Lecture 3

TOPOLOGICAL CONSTRUCTIONS

In this lecture, we study the basic constructions used in topology. These constructions transform one or several given topological spaces into a new topological space. Starting with the simplest topological spaces and using these constructions, we can create more and more complicated spaces, including those which are the main objects of study in topology.

3.1. Disjoint Union

The *disjoint union* of two topological spaces X and Y, in the case when the two sets X and Y do not intersect, is the union of the sets X and Y with the following topology: a set W in $X \cup Y$ is open if the sets $W \cap X$ and $W \cap Y$ are open in X and Y, respectively; if the two sets X and Y intersect, the definition is a little trickier: first we artificially make them nonintersecting by considering, instead of the set Y, the same set of elements but marked, say, with a star, i.e., $Y^* := \{(y, *) \mid y \in Y\}$, and then proceed as before, declaring that a set W in $X \cup Y^*$ is open if the sets $W \cap X$ and $W \cap Y^*$ are open in X and Y^*, respectively. In both cases we obtain a topological space denoted by $X \sqcup Y$.

This choice of topology ensures that both natural inclusions $X \hookrightarrow X \cup Y$ $(x \mapsto x)$ and $Y \hookrightarrow X \cup Y$ $(y \mapsto y)$ are continuous maps.

It is easy to see that the subsets X and Y (we do not explicitly write the stars (if any) in Y^*, but consider them implicitly present) are both open and closed in $X \sqcup Y$, so that the set $X \sqcup Y$ is not connected (provided both X and Y are nonempty).

3.2. Cartesian Product

Roughly speaking, the Cartesian product of two spaces is obtained by putting a copy of one of the spaces at each point of the other space.

More precisely, let X and Y be topological spaces; consider the set of pairs $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ and make $X \times Y$ into a topological space by defining its base: a set $W \subset X \times Y$ belongs to the base if it has the form $W = U \times V$, where U is an open set in X and V is open in Y. It is easy to check that in this way we obtain a topological space, which is called the *Cartesian product* of the spaces X and Y.

This choice of topology ensures that both natural projections $X \times Y \to X$ $((x, y) \mapsto x)$ and $X \times Y \to Y$ $((x, y) \mapsto y)$ are continuous maps.

Classical examples are: (i) the Cartesian product of two closed intervals is the square; (ii) the Cartesian product of two circles is the torus; (iii) the Cartesian product of two real lines $\mathbb{R}$ is the plane $\mathbb{R}^2$.

Theorem 3.1. *The Cartesian product of the n-disk and the m-disk is the $(n+m)$-disk. The Cartesian product of $\mathbb{R}^n$ and $\mathbb{R}^m$ is $\mathbb{R}^{n+m}$.*

The proof is absolutely straightforward.

3.3. Quotient Spaces

Roughly speaking, a quotient space is obtained from a given space by identifying the points of certain subsets of the given space ("dividing" our space by these subsets).

More precisely let X be a topological space and let $\sim$ be an equivalence relation on the set X; we then consider the equivalence classes with respect to this relation as points of the quotient set $X/_\sim$ and introduce a topology in this set by declaring open any subset $U \subset X/_\sim$ such that $U^* := \{x \in \xi \mid \xi \in U\}$ is open in X. The topological space thus obtained is denoted by $X/_\sim$.

Suppose X and Y are topological spaces, A and B are closed subspaces of X and Y, respectively, and $f : A \to B$ is a continuous map. (The particular case in which f is a homeomorphism is often considered.) In the disjoint union of X and Y, we identify all points of each set in the family

$$\mathcal{F}_b := \{b \sqcup f^{-1}(b) \mid b \in B\}.$$

Then we denote the quotient space $(X \cup Y)/\sim$, where $\sim$ is the equivalence relation identifying points in each of the sets $\mathcal{F}_b$, $b \in B$, $X \cup_f Y$ and say that this space is obtained by *attaching* (or *gluing*) Y to X along f.

3.5. Cone, Suspension, and Join

(i) Roughly speaking, the cone over a space is obtained by joining a fixed point by line segments with all the points of the space. More precisely, let X be a topological space; consider the Cartesian product $X \times [0,1]$ (called the *cylinder* over X) and on it, the equivalence relation $(x,1) \sim (y,1)$ for any x, $y \in X$; we define the *cone over* X as the quotient space of the cylinder by the equivalence relation $\sim$:

$$C(X) := (X \times [0,1])/_\sim.$$

Note that all the points (x,t) with $t = 1$ are identified into one point, called the *vertex* of the cone. By definition, the cone over the empty set is one point. The cone over a point is a line segment, the cone over the circle is homeomorphic to the disk (although it is more natural to think of it as the lateral surface of the ordinary circular cone).

(ii) Roughly speaking, the suspension over a topological space is obtained by joining two fixed points by segments with all the points of the given space. Another heuristic way of saying this is that the suspension is a double cone (on "different sides") over that space.

More precisely, let X be a topological space; consider the Cartesian product $X \times [-1,1]$ and on it, the equivalence relation

$$(x,1) \approx (y,1) \quad \text{and} \quad (x,-1) \approx (y,-1)$$

for any $x, y \in X$; now define the *suspension over* X as the quotient space of the cylinder $X \times [-1,1]$ by the equivalence relation $\approx$:

$$\Sigma(X) := (X \times [-1,1])/_\approx.$$

By definition, the suspension over the empty set is the two point set $\mathbb{S}^0$. The suspension over the two point set is homeomorphic to the circle, that over the circle is homeomorphic to the 2-sphere.

The notion of suspension is extremely important in topology, particularly in algebraic topology (surprisingly, it is much more important than that of the cone).

(iii) Roughly speaking, the join of two spaces is obtained by joining each pair of points from the two spaces by a segment.

More precisely, suppose that X and Y are topological spaces; consider the Cartesianproduct $X \times [-1,1] \times Y$ and identify (via an equivalence relation that will be denoted by $\equiv$) all pairs of points of the form $(x_1, 1, y) \equiv (x_2, 1, y)$ as well as all pairs of the form $(x, -1, y_1) \equiv (x, -1, y_2)$. The topological space $X * Y$ thus obtained,

$$X * Y := (X \times [-1,1] \times Y)/_{\equiv},$$

is called the *join* of the spaces X and Y.

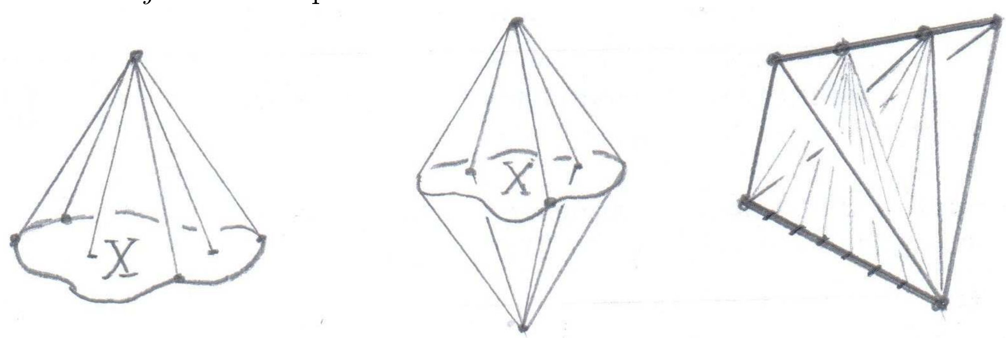

FIGURE 3.1. Cone and suspension. Join of two closed intervals

Theorem 3.2. *The cone over the n-sphere is the $(n+1)$-disk and the cone over the n-disk is the $(n+1)$-disk. The suspension over the n-sphere is the $(n+1)$-sphere and the suspension over the n-disk is the $(n+1)$-disk. The join of the n-disk and the m-disk is the $(n+m+1)$-disk. The join of the n-sphere and the m-sphere is the $(n+m+1)$-sphere.*

The proof is not difficult: one performs the construction in a Euclidean space of the appropriate dimension; in each case the corresponding homeomorphism is not hard to construct, although for large values of n and m it is difficult to visualize. The simplest (and only really "visual") nontrivial example is the join of two segments (which is the tetrahedron, otherwise known as the 3-simplex); it is shown in Figure 3.1.

3.6. Simplicial spaces

A 0-simplex is a point, a 1-simplex is a closed interval, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on. More generally and precisely, we define an *n-dimensional simplex* σ_n (*n-simplex* for short) as a topological space supplied with a homeomorphism

$$h : \sigma_n \to \Delta^n = [e_0, e_1, \ldots, e_n],$$

where Δ^n is the convex hull of the set of $n+1$ points consisting of the origin $0 = e_0$ and the endpoints $e_1, \dots, e_n$ of the basis unit vectors of Euclidean space $\mathbb{R}^n$. The n-simplex is of course homeomorphic to the n-disk $\mathbb{D}^n$, but it has a richer structure coming from the homeomorphism h. Namely, for any $i, 0 \le i \le n$, it has a set of *i-faces*, each i-face is the preimage under h of the convex hull in $\mathbb{R}^n$ of i points from the set $\{e_0, e_1, \dots, e_n\}$. The 0-faces of an n-simplex are called *vertices*, and we often write

$$\sigma_n = [0, 1, \dots, n],$$

where by abuse of notation i, $i = 0, 1, \dots, n$, denotes the vertex $h^{-1}(e_i)$.

Thus the 3-simplex possesses four 2-faces (triangles), six 1-faces (edges) and four 0-faces (vertices). By convention, we agree that the empty set is regarded as the (-1)-dimensional simplex. Note that the 3-simplex (as well as its faces), inherits a linear structure from $\mathbb{R}^3$ by the homeomorphism $h : \sigma_3 \to \Delta^3 \subset \mathbb{R}^3$.

We now define a *finite simplicial space* X (also called *finite simplicial complex*) as the space obtained from the disjoint union of a finite set of simplices by gluing some of their faces together by homeomorphisms; it is assumed that the attaching homeomorphisms respect the linear structure of the faces (so that after the gluing is performed, all the simplices have a coherent linear structure). In this course, we will not consider the more general notion of simplicial space with a possibly infinite number of simplices, and so will often drop the adjective finite when speaking of finite simplicial spaces. By the *dimension* of a simplicial space X we mean the dimension of the simplices of the highest dimension in X and we often write it in the form of a superscript, writing X^n for an n-dimensional simplicial space.

A more geometric way of defining a simplicial space is to represent it as a subset of some Euclidean space, with the simplices being rectilinear geometric subsets of the space. Figure 3.2 shows two such examples of simplicial spaces, represented as lying in $\mathbb{R}^3$: a 2-sphere and a funny 3-dimensional simplicial space.

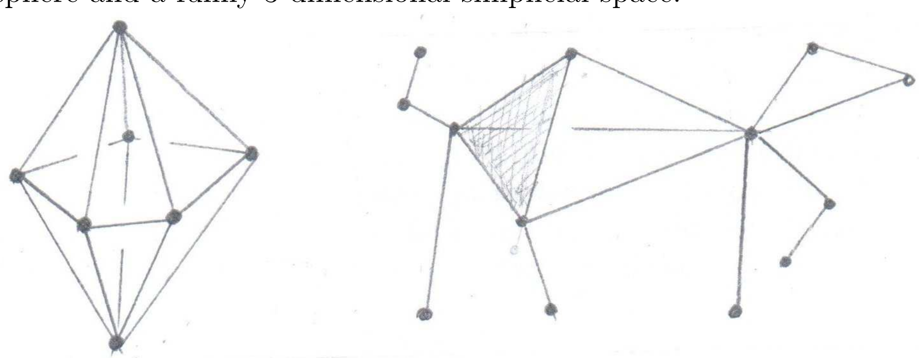

FIGURE 3.2. Two simplicial spaces as subsets of $\mathbb{R}^3$

As the following theorem claims, any finite simplicial space X can be represented as a subset of some Euclidean space $\mathbb{R}^N$ in the sense specified above – one then says that X is *piecewise-linearly embedded* (PL-*embedded* for short) in $\mathbb{R}^N$.

Theorem 3.3. *Any finite n-dimensional simplicial space X^n can be PL-embedded in $\mathbb{R}^{2n+1}$.*

We shall not use this theorem and therefore omit its proof. The reader may wonder where the exponent $2n+1$ comes from; there are examples of 1 -dimensional simplicial spaces (e.g. the so-called $K_{3,3}$ space) that cannot be embedded in $\mathbb{R}^2$.

3.7. CW Spaces

Roughly speaking, a CW-space is a space obtained by inductively attaching k-disks ($k = 0, 1, 2, \dots$) along their boundaries to the $(k-1)$-dimensional part of the previously constructed space via continuous maps of their boundaries (these maps, as well as their images, are called k-cells).

The formal definition of CW-space (also called CW-complex) is the following. Let X be a Hausdorff topological space such that

$$X = \bigcup_{i=0}^{\infty} X^i,$$

where X^0 is a discrete space and the space X^{i+1} is obtained by attaching the disjoint union of $(i+1)$-dimensional closed discs $\sqcup_{\alpha \in A} D_\alpha^{i+1}$ to X^i along a continuous map $\sqcup_{\alpha \in A} S_\alpha^i \to X^i$, where $S_\alpha^i = \partial D_\alpha^{i+1}$. Let us call the image of D_α^{i+1} and the image of the interior of D_α^{i+1} under the natural map to $X^{i+1} \hookrightarrow X$ *closed cell* and *open cell*, respectively. The space X is called a *CW-space* (or *CW-complex*) if the two following conditions hold:

(C) any closed cell intersects a finite number of open cells;

(W) a set $C \subset X$ is closed iff any intersection of C with a closed cell is closed.

"C" is the abbreviation for "Closure Finite", "W" is the abbreviation for "Weak Topology". If the number of cells is finite, then conditions (C) and (W) hold automatically. Since we will only be considering finite cell spaces in this course, you can forget about conditions (C) and (W).

Note that any simplicial space can be considered as a CW-space (how?). Simplicial spaces are easier to visualize than CW-spaces, because simplices are simpler than cells, but CW-spaces are more economical. For example, the 77-dimensional sphere has a CW-space structure with only two cells, whereas the simplest simplicial structure of that sphere has hundreds of simplices of dimensions $0, 1, 2, \dots, 77$.

3.8. Exercises

3.1. Prove that $\mathbb{D}^n/\partial\mathbb{D}^n \approx \mathbb{S}^n$.

3.2. Prove that the space $\mathbb{S}^1\times\mathbb{S}^1$ is homeomorphic to the space obtained by the following identification of points of the square $0 \le x \le 1$, $0 \le y \le 1$ belonging to its sides: $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$. (This space is called the torus.)

3.3. Let $I = [0,1]$. Prove that the space $\mathbb{S}^1\times I$ is not homeomorphic to the Möbius band.

3.4. Prove that the following spaces (supplied with the natural topology) are homeomorphic:
(a) the set of lines in $\mathbb{R}^{n+1}$ passing through the origin;
(b) the set of hyperplanes in $\mathbb{R}^{n+1}$ passing through the origin;
(c) the sphere $\mathbb{S}^n$ with identified diametrically opposite points (every pair of diametrically opposite points is identified);
(d) the disc $\mathbb{D}^n$ with identified diametrically opposite points of the boundary sphere $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$.

3.5. Prove that the following spaces are homeomorphic:
(a) the set of complex lines in $\mathbb{C}^{n+1}$ passing through the origin;
(b) the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ with identified points of the form λx for every $\lambda \in \mathbb{C}$, $|\lambda| = 1$ (for any fixed point $x \in \mathbb{S}^{2n+1}$);
(c) the disc $\mathbb{D}^{2n}\subset\mathbb{C}^n$ with points of the boundary sphere $\mathbb{S}^{2n-1}=\partial\mathbb{D}^{2n}$ of the form λx for every $\lambda\in\mathbb{C}$, $|\lambda|=1$ identified for any fixed point $x\in\mathbb{S}^{2n-1}$.

3.6. Prove that $C(\mathbb{D}^n) \approx \mathbb{D}^{n+1}$ and $\Sigma(\mathbb{D}^n) \approx \mathbb{D}^{n+1}$. (Here and below $\approx$ denotes homeomorphisms)

3.7. Prove that $\mathbb{R}P^1 \approx \mathbb{S}^1$ and $\mathbb{C}P^1 \approx \mathbb{S}^2$.

3.8. Prove that $C(\mathbb{S}^n) \approx \mathbb{D}^{n+1}$ and $\Sigma\mathbb{S}^n \approx \mathbb{S}^{n+1}$.

3.9. Is it true (for arbitrary CW-spces) that (a) $X * Y \approx Y * X$; (b) $(X * Y) * Z \approx X * (Y * Z)$; (c) $C(X * Y) \approx C(X) * Y$; (d) $\Sigma(X * Y) \approx \Sigma(X) * Y$?

3.10. Prove that $\mathbb{S}^n * \mathbb{S}^m \approx \mathbb{S}^{n+m+1}$.

3.11. Prove that $\mathbb{S}^{n+m-1} \setminus \mathbb{S}^{n-1} \approx \mathbb{R}^n \times \mathbb{S}^{m-1}$. (We suppose that the position of $\mathbb{S}^{n-1}$ in $\mathbb{S}^{n+m-1}$ is standard.)

3.12. Prove that (a) the sphere $\mathbb{S}^2$; (b) the torus $\mathbb{T}^2$; (c) the real projective space $\mathbb{R}P^n$; (d) the complex projective space $\mathbb{C}P^n$ are CW-spaces.

3.13. Find an example of a space consisting of cells that satisfies the W-axiom, and does not satisfy the C-axiom and vice versa.