{Hw}$.

The basic example of a weight structure is given by the stupid filtration on the homotopy category of complexes over an arbitrary additive category B. Its heart is the (easily described) idempotent completion of B in $K(B)$.

Any weight structure yields a weight complex functor $t:\underline{C}\to K_w(\underline{Hw})$; here $K_w(\underline{Hw})$ is a certain factor of $K(\underline{Hw})$ (we 'kill' morphisms of the form $df+gd$ for f,g being collections of arrows that shift degrees by -1; this does not change isomorphism classes of objects in $K(\underline{Hw})$). This functor has several nice properties; in particular, it is 'usually' conservative (at least, this is the case when w is *bounded* i.e. $\cap\underline{C}^{w\le 0}[i]=\cap\underline{C}^{w\ge 0}[i]=\{0\}$). Moreover, one can 'often' replace $K_w(\underline{Hw})$ by $K(\underline{Hw})$ (then t will be exact); this is the case for motives and spectra.

If $H:\underline{C}\to\underline{A}$ is a cohomological functor, then for any $X\in Obj\underline{C}$ one has a spectral sequence $T(H,X)$ with $E_1^{pq}=H(X^{-p}[-q])$; here X^i are the terms of (any choice of) the weight complex of X. It (weakly) converges to $H(X[-p-q])$; it is $\underline{C}$-functorial in X starting from E_2. In particular, one obtains a functorial ('weight') filtration on $H(X)$.

Now we describe the relation of weight structures with t-structures.

Let $\Phi:\underline{C}^{op}\times\underline{D}\to\underline{A}$ be a *nice duality* of triangulated categories (see Definition 2.5.1 of [2]). Suppose also that $\underline{C}$ is endowed with a weight structure w, $\underline{D}$ is endowed with a t-structure t, and w is *orthogonal* to t (with respect to Φ).

The easiest (but not the only one existing) example of a duality is: $\underline{D}=\underline{C}$, $\underline{A}=Ab$, $\Phi(X,Y)=\underline{C}(X,Y)$ for $X,Y\in Obj\underline{C}$. In this case t is orthogonal to w whenever $\underline{C}^{w\le 0}=\underline{C}^{t\le 0}$; we will say that t is adjacent to w.

For some $Y\in Obj\underline{D}$ we consider the functor $H=\Phi(-,Y)$. Then one has a functorial description of $T(H,-)$ (starting from E_2) in terms of t-truncations of Y; see Theorem 2.6.1 of [3]. This is a powerful tool for comparing spectral sequences (in this situation); it does not require constructing any complexes (and filtrations for them) in contrast to the method of Paranjape (probably, originating from Deligne).

Also, $\underline{Hw}$ is 'dual' to the heart of t in a very interesting sense. A functor right adjoint to a t-exact functor $F:\underline{C}\to\underline{D}$ (with respect to some t for $\underline{C}$ and t' for $\underline{D}$) is weight-exact (in the natural sense) with respect to the weight structures adjacent to t' and t (if those exist); the converse is also true.

If w is bounded and $\underline{Hw}$ is idempotent complete, then $\underline{C}$ is idempotent complete also and $K_0(\underline{C}) \cong K_0(\underline{Hw})$.

Weight structures also descend to localizations, and can be glued (under certain conditions) in ways that are similar to the corresponding ones for t-structures.

5 Applications to motives

We have two main 'motivic' weight structures (that actually belong to a single series of those). They correspond to (Chow)-weight and coniveau spectral sequences, respectively. Note that both of these spectral sequences were 'classically' defined only for cohomology of varieties; still our approach allows to define them for arbitrary Voevodsky's motives, and also yields their motivic functoriality (which is very far from being obvious for both of them if one uses their 'classical' definitions!).

We use some notation from [7].

The first ('motivic') weight structure is w_{Chow}; it is defined on $DM_{gm}^{eff} \subset DM_{gm}$, its heart is $Chow^{eff} \subset Chow$. So, Voevodsky's motives could be 'sliced into pieces' that are Chow motives 'canonically up to homotopy equivalence'. Note here: the corresponding weight complex functor $t: DM_{gm}^{eff} \to K^b(Chow^{eff})$ is conservative, whereas DM_{gm}^{eff} is very far from being isomorphic to $K^b(Chow^{eff})$.

The weight spectral sequence with respect to w_{Chow} is isomorphic to the Deligne's ones for H being étale or singular cohomology of varieties. Yet note that $T_{w_{Chow}}(H, -)$ is defined for any H (including motivic cohomology and singular cohomology with integral coefficients!) and is DM_{gm}^{eff}-functorial starting from E_2!

There exists a *Chow t-structure* t_{Chow} for DM_{-}^{eff} whose heart is $\mathrm{AddFun}(Chow^{eff}, Ab)$. t_{Chow} is adjacent to the Chow weight structure for DM_{-}^{eff}; it is related with unramified cohomology.

w_{Chow} is also closely related with the 'usual expectations from weights for Voevodsky's motives'; see §8.6 of [2].

The second 'motivic' weight structure is the Gersten weight structure w defined on the category $\mathfrak{D}^s \supset DM_{gm}^{eff}$ (for a countable k). Here $\mathfrak{D}^s$ is a full triangulated subcategory of a certain category $\mathfrak{D}$ of *comotives*.

The idea is that w should be orthogonal to the homotopy t-structure on DM_{-}^{eff} (recall that the latter is the restriction of the canonical t-structure of the derived category of Nisnevich sheaves with transfers). So, $\underline{Hw}$ is 'generated' by comotives of function fields over k (note that these are Nisnevich points); in particular, it cannot be defined on DM_{gm}^{eff} (or DM_{-}^{eff}).

The problem with $DM_{-}^{eff} \supset DM_{gm}^{eff}$ is that there are no 'nice' homotopy limits in them. In order to have these limits one needs 'nice' (small) products; one also needs the objects of DM_{gm}^{eff} to be cocompact (in this 'category of homotopy limits'). DM_{-}^{eff} definitely does not satisfy these conditions. Instead in §5 of [3] the category $\mathfrak{D}$ of comotives was constructed; there exists a nice duality $\mathfrak{D}^{op} \times DM_-^{eff} \to Ab$. In $\mathfrak{D}^s \subset \mathfrak{D}$ the (co)motif of any smooth variety can be 'decomposed' (in the sense of Postnikov towers) into Tate twists of comotives of its points (using a triangulated analogue of the usual Cousin complex construction).

The general theory of weight spectral sequences yields $T_w(G, X)$ for any cohomological functors $G : \mathfrak{D}^s \to \underline{A}$; the problem here is that $\mathfrak{D}^s$ is a 'large' and rather 'mysterious' category. Yet, any $H : DM_{gm}^{eff} \to \underline{A}$ has a 'nice' extension to $\mathfrak{D}^s$ (and also to $\mathfrak{D} \supset \mathfrak{D}^s$) if $\underline{A}$ satisfies AB5 (see Proposition 4.3.1 of [3]). So, we can consider weight spectral sequences $T = T_w(H, X)$ for any such H and any $X \in Obj DM_{gm}^{eff}$ or $X \in Obj\mathfrak{D}^s$. It turns out that for X being the motif of a smooth variety, T is isomorphic to the coniveau spectral sequence (corresponding to H) starting from E_2. So, we call T a coniveau spectral sequence for any X. Thus (very similarly to the case of Chow-weight spectral sequences) I vastly generalized coniveau spectral sequences, and proved that they are motivically functorial.

As well as for 'classical' coniveau spectral sequences, if H is represented by an object of DM_-^{eff}, $T_w(H, X)$ could be described in terms of cohomology of X with coefficients in the homotopy t-truncations of H; this fact extends the related results of Bloch-Ogus and Paranjape. A related result is: torsion motivic cohomology of motives can be expressed in terms of étale cohomology (in a certain way; here the recently proved Beilinson-Lichtenbaum conjecture is used).

Also, I proved a collection of direct summand results. In particular, the comotif of a smooth semi-local scheme (or any *primitive* smooth scheme) is a direct summand of the comotif of its generic fibre; comotives of fields contain as direct summands twisted comotives of their residue fields (for any geometric valuations). Hence similar results hold for any cohomology of (semi-local) schemes mentioned.

Besides, w could be restricted to the category $DAT \subset DM_{gm}^{eff}$ of so-called Artin-Tate motives (this is the category generated by Tate twists of Artin motives). This yields 'economic' descriptions of coniveau spectral sequences for such motives (starting from E_2).

I also constructed (non-explicitly) a whole series of weight structures for the category of comotives. This series is indexed by a single integral parameter; all the structures induce the same weight structure on the category of birational comotives (i.e. the localization of $\mathfrak{D}$ by $\mathfrak{D}(1)$), and for the i-th weight structure tensoring by $\mathbb{Z}(1)[i]$ is *weight-exact* (i.e. $\underline{C}^{w_i \le 0}(1)[i] \subset \underline{C}^{w_i \le 0}$ and $\underline{C}^{w_i \ge 0}(1)[i] \subset \underline{C}^{w_i \ge 0}$). In particular, for $i = 2$ one obtains the Chow weight structure (for $\mathfrak{D}$); for $i = 1$ one obtains the Gersten weight structure. It is quite amazing that spectral sequences that are so distinct from the geometric point of view differ just by [1] (in this description)! Possibly, other members of this series could be also interesting (especially the one corresponding to $i = 0$).

6 Pedagogical activity

In 2007–2009 I led student's practice in higher algebra and number theory (in St. Petersburg State University). Besides I actively participated in the composition of two books of problems: a one in Number theory and a one in Field theory. The first one is is published now.

7 Conferences

During 2007–2009 I made talks (on motives and weight structures) at the following international conferences:

1. Arithmetic Geometry, Saint-Petersburg, 13–19.06.2007.

2. International Algebraic Conference dedicated to the 100th anniversary of D. K. Faddeev, Saint-Petersburg, 24–29.09.2007.

3. Young Mathematics in Russia, Moscow, 12–13.01.2009.

4. Workshop "Finiteness for Motives and Motivic Cohomology Regensburg, 9–13.02.2009.

5. Workshop on Motivic Homotopy Theory, Münster, 27–31.07.2009.

6. Algebraic Conference dedicated to the 60th Anniversary of A. I. Generalov, St. Petersburg, 2–3.09.2009.

I also participated in seminars in: St. Petersburg State University, University Paris 13, Max Planck Institut für Mathematik, University of Salamanca, and Institut de Mathématiques de Jussieu.

Список литературы

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