7. FUNDAMENTAL GROUP.

Problem 1. (a) Write out a formula for a homotopy which proves that multiplication in the fundamental group is associative. (b) Let the topological space X be a topological group (i.e. there is a multiplication with the usual group properties; and multiplication and taking an inverse are continuous maps). Prove that $\pi_1(X)$ is commutative.

Problem 2. For the following spaces construct their cell decompositions and compute (write down presentations of) their fundamental groups: (a) a wedge product of n circles; (b) a connected sum of g tori (of dimension 2); (c) a connected sum of g copies of $\mathbb{R}P^2$; (d) a genus g surface with n holes; (e) a connected sum of g copies of $\mathbb{R}P^2$ with n holes; (f) $\mathbb{R}^3 \setminus \{(x, y, 0) \mid x^2 + y^2 = 1\}$.

Problem 3. (a) Prove that if $G = \pi_1(g\mathbb{T}^2)$, then $G/G' = \mathbb{Z}^{2g}$. (Here G' is a *commutant*, i.e. G' is a subgroup generated by all elements of the type $aba^{-1}b^{-1}$ for a, $b \in G$.) (b) Prove that if $G = \pi_1(g\mathbb{R}P^2)$, then $G/G' \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.

Problem 4. Prove that the boundaries of the following surfaces are not their retracts: (a) a Moebius band; (b) a handle (a torus with one hole); (c) a Klein bottle with one hole.

Problem 5. Let X_n be a set of all cardinality n subsets $U \subset \mathbb{R}^2$. $\pi_1(X_n) \stackrel{\text{def}}{=} B_n$ is called a braid group on n strands. (a) Prove that X_2 is homotopy equivalent to a circle, and therefore $B_2 = \mathbb{Z}$. (b) Prove that for all $n \geq 2$ the group B_n is infinite. (c) Prove that for all $n \geq 3$ the group B_n is not commutative.

Problem 6. (a) Prove that X_3 is homotopy equivalent to the set $\{(p, q) \in \mathbb{C}^2 \mid p^2 + q^3 \neq 0\}$. (b) Prove that X_3 is homotopy equivalent to the set $\{(p, q) \in \mathbb{C}^2 \mid p^2 + q^3 \neq 0, |p|^2 + |q|^2 = 1\}$. (c) Prove that B_3 is isomorphic to a fundamental group of $\mathbb{R}^3 \setminus K$ where K is the trefoil knot.

Hint. For 6(a): the space X_n is homeomorphic to the set of degree n complex polynomials without multiple roots. Now consider cubic polynomials $t^3 + pt + q$. For 6(c): a trefoil knot is defined as a curve on a standard torus $\mathbb{T}^2 \subset \mathbb{R}^3$ making 2 turns along the meridian and 3 turns along the parallel of the torus.