3. π_n AND THE HOPF BUNDLE

Problem 1. (a) Let X be a CW-complex. Prove that $\pi_k(X) = \pi_k(\mathrm{sk}_{k+1}(X))$. (b) Prove that $\pi_k(S^n)$ is trivial for any $k < n$.

Problem 2. Prove that (a) $\pi_1(S^1 \vee S^n) = \mathbb{Z}$, (b) $\pi_n(S^1 \vee S^n)$ is a direct sum of countably many copies of the group $\mathbb{Z}$. (c) Describe the natural action of π_1 on π_n.

Let $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$. The map $p : S^3 \to \mathbb{C}P^1$ given by the formula $p(z, w) = [z : w]$ is called the Hopf bundle.

Problem 3. Prove that the Hopf bundle is indeed a fiber bundle with the fiber S^1. In particular, for any $[z : w] \in \mathbb{C}P^1$ the set $p^{-1}([z : w])$ is homeomorphic to a circle.

Problem 4. (a) Prove that the sets $A = \{[z : w] \in \mathbb{C}P^1 \mid |w/z| \le 1\}$ and $B = \{[z : w] \in \mathbb{C}P^1 \mid |w/z| \ge 1\}$ are homeomorphic to disks with the common boundary $C = \{[z : w] \in \mathbb{C}P^1 \mid |w/z| = 1\}$. Prove that $\mathbb{C}P^1$ is homeomorphic to S^2. (b) Prove that $p^{-1}(A)$ and $p^{-1}(B)$ are homeomorphic to solid tori $S^1 \times D^2$ and $p^{-1}(C)$ is homeomorphic to the torus $S^1 \times S^1$.

Problem 5. (a) Prove that there exists a homeomorphism between the 2-disk D and the set $Q = \{(z, w) \in S^3 \mid w \in [0, 1]\}$ mapping ∂D to $p^{-1}([1 : 0]) \subset Q$. (b) Prove that for any $a \ne [1 : 0]$ the intersection $Q \cap p^{-1}(a)$ contains exactly one point.

$\mathrm{SU}(2)$ is the group of matrices $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ where $|a|^2 + |b^2| = 1$.

Problem 6. (a) Prove that the standard (linear) action of $\mathrm{SU}(2)$ on $\mathbb{C}^2$ maps $S^3 \subset \mathbb{C}^2$ to itself. (b) Prove that for any $A \in \mathrm{SU}(2)$ and $x \in S^3$ one has $p(Ax) = Ap(x)$ where $Ap(x)$ means the standard (projective) action of A on $p(x) \in \mathbb{C}P^1$. (c) Prove that the action of $\mathrm{SU}(2)$ on $\mathbb{C}P^1$ is transitive: for any $u, v \in \mathbb{C}P^1$ there is $A \in \mathrm{SU}(2)$ such that $A(u) = v$. (d) Prove using 6(c) and 5(b) that any two fibers of the Hopf bundle are like two links of an anchor chain: for any $u \in \mathbb{C}P^1$ there exists $D_u \subset S^3$ homeomorphic to the 2-disk D, such that the homeomorphism maps $p^{-1}(u) \subset D_u$ to ∂D and for any $v \ne u$ the intersection $D_u \cap p^{-1}(v)$ is one point.

Let $e_0 \stackrel{\mathrm{def}}{=} \{[1 : 0 : 0]\} \subset \mathbb{C}P^2$, $e_1 \stackrel{\mathrm{def}}{=} \{[x_0 : 1 : 0]\} \subset \mathbb{C}P^2$, $e_2 \stackrel{\mathrm{def}}{=} \{[x_0 : x_1 : 1]\} \subset \mathbb{C}P^2$ be affine charts in $\mathbb{C}P^2$.

Problem 7. (a) Define in $\mathbb{C}P^2$ a cell space where e_0, e_1 and e_2 are cells of dimension 0, 2 and 4, respectively. Prove that $e_0 \cup e_1$ is homeomorphic to S^2. (b) Let $\chi_2 : D^4 \to S^2$ be the characteristic map of the cell e_2. Prove that $\chi_2|_{\partial D^4} : S^3 \to S^2$ is the Hopf bundle.