I.4. Two towns A and B are connected by two roads. Two travellers can walk along these roads from A to B so that the distance between them at any moment is less than or equal to 1 km. Can one traveller walk from A to B and the other from B to A (using these roads) so that the distance between them at any moment is greater than 1 km?

* * *

Suppose $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The *distance* from the point x to the subset A is equal to $d(x, A) = \inf_{a \in A} \|x - a\|$.

I.5. Prove that the function $f(x) = d(x, A)$ is continuous for any subset $A \subset \mathbb{R}^n$. ?

I.6. Prove that if the set A is closed, then $f(x) = d(x, A)$ is positive for any $x \notin A$.

I.7. Let X be the subset of $\mathbb{R}^2$ given by the equation $xy = 0$ (X is the union of two lines). Give an example of a neighborhood: a) of the point $(0, 0)$; b) of the point $(0, 1)$.

I.8. Let A and B be two subsets of the set X defined in Exercise I.7. Suppose that A and B are homeomorphic and A is open in X. Is it true that B is also open in X?

I.9. Find the set of points x in $\mathbb{R}^2$ such that $d(x, A) = 1; 2; 3$, where the set A is given by the formula:

(a) $x^2 + y^2 = 0$; (b) $x^2 + y^2 = 2$;

(c)* $x^2 + 2y^2 = 2$; (d) the square of area two.

* * *

A *graph* G is a set of points, called *vertices*, some pairs of which are joined by *edges*.

The *degree* of a vertex V of a graph G is the number of edges outgoing from V. A graph G is *connected* if for any two vertices of G there is a path from one vertex to the other along some edges. Sometimes it is useful to consider graphs with *loops* (a loop is an edge with coinciding end points) and *double edges* (double edges are different edges with a common pair of vertices). A *cycle* is a collection of pairwise different vertices $v_1, \ldots, v_n$ that are joined by edges $v_1v_2, v_2v_3, \ldots, v_nv_1$.

A graph G is *planar* if it can be disposed on the plane so that its edges have no internal common points.

A planar graph $G \subset \mathbb{R}^2$, besides vertices and edges, has *faces*, i.e., the connected components of $\mathbb{R}^2 \setminus G$.

I.10. Is it possible to build direct roads between 53 towns such that any town is connected exactly with 3 other towns?

I.11. Suppose the degrees of all vertices of a connected graph G are all even. Then there exists a path that traverses each edge of G exactly once.

I.12. Prove that any connected planar graph (without loops and double edges) has a vertex of degree not greater than 5. ?

I.13. Prove that one can color the vertices of any planar graph (without loops) using five colors so that the ends of any edge have different colors.

Let K_n be the graph consisting of n vertices pairwise joined by edges. Let $K_{n,m}$ be the graph consisting of $n+m$ vertices divided into two parts (n vertices in one part and m vertices in the other), the edges of $K_{n,m}$ joining each pair of vertices from different parts.

I.14. Prove that the graphs $K_{3,3}$ and K_5 are not planar.

I.15. (a) Let G be a planar graph such that any face of G is bounded by an even number of edges. Prove that one can color the vertices of G using two colors so that the ends of any edge have different colors.

(b) Let γ be a smooth closed curve with transversal self-intersections. Prove that γ divides the plane into domains so that one can color those domains using two colors (two domains with a common edge must be of different colors).

I.16 (Polygonal Jordan Theorem). Let C be a closed non-self-intersecting broken line (with a finite number of segments) on $\mathbb{R}^2$. Prove that $\mathbb{R}^2 \setminus C$ consists of two connected components and the boundary of each component is C.

I.17. Let a, b, c, d be points of a closed non-self-intersecting broken line C ordered as indicated. Suppose that points a and c are joined by a broken line L_1, points b and d are joined by a broken line L_2 and both broken lines belong to the same connected component defined by C. Prove that L_1 and L_2 have a common point.

I.18. Let G be a *tree*, i.e., a connected graph without cycles. Prove that $v(G) = e(G) + 1$, where $v(G)$ is the number of vertices and $e(G)$ is the number of edges.

I.19 (Euler Formula). Let G be a polygonal planar graph consisting of s connected components each of which is not an isolated vertex. Let G have v vertices and e edges. Using Exercises I.16, I.18 and induction prove that for any embedding of G in the plane the number of faces f is equal to $f = 1 + s - v + e$.

I.20. (a) Suppose G is a planar graph without isolated vertices, v_i is the number of its vertices of degree i, f_i is the number of faces with i edges. Prove that $\sum_i (4-i)v_i + \sum_j (4-j)f_j = 4(1+s) \geqslant 8$, where s is the number of connected components of G.

(b) Prove that if all faces are quadrilaterals, then $3v_1 + 2v_2 + v_3 \geqslant 8$.

(c) Prove that if the boundary of any face is a cycle containing no less than n edges, then $e \leqslant n(v-2)/(n-2)$.

I.21. Find and deduce the Euler Formula for convex polyhedra from the Euler formula for planar graphs. (The Euler Formula for convex polyhedra is a relation between numbers of vertices, edges and faces.)

I.22. With the help of Exercise I.20 (c), give another proof of the nonplanarity of the graphs K_5 and $K_{3,3}$.