Lecture 2

ABSTRACT TOPOLOGICAL SPACES

In this lecture, we move from the topological study of concrete geometrical figures (subsets of $\mathbb{R}^n$) to the axiomatic study of abstract topological spaces. What is remarkable about this approach is the simplicity of the underlying axioms (based on the notion of open set, now an undefined concept in the axiomatics), which nevertheless allow to generalize the deep theorems about subsets of $\mathbb{R}^n$ proved in the previous lecture to subsets of any abstract topological space, reproducing the proofs practically word for word.

2.1. Topological spaces

By definition, an (*abstract*) *topological space* $(X, \mathcal{T} = \{U_\alpha\})$ is a set X of arbitrary elements $x \in X$ (called *points*) and a family $\mathcal{T} = \{U_\alpha\}$ (called the *topology of the space* X) of subsets of X (called *open sets*) such that

(1) X and $\emptyset$ are open;

(2) if U and V are open, then $U \cap V$ is open;

(3) if $\{V_\beta\}$ is any collection of open sets, then the set $\cup_\beta V_\beta$ is open.

Any set $X \subset \mathbb{R}^n$ is a topological space if the family of open sets is defined as in Section 1.1. (The proof is a straightforward exercise.) All the definitions from Sections 1.2-1.4 are valid for any topological space (and not only for subsets of $\mathbb{R}^n$), because they only use the notion of open set. All the theorems (and their proofs) from the previous lecture are also valid. At this point the reader should read through these proofs again and check that, indeed, only the properties of open sets appearing in the axioms are used.

In order to define a topological space, we don't have to specify *all* the open sets: there is a more "economical" way of defining the topology. For a topological space X, $\mathcal{T}$, we say that a subset $\mathcal{T}_0 \subset \mathcal{T} = \{U_\alpha\}$) is a *base of the topology* of X, $\mathcal{T}$ if for any open set $U \in \mathcal{T}$ there exists a collection $\{V_\beta\}$ of open sets in $\mathcal{T}_0$ such that $U = \bigcup_\beta V_\beta$.

Clearly, any base of the topology uniquely determines the whole topology (how?). For example, the set of all open balls in $\mathbb{R}^n$ is a base of the standard topology of Euclidean space.

Examples. (1) Any set D becomes a topological space if it is supplied with the *discrete topology*, i.e., if any set is declared open. Obviously, a topology is discrete if and only if any point is an open set.

(2) Any set X supplied with only two open sets (the empty set and X itself) is a topological space with the *trivial topology*.

(3) Any metric space M (see the definition in the next section) is a topological space in the *metric topology*, which is given by the base of all open balls in M $O_r(m) := \{m' : d(m', m) < r\}$, where d is the distance function in M .

(4) The space $\mathcal{C}[0, 1]$ of continuous real-valued functions on the closed interval $[0, 1] \subset \mathbb{R}$ has a standard topology given by the base of open balls $O_r(f) := \{g : \sup_x(|g(x) - f(x)| < r)\}$.

Many more nontrivial examples will be given at the end of this lecture, in the exercise class and in subsequent lectures.

2.2. Metric spaces

A *metric space* is a set M supplied with a *metric* (or *distance function*), i.e., a function $d\colon M \times M \to \mathbb{R}$ such that

(1) for all $x,\ y \in M$, $d(x, y) \geq 0$ (*nonnegativity*);
(2) for all $x,\ y \in M$, $d(x, y) = 0$ iff $x = y$; (*identity*);
(3) for all $x,\ y \in M$, $d(x, y) = d(y, x)$ (*symmetry*);
(4) for all $x,\ y,\ z \in M$, $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

The most popular example of a metric space is Euclidean space $\mathbb{R}^n$ (and its subsets) with the standard metric:

$$d(p, q) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}, \quad \text{where } p = (x_1, \dots, x_n), q = (y_1, \dots, y_n).$$

Other less familiar examples will appear in the exercise classes.

As we mentioned above, any metric space (M, d) becomes a topological space in the metric topology. Conversely, it is *not* true that any topological space $(X, \mathcal{T})$ has a metric (i.e., possesses a distance function for which the metric topology coincides with $\mathcal{T}$). Until the middle of the 20th century one of the main problems of topology was to find necessary and sufficient conditions for a topological space $X, \mathcal{T})$ to be *metrizable*, i.e., for X to have a metric such that the corresponding metric topology coincides with $\mathcal{T}$.

2.3. Induced Topology

If A is a subset of a topological space X, then A acquires a topological structure in a natural way: the topology on A is *induced* from X if we declare all the intersections of open sets of X with A to be the open sets of A. It is easy to check that A with the induced topology is indeed a topological space (i.e., satisfies axioms (1)—(3) from Section 2.1).

It is important to note that open sets in the induced topology of A are not necessarily open in X (in fact, in most cases they are not).

Whenever we consider a subset of a topological space, we will always regard it as a topological space in the induced topology without explicit mention. Speaking of open sets, however, one should always make clear with respect to what set or subset openness is understood. Thus the open interval $(0, 1)$ is open on the real line but not in the plane.

2.4. Connectedness

In the previous lecture, we defined path connectedness of subsets of $\mathbb{R}^n$; that definition remains valid, word for word, for topological spaces. Intuitively, pathconnectedness of a topological space means that you can move continuously within the space from any point to any other point. But there is another definition of connectedness based on the idea that a connected set is "a set that consists of one piece". The rigorous formalization of the idea of "consisting of one piece" is as follows.

A topological space X is called *connected* if it is not the union of two open, closed, nonempty, and nonintersecting sets, i.e., $X = A \cup B$, where A and B are both open, closed, and nonempty, implies $A \cap B \neq \emptyset$.

What is the relationship between the notions of connectedness and path connectedness?

Theorem 2.4. *Any path connected topological space is connected, but there exist connected topological spaces that are not path connected.*

Proof. Suppose that the space X is path connected. Arguing by contradiction, let us assume that it is the disjoint union of two open and closed nonempty sets A and B. Let $a \in A$, $b \in B$. Then there exists a continuous map $f\colon [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$. Denote $A_0 := f^{-1}(A)$ and $B_0 := f^{-1}(B)$. These two sets are disjoint, open (as inverse images of open sets) and cover the closed interval $[0, 1]$ (because $f([0, 1]) \subset X = A \cup B$). We know that $1 \in B_0$. Let ξ be the least upper bound of A_0. If $\xi \in A_0$, then A_0 cannot be open, so ξ belongs to B_0; but then B_0 cannot be open. A contradiction.

Concerning the converse statement, see Exercise 2.1. □

Connectedness, like path connectedness, is not only a topological property – it is preserved by *any* continuous maps (not only by homeomorphisms).

Theorem 2.5. *The continuous image of a connected set is connected, i.e., if a map $f\colon X \to Y$ is continuous and $X \subset \mathbb{R}^n$ is connected, then $f(X)$ is connected.*

Proof. We argue by contradiction: suppose that X is connected, but $f(X)$ is not. Then $f(X) = A \cup B$, where both A and B are both closed and open, and don't intersect. Denote $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$. Then $X = A \cup B$, $A \cap B = \emptyset$, both A and B are open (as preimages of open sets) and closed (as complements to open sets). But this means that X is not connected – a contradiction. □

Roughly speaking, a connected component of a nonconnected set is just one of its many "pieces". The formal definition is this: a *connected component* of a not necessarily connected space X is any connected subset of X not contained in a larger connected subset of X. It is easy to prove that any connected component of a space X is both open and closed in X.

2.5. Separability

An important type of property for topological spaces comes from various separability axioms, which specify how well it is possible to "separate" points and/or sets (i.e., put them into nonintersecting neighborhoods). We only define one such property, the most natural and classical one: a topological space is said to be a *Hausdorff space* if any two distinct points possess nonintersecting neighborhoods. Obviously, Euclidean space and any of its subsets are Hausdorff, as are indeed any metric spaces (why?). The sad fact that there exist non-Hausdorff spaces will be considered in the exercise class.

2.6. More examples of topological spaces

In this section, we list twelve classical mathematical objects (not necessarily familiar to you) coming from completely different areas of mathematics. All of them are topological spaces. In the exercise class (and in doing the homework assignments), you will learn how to define their topology (by introducing an appropriate base). You will perhaps be surprised to learn that certain objects from different parts of mathematics and physics, which at first glance have nothing in common, turn out to be topologically equivalent (homeomorphic).

We begin with examples coming from algebra.

(1) The group $\mathrm{Mat}(n, n)$ of all nondegenerate $n \times n$ matrices.

(2) The group $\mathrm{O}(n)$ of all orthogonal transformations of $\mathbb{R}^n$.

(3) The set of all polynomials of degree n with leading coefficient 1.

The next examples come from geometry.

(4) The real projective space $\mathbb{R}P^n$ of dimension n.

(5) The Grassmanian $G(k, n)$, i.e., the set of k-dimensional planes containing the origin in n-dimensional affine space.

(6) The hyperbolic plane.

The next example comes from complex analysis.

(7) The Riemann sphere $\mathbb{C}^*$ and, more generally, Riemann surfaces.

Here are some examples from classical mechanics.

(8) The configuration space of a solid rotating about a fixed point in 3-space.

(9) The configuration space of a rectilinear rod rotating in 3-space about (a) one of its extremities, (b) its midpoint.

Here are two from algebraic geometry.

(10) The set of solutions $p = (x_1, \ldots, x_9) \in \mathbb{R}^9$ of the following system of 6 equations:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 1, & x_1x_4 + x_2x_5 + x_3x_6 &= 0, \\ x_4^2 + x_5^2 + x_6^2 &= 1, & x_1x_7 + x_2x_8 + x_3x_9 &= 0, \\ x_7^2 + x_8^2 + x_9^2 &= 1, & x_4x_7 + x_5x_8 + x_6x_9 &= 0. \end{aligned}$$

(11) Any affine variety in the *Zariski topology* is a topological space.

In conclusion, an example from dynamical systems (differential equations).

(12) The phase space of billiards on the disk.

2.7. Exercises.

2.1. Prove that any constant map is continuous.

2.2. For any subsets A, $B \subset \mathbb{R}^n$, define the *distance* between A and B by putting $d(A, B) := \inf\{\|a - b\| a \in A,\, b \in B\}$.

(a) Is it true that $d(A, C) \leq d(A, B) + d(B, C)$?

(b) Let $A \subset \mathbb{R}^n$ be a closed subset, let $C \subset \mathbb{R}^n$ be a compact subset. Prove that there exists a point $c_0 \in C$ such that $d(A, C) = d(A, c_0)$. Further, prove that if the set A is also compact, then there exists a point $a_0 \in A$ such that $d(A, C) = d(a_0, c_0)$.

2.3. Prove that any closed subspace of a compact space is compact.

2.4. Prove that the topology of $\mathbb{R}^n$ has a countable base (i.e., a base consisting of a countable family of open sets).

2.5. Introduce a "natural" topology on

(a) the set $\mathrm{Mat}(m, n)$ of matrices of size $n \times m$;

(b) the real projective space $\mathbb{R}P(n)$ of dimension n;

(c) the Grassmannian $G(k, n)$, i.e., the set of $k$dimensional planes containing the origin of $n$dimensional affine space;

(d) the set of solutions $p = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ of the following system of two equations: $x_1^2 + x_2^3 + x_3^4 + x_4^5 = 1$ and $x_1 x_2 x_3 x_4 = -1$;

(e) the set of all polynomials of degree n with leading coefficient 1.

2.6. (a) Is the topological space $GL(n)$ connected?

(b) Prove that the topological space $SO(3)$ is connected.

(c) Prove that the topological space $GL(3)$ consists of two connected components.

2.7. (a) Prove that $d(x, y) = \max\{|x_i - y_i|, i = 1, ..., n\}$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, is a metric in $\mathbb{R}^n$.

(b) Prove that $d(x, y) = \sum\limits_{i=1}^{n} |x_i - y_i|$ is a metric in $\mathbb{R}^n$.

(c) Draw some ε-neighborhood of the point $(0, 0, \ldots, 0)$ in the metrics defined in (a) and (b).

2.8. Prove that any metric space is Hausdorf and construct an example of a non-Hausdorff space.

2.9. Let X be a Hausdorff space. Prove that for any two distinct points x, $y \in X$ there exists a neighborhood $U \ni x$ such that its closure does not contain the point y.

2.10. Let C be a compact subspace of a Hausdorff space X. Let $x \in X \setminus C$. Prove that the point x and the set C have disjoint neighborhoods.

2.12. Prove that any two disjoint compact subsets of a Hausdorff space have disjoint (open) neighborhoods.