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PROBLEMS (14.03.2012)

VI.1. Describe a homotopy that takes the immersed curve with two self-intersections forming two "outside" little loops shown in Fig.1 to the circle.

VI.2. Describe a homotopy that takes the immersed curve with two self-intersections, one of which is an "outside" little loop and the other is an "inside" little loop, shown in Fig.2 to the circle.

VI.3. Recall that a *simple loop* ω is a part of an immersed curve γ such that:
(1) ω starts and ends at a double point of γ;
(2) ω is not self-intersecting (however, it can intersects other parts of γ).
Prove that any immersed curve with self-intersections has a simple loop.

VI.4. Describe a homotopy that changes the simple loop ω of the immersed curve γ in Fig.3 (and does not change the rest of γ) so that after this homotopy we obtain a new simple loop ω' that does not intersect the rest of γ.

VI.5. Using the previous problems, describe a proof of the Whitney Theorem for immersed curves in the plane.

VI.6. Prove the Whitney Theorem for immersed curves in the sphere.

VI.7. Can an immersed curve with two self-intersections forming two "inside" little loops be regularly homotopic to the circle?

VI.8. Can an immersed curve with 17 self-intersections forming 17 "outside" little loops be regularly homotopic to the circle?

The next two problems yield an analytic proof of the Whitney Theorem in the general case.

VI.9. Prove that there exists a regular homotopy of a curve γ to a curve γ' of length 1 such that the following conditions hold:
- $\gamma'(0) = 0$;
- at the point t=0 the velocity vector of the curve γ' is equal to $(1, 0)$;
- at every point the velocity vector of the curve γ' is of length 1.

VI.10. Let γ_0 and γ_1 be smooth curves such that $w(\gamma_0) = w(\gamma_1) = N$. Suppose that for these curves the conditions of Problem VI.9. are satisfied. Write the velocity vectors of the curves γ_0 and γ_1 in the form $v_0(s) = e^{i\phi_0(s)}$ and $v_1(s) = e^{i\phi_1(s)}$, where $\phi_0(0) = \phi_1(0) = 0$ and $\phi_0(1) = \phi_1(1) = 2\pi N$. Set $\phi_t(s) = (1 - t)\,\phi_0(s) + t\,\phi_1(s)$ and consider the curve $\widetilde{\gamma}_t$ with the velocity vector $v_t(s) = e^{i\phi_t(s)}$:

$$\widetilde{\gamma}_t(s) = \int_0^s e^{i\phi_t(\tau)}\,d\tau.$$

For $t \neq 0,\ 1$ the curve $\widetilde{\gamma}_t$ is not necessary closed. With a help of this curve construct a smooth closed curve with nonzero velocity vectors and conclude the proof of Whitney's theorem. [*Hint.* Consider the curve

$$\gamma_t(s) = \widetilde{\gamma}_t(s) - s\widetilde{\gamma}_t(1) = \int_0^s e^{i\phi_t(\tau)}\,d\tau - s\int_0^1 e^{i\phi_t(\tau)}\,d\tau.].$$

VI.11. For an oriented immersed curve γ indicate a rule for assigning ± 1 to each self-intersection point so that the sum of all such numbers equals $w(\gamma)$.

VI.12. Show that the degree of a point P with respect to an immersed curve γ may be computed by joining P to some point O far away from the curve by a smooth curve transversal to γ and using the "signs" of the intersections of the two curves.