8. COVERING SPACES.

Problem 1. (a) Construct a 2-fold covering $p : C \to M$ of the Möbius band M by a cylinder $C = \mathbb{R} \times S^1$. Describe the preimage $p^{-1}(U)$ where U is a thin neighbourhood of the middle line of M. (b) Describe the homomorphism $p_* : \pi_1(C) \to \pi_1(M)$. (c) Construct a 2-fold covering $q : S^2 \to \mathbb{R}P^2$ and compute $\pi_1(\mathbb{R}P^2)$. Represent $\mathbb{R}P^2 = D/\sim$ where D is a disk and $\sim$ glues the opposite points of its boundary ∂D. Let V be a thin neighbourhood of ∂D. Prove that $\mathcal{M} = q^{-1}(V)$ is a Möbius band and the restriction of q to $\mathcal{M}$ is a covering equivalent to p of Problem 1(a).

Problem 2. List all the coverings of (a) $\mathbb{R}P^2$, (b) S^1, (c) $\mathbb{T}^2 = S^1 \times S^1$. For every two coverings indicate whether a morphism between them exists.

Problem 3. (a) Prove that if the base surface of a covering $p : S_1 \to S_2$ is orientable, then so is the covering surface S_1. (b) Prove that for any nonorientable surface S_2 there exists its 2-fold covering $p : S_1 \to S_2$ where S_1 is orientable. (c) Prove that if a genus g_1 surface admits an n-fold covering by a genus g_2 surface if and only if $g_2 - 1 = n(g_1 - 1)$.

Problem 4. Prove that for any $n \geq 2$ the wedge product of two circles can be covered by the wedge product of n circles.

Problem 5. (a) Construct a 2-fold covering $p : \mathbb{T}^2 \to K$ where K is a Klein bottle. (b) Prove that $\pi_1(K)$ is generated by two elements a and b with a single relation $abab^{-1} = 1$. (c) Describe an index 2 subgroup $p_*(\pi_1(\mathbb{T}^2)) \subset \pi_1(K)$.

Remark. Problem 5(b) follows from the cell decomposition of the Klein bottle. It is interesting, though, that it can be solved independently using a cover from Problem 5(a).

Problem 6. Let $p : E \to B$ be a covering with arcwise connected and localy simply connected spaces E and B. Let $b \in B$ and $e \in E$ be marked points, and $p(e) = b$. Prove that the subgroup $p_*(\pi_1(E, e)) \subset \pi_1(B, e)$ is *not* normal if and only if there is a loop $\gamma : [0, 1] \to B$, $\gamma(0) = \gamma(1) = b$, such that $\gamma = p \circ \Gamma_1 = p \circ \Gamma_2$ where $\Gamma_1 : [0, 1] \to E$ is a loop and $\Gamma_2 : [0, 1] \to E$ is a path but not a loop ($\Gamma_2(0) \neq \Gamma_2(1)$).

A *graph* is a topological space obtained by gluing some set A of segments by their ends. The elements of the set A are called edges of the graph, the equivalence classes of the ends are called vertices. The number of edge ends glued to obtain a given vertex is called its valency.

Problem 7. (a) Let a graph Γ be locally finite, that is, every its vertex has a finite valency. Prove that a subspace $X \subset \Gamma$ is compact if and only if it is closed and lies in a union of finitely many edges. (b) Prove that the graph Γ_a drawn at Fig. 1a (an infinite tree) is simply connected. (c) Construct a covering $p_a : \Gamma_a \to S^1 \vee S^1$. Prove using Problem 7(b) that $\pi_1(S^1 \vee S^1)$ is a free group $\mathcal{F}_2$ with two generators.

Problem 8. (a) Construct a covering $p_b : \Gamma_b \to S^1 \vee S^1$ where the graph Γ_b is drawn at Fig. 1b. Compute the subgroup $(p_b)_*(\pi_1(\Gamma_b)) \subset \mathcal{F}_2$. Is it a normal subgroup? Describe $\pi_1(\Gamma_b)$ as a group. (b) The same questions for the graph Γ_c from Fig. 1c.

Problem 9. (a) A topological space Γ is a connected n-fold covering of a wedge product of k circles. Prove that Γ is homeomorphic to a finite graph. Find the number of vertices in the graph. (b) Prove that every connected

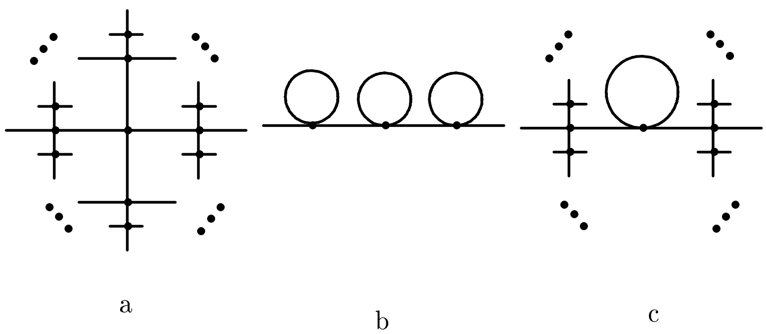

FIGURE 1. Covering spaces for $S^1 \vee S^1$

finite graph is homotopy equivalent to a wedge product of k circles. Express k via the number of edges and the number of vertices of the graph. (c) Prove using Problems 9(a) and 9(b) that if a group $G \subset \mathcal{F}_k$ is a subgroup of a finite index n then G is isomorphic to a free group $\mathcal{F}_p$. Express p via k and n. Try also to find a "purely algebraic" proof of the statement.

Problem 10. Construct a universal covering of the wedge product $S^1 \vee S^2$. What is $\pi_1(S^1 \vee S^2)$?

Problem 11. Construct the universal covering of the sphere with g handles, where $g \geq 2$.

Problem 12. Construct the universal covering of the sphere with n holes, $n \geq 2$.

Problem 13. (a) Let X and Y be finite CW complexes. Describe $\pi_1(X \vee Y)$ in terms of $\pi_1(X)$ and $\pi_1(Y)$. (b) Let $\tilde{X} \to X$ and $\tilde{Y} \to Y$ be the unversal coverings. Construct the universal covering of $X \vee Y$. Compare the result with Problems 10 and 7(c).