5. HOMOTOPIES

Problem 1. Prove that the following spaces are homotopy equivalent: (a) the sphere S^2 with two points identified; (b) the union of the sphere S^2 and one of its diameters; (c) $S^1 \vee S^2$.

Problem 2. (a) Prove that the spaces $\mathbb{R}^3 \setminus S^1$ and $S^2 \vee S^1$ are homotopy equivalent; here $S^1 \subset \mathbb{R}^3$ is the standard circle lying in the xy coordinate plane. (b) Let X be the space $\mathbb{R}^3$ from which n copies of S^1 lying in parallel planes are deleted. Prove that X is homotopy equivalent to the wedge product of n copies of the space $S^2 \vee S^1$.

Problem 3. Let $L = \{(x, y, 0) \mid x^2 + y^2 = 1\} \cup \{(0, y, z) \mid (y-1)^2 + z^2 = 1\} \subset \mathbb{R}^3$ be a union of two circles linked in the simplest way. Prove that the spaces $\mathbb{R}^3 \setminus L$ and $S^2 \vee \mathbb{T}^2$ are homotopy equivalent.

For every continuous map $f : S^1 \to S^1$ one can define an integer $\deg(f)$ so that the following statements are true:

- Maps f_1 and f_2 are homotopic if and only if $\deg(f_1) = \deg(f_2)$.
- If $f_n : S^1 \to S^1$ ($n \in \mathbb{Z}$) is the map defined by the formula $f(\varphi) = n\varphi$ (where φ is the central angle on the circle) then $\deg(f) = n$. In particular, if f is a constant map then $\deg(f) = 0$; if f is the identity map then $\deg(f) = 1$.

Problem 4. Let $f, g : S^1 \to S^1$ be continuous maps. Prove that $\deg(f \circ g) = \deg(f)\deg(g)$.

Problem 5. (a) Prove that if a map $f : X \to S^n$ is not onto, then f is homotopic to a constant map. (b) Prove that if for a map $f : S^1 \to S^1$ there exists $c \in S^1$ such that $f^{-1}(c)$ consists of n points then $|\deg(f)| \le n$.

Problem 6 (a hint for Problem 7)**.** (a) Let $f_0 \sim f_1$ and $g_0 \sim g_1$ where $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$ are continuous maps. Prove that $g_0 \circ f_0 \sim g_1 \sim f_1$. (b) Let $X_1 \approx X_2$ and $Y_1 \approx Y_2$. Define a one-to-one correspondence $\mathrm{Eq}_{X_1,X_2}^{Y_1,Y_2}$ between the homotopy classes of continuous maps $X_1 \to Y_1$ and $X_2 \to Y_2$. Check the following properties of the correspondence defined: if $[f_2] = \mathrm{Eq}_{X_1,X_2}^{Y_1,Y_2}([f_1])$ and $[f_3] = \mathrm{Eq}_{X_2,X_3}^{Y_2,Y_3}([f_2])$ then $[f_3] = \mathrm{Eq}_{X_1,X_3}^{Y_1,Y_3}([f_1])$ ($[f]$ means a homotopy class of the map f), and if $[f_2] = \mathrm{Eq}_{X_1,X_2}^{Y_1,Y_2}([f_1])$ and $[g_2] = \mathrm{Eq}_{Y_1,Y_2}^{Z_1,Z_2}([g_1])$ then $[g_2 \circ f_2] = \mathrm{Eq}_{X_1,X_2}^{Z_1,Z_2}([g_1 \circ f_1])$.

Problem 7. Let X_n be a wedge product of n circles, $f_{n,k} : S^1 \to X_n$ be a map sending S^1 homeomorphically to the k-th circle of the wedge, and $g_{n,k} : X_k \to S^1$ be a map sending the k-th circle of X_n homeomorphically to S^1 and all the other circles, to a point. (a) Prove that $f_{n,k}$ and $g_{n,k}$ are not homotopic to the map to a point. (b) Prove that X_n for $n > 1$ is not homotopy equivalent to a circle. (c) Prove that X_{n_1} is not homotopy equivalent to X_{n_2} is $n_1 \ne n_2$.

Hint. Consider a composition $g_{n,k_1} \circ f_{n,k_2} : S^1 \to S^1$ and use the results of Problems 4 and 6.

Problem 8. (a) Prove that the map $f : [0, 1) \to \{z \in \mathbb{C} \mid |z| = 1\} = S^1$ defined by the formula $f(t) = \exp(2\pi i t)$ is continuous and one-to-one, but is not a homeomorphism. (b) Prove that a continuous one-to-one map of compact topological spaces is a homeomorphism.

Problem 9. Prove that the sphere with g handles from which a point has been removed is homotopy equivalent to the wedge product of n copies of the circle and find n. (a) Prove that the sphere with g_1 handles is not homeomorphic to a sphere with g_2 handles if $g_1 \ne g_2$.

Problem 10. Prove that $A \subset X$ is a retract of X if and only if any continuous map $f : A \to Y$ can be extended to X.

Problem 11. Prove that if any continuous map $X \to X$ has a fixed point and $A \subset X$ is a retract of X, then any continuous map $A \to A$ also has a fixed point.

Brouwer Fixed Point Theorem says that any continuous map $f : D^n \to D^n$ has a fixed point.

Problem 12. Prove that the following assertions are equivalent to the Brouwer theorem: (a) There is no retraction $r : D^n \to S^{n-1}$. (b) Let $v(x)$ be a continuous vector field on D^n such that $v(x) = x$ for any point $x \in \partial D^n = S^{n-1}$. Then $v(x) = 0$ for some point $x \in D^n$.

Problem 13. Is there a continuous map $f : X \to X$ without fixed points if X is (a) a 2-sphere with n holes; (b) a 2-sphere with g handles and a hole?