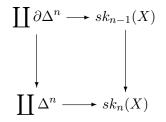
Problems

April 12, 2012

- 1. Describe explicitly (co)products and (co)equalizers in Set, Top, Mod_A , SSet and DGMod_A. Explain how to compute a (co)limit of an arbitrary small diagram in all these categories.
- 2. For $X \in \mathbf{SSet}$ denote by $sk_n(X)$ the minimal simplicial subset of X containing all simplices of X of degree n and less (put $sk_{-1}(X) = \emptyset$).
 - Prove that $sk_n(X)$ is isomorphic to the colimit of the following functor $F : (\Delta_X^{\leq n})_N \to \mathbf{SSet}$. Here $(\Delta_X^{\leq n})_N$ is the subcategory of the category of elements Δ_X of X (see lectures) containing all non-degenerate simplices of degree n and less, and F sends a non-degenerate simplex $\Delta^k \to X$ to $\Delta^k \in \mathbf{SSet}$.
 - Prove that for any $n \in \mathbb{N}$ there is a pushout diagram



What is the set over which the coproduct on the left is taken?

- Prove that the realisation $|\partial \Delta^n|$ is isomorphic to S^n in **Top**.
- Prove that |X| is a CW-complex.
- 3. In Δ , define¹ $\partial_i : [n-1] \to [n]$ to be the unique injective monotone map not containing *i* in its image. Denote also by $\sigma_i : [n+1] \to [n]$ the unique surjective monotone map which maps *i* and i+1 in [n+1] to *i* in [n]. For a simplicial set *X*, denote by $d_i = X(\partial_i) : X(n) \to X(n-1)$ and $s_i = X(\sigma_i) : X(n) \to X(n+1)$. These are called, respectively, *i*-th face and degeneracy maps.
 - Prove the identities

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \, (i < j), \\ d_i s_j &= d_{j-1} s_i \, (i < j), \\ d_j s_j &= 1 = d_{j+1} s_j \\ d_i s_j &= s_j d_{i-1} \, (i > j+1) \\ s_i s_j &= s_{j+1} s_i \, (i \le j) \end{aligned}$$

 $^{{}^{1}[}n]$ is omitted from the notation

- Prove that any injective (surjective) map in Δ can be written as a composition of ∂_i (σ_i). Consequently, prove that any map in Δ can be written as a composition of σ_i followed by ∂_i .
- Prove that $\partial \Delta^n$ is isomorphic to the colimit of

$$\coprod_{0 \le i < j \le n} \Delta^{n-2} \Longrightarrow \coprod_{0 \le i \le n} \Delta^{n-1}$$

(what are the two maps in this diagram?)

• Let A[-]: Set $\to \operatorname{Mod}_A$ denote the free A-module functor. For a simplicial set X, define $A[X]^{-i} = A[X(i)]$ and

$$d_X^{-i}: A[X(i)] \to A[X(i-1)]$$

to be the sum $\sum_{j} (-1)^{j} d_{j}$. Prove that this gives a functor from **SSet** to **DGMod**^{≤ 0}.

- 4. A groupoid is a category \mathcal{C} such that any morphism in Mor \mathcal{C} is an isomorphism
 - Prove that for any category \mathcal{D} and X in **SSet**, a map $f: X \to N(\mathcal{D})$ is determined by f_0, f_1 and f_2 $(f_n: X(n) \to N(\mathcal{D})(n))$.
 - Prove that for a groupoid \mathcal{C} , the nerve $N(\mathcal{C})$ is fibrant in the standard model structure on **SSet**.
- 5. Let \mathcal{M} be a model category. Prove Whitehead's theorem A morphism $f : X \to Y$ between fibrantcofibrant objects is a weak equivalence if and only if it is an isomorphism in $\pi \mathcal{M}_{cf}$. (Hint: for 'only if' part, it is enough to prove it only for trivial (co)fibrations. For 'if' part, it might be useful to factor $f = p \circ i$, so that *i* is a trivial cofibration and *p* is a fibration, and then try to show that *p* is a weak equivalence.)
- 6. Prove Ken Brown's lemma: let \mathcal{M} be a model category. If a functor $F : \mathcal{M} \to \mathcal{D}$ takes trivial cofibrations between cofibrant objects to isomorphisms in \mathcal{D} , then F takes all weak equivalences between cofibrant objects to isomorphisms in \mathcal{D} and the left derived functor $\mathbb{L}F$ exists. (Hint: if $f: A \to B$ is a weak equivalence between cofibrant objects, factor the induced map $A \coprod B \xrightarrow{f \sqcup id_B} B$ as a cofibration followed by trivial fibration. Also, remember that cofibrations are stable under pushout.)
- 7. Let

$$\mathcal{M} \xrightarrow{F} \mathcal{N}$$

be a Quillen adjunction. Prove that the following are equivalent

• The pair

$$\operatorname{Ho}(\mathcal{M}) \xrightarrow{\mathbb{L}F} \operatorname{Ho}(\mathcal{N})$$

are inverse to each other up to natural isomorphism (that is, $(\mathbb{L}F, \mathbb{R}G)$ is an adjoint equivalence of categories).

• For any cofibrant object X of \mathcal{M} and any fibrant object Y of \mathcal{N} a morphism $F(X) \to Y$ is a weak equivalence in \mathcal{N} if and only if the adjoint morphism $X \to G(Y)$ is a weak equivalence in \mathcal{M} .