Problems IV (29.02.2012)

The following facts are considered known in this problem set:

- The Euler characteristic of a compact surface S is defined as $V - E + F$ where V is the number of vertices and E is the number of edges of a graph Γ embedded into S so that $S \setminus \Gamma$ is homeomorphic to a disjoint union of F disks (faces). The Euler characteristics depends on the surface S only and does not depend on the graph Γ.
- A closed surface (i.e. a compact surface without boundary) is homeomorphic to either a sphere with g handles (a connected sum of a sphere with g tori) or to a sphere with g Möbius bands attached (a connected sum of a sphere with g copies of $\mathbb{R}P^2$). A compact surface with boundary is homeomorphic to a closed surface with a finite number of disks deleted.

IV.1. Find $\chi(\mathrm{Kl})$, where Kl is the Klein bottle.

IV.2. Prove that $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$.

IV.3. Prove that $\chi(mT^2) = 2 - 2m$ and $\chi(n\mathbb{R}P^2) = 2 - n$.

IV.4. Determine the type of the surface shown in Fig. 16 [page 31 of the Prasolov–Sossinsky "Topology-I" book].
(a) for $n = 3$;
(b) for arbitrary $n \geq 2$.

IV.5. Prove that the surface shown in Fig. 17 [page 31 of the Prasolov–Sossinsky "Topology-I" book] is homeomorphic to the torus from which a disk has been removed.

IV.6. Consider the quotient space $(S^1 \times S^1)/(x, y) \sim (y, x)$. Prove that this space is a surface. Which one?

IV.7. (a) Prove that any closed nonorientable surface is homeomorphic to one of the surfaces in the following list: $\mathbb{R}P^2$ (projective plane), $\mathbb{R}P^2 \# \mathbb{R}P^2$ (Klein bottle), $\ldots \mathbb{R}P^2 \# \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2, \ldots$.
(b)* Any two distinct surfaces in the list are not homeomorphic.

IV.8. Prove that a closed orientable surface is not homeomorphic to a closed nonorientable surface.

IV.9. What orientable surfaces can be obtained by identifying the sides of a regular octagon? For a given g find how many are there ways to identify the sides so as to obtain a sphere with g handles.

IV.10. Prove that on the sphere with g handles the maximal number of nonintersecting closed curves not dividing this surface is equal to g.

IV.11. Draw four closed curves issuing from a common point on the sphere with two handles so that cutting along these curves produces an octagon (a topological disk).

IV.12. Prove that the standard circle can be spanned by a Möbius band, i.e. there exists a subset $M \subset \mathbb{R}^3$ of the 3-space homeomorphic to the Möbius band and such that its boundary ∂M is a circle (lying in some plane).

IV.13. Prove that the boundary of $\mathrm{Mb} \times [0, 1]$, where Mb is the Möbius band, is the Klein bottle.

IV.14. Give an example of two surfaces-with-boundary M and N that are not homeomorphic but $M \times [0, 1]$ and $N \times [0, 1]$ are homeomorphic.

IV.15. Two three-dimensional disks $\mathbb{D}^3_i$, $i = 1, 2$, are glued together by a homeomorphism h of their boundary spheres. Find the resulting quotient space $\mathbb{D}^3_1 \cup_h \mathbb{D}^3_2$ if
(a) the homeomorphism h is the identity;
(b)* the homeomorphism h is the symmetry w.r.t. the equatorial plane?

IV.16. * Two solid tori $\mathbb{D}^2 \times \mathbb{S}^1$ and $\mathbb{S}^1 \times \mathbb{D}^2$ with coordinates (r, φ, ψ) and (ψ, r, φ), where (r, φ) are polar coordinates in $\mathbb{D}^2$, are glued together by identifying their boundaries according to the rule $(1, \varphi, \psi) \sim (\varphi, 1, \psi)$. Prove that the quotient space obtained is $\mathbb{S}^3$.