2. TOPOLOGICAL AND METRIC SPACES

Problem 1. Prove that (a) a closed subspace of a compact topological space is compact; (b) a compact subspace of a Hausdorff space is closed; (c) the image of a compact space under a continuous map is compact.

Problem 2. Let $A \subset \mathbb{R}^n$ be a closed subset, let $C \subset \mathbb{R}^n$ be a compact subset. Prove that there exists a point $c_0 \in C$ such that $d(A, C) = d(A, c_0)$. Further, prove that if the set A is also compact, then there exists a point $a_0 \in A$ such that $d(A, C) = d(a_0, c_0)$. Show that if A and C are closed but not compact then both statements may be false.

Hint. A subset $K \subset \mathbb{R}$ is compact if and only if it is bounded and closed. Now combine 1(c) with Problem 1.5.

Problem 3. (a) Prove that the topological space $\Delta \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1, \dots, x_n \geq 0, x_1 + \dots + x_n = 1\}$ is compact and the function $f : \Delta \to \mathbb{R}$ given by the formula $f(x_1, \dots, x_n) = x_1 \dots x_n$ is continuous. (The topology of Δ is inherited from $\mathbb{R}^n$.) (b) Prove that the function f achieves its maximum at the point $(1/n, 1/n, \dots, 1/n)$. Prove that for all $z_1, \dots, z_n \geq 0$ the inequality $\sqrt[n]{z_1 \dots z_n} \leq (z_1 + \dots + z_n)/n$ holds.

Problem 4. Prove that (a) the image of a connected space under a continuous map is connected; (b) the same, for path connectedness; (c) an open subset in $\mathbb{R}^n$ is connected if and only if it is path connected.

Problem 5. (a) Prove that the topological space $\mathrm{SO}(3)$ is path connected. (b) Prove that the topological space $\mathrm{GL}(n, \mathbb{R})$ is not path connected. (c) Prove that the topological space $\mathrm{GL}(n, \mathbb{R})$ is a disjoint union of two path connected components.

Problem 6. A set $A \subset \mathbb{R}^2$ is a union $B \cup C$ where B is a unit circle centered at the origin, and C is given in polar coordinates (r, φ) by the equation $r = \varphi/(1 + \varphi)$, $0 \leq \varphi < \infty$. Prove that A is connected but not path connected.

Problem 7. Prove that (a) $d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$, (b) $d(x, y) = \sum_{1 \leq i \leq n} |x_i - y_i|$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, is a metric in $\mathbb{R}^n$.

Problem 8. For a rational number $x \in \mathbb{Q}$, $x \neq 0$, denote by $\|x\|_2$ the number 2^{-k} where k is an integer (positive, negative or zero) such that $x = 2^k \frac{m}{n}$ where $m, n \in \mathbb{Z}$ are odd. Take also $\|0\|_2 = 0$ by definition. (a) Prove that $d(x, y) = \|x - y\|_2$ is a metric in $\mathbb{Q}$; denote the metric space obtained $\mathbb{Q}_2$. (b) Is $\mathbb{Q}_2$ compact? (c) Is the subspace $\mathbb{Z} \subset \mathbb{Q}_2$ compact? (d) What subsets of $\mathbb{Q}_2$ are connected?

Let $\langle l_1, l_2, \dots, l_{n-1}; d\rangle$ be a plane hinge mechanism that consists of n rods, one of which is fixed and the other rods (together with the fixed rod) form a closed polygonal line, with fixed rod of length d and moving rods of lengths l_1, l_2, …, l_{n-1} numbered successively.

Problem 9. Find the configuration spaces of the following quadrangles: (a) $\langle 1, 1, 1; 2.9\rangle$; (b) $\langle 1, 1, 1; 1\rangle$; (c) $\langle 2, 3, 2; 3\rangle$.