2. CW-COMPLEXES

Problem 1. Let C be the union of all circles of center $(1/n, 0)$ in the xy-plane and radius $1/n$. Prove that C is not homeomorphic to a CW-space.

Problem 2. (a) Show that the Cartesian product, the cone, the suspension, and the join of finite CW-complexes have natural CW-complexes structures. (b*) Show that it may be not true for a product of infinite CW-complexes.

Problem 3. Prove that any finite CW-space of dimension n can be embedded into $\mathbb{R}^N$ where $N = (n+1)(n+2)/2$.

Problem 4. (a) Prove that a CW-space X is connected $\Leftrightarrow$ it is arcwise connected $\Leftrightarrow$ its 1-skeleton is arcwise connected. (b) Prove that a CW-space is compact $\Leftrightarrow$ it contains finitely many cells.

Problem 5. (a) Prove that $\pi_n(X) = \pi_n(\mathrm{sk}_{n+1}(X))$. (b) Prove that $\pi_k(S^n) = 0$ if $k < n$.

Problem 6. Let S^∞ be a set of sequences $(x_1, x_2, \dots)$ where the terms are real numbers, all of them except a finite number being zero, and $\sum_{i=1}^\infty x_i^2 = 1$. (a) Construct a CW-structure in S^∞. (b) Prove that the identity map $\mathrm{id} : S^\infty \to S^\infty$ is homotopic to the map $\iota : S^\infty \to S^\infty$ defined by the formula $\iota(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$. (c) Prove that S^∞ is contractible.

Problem 7. Find a CW-structure in the classical surfaces: (a) a torus $S^1 \times S^1$, (b) a sphere with g handles, (c) a Klein bottle, (d) a projective plane, (e) a Klein bottle and a projective plane with g handles, (f) all the examples above with n holes cut out.

Problem 8. Find CW-structures in the spaces X and Y such that the map f is cellular: (a) $X = \mathbb{R}^2 \setminus \{0\}$. $f : X \to S^1$ is the standard retraction. (b) $X = \mathbb{R}P^n$, $f : S^n \to X$ is the 2-fold covering. (c) $X = \mathbb{C}P^n$. $f : S^{2n+1} \to \mathbb{C}P^n$ is the generalized Hopf bundle. (d) $X = S^\infty$, $Y = \mathbb{C}P^\infty$ (define it!), $f : S^\infty \to \mathbb{C}P^\infty$ is the infinity-dimensional Hopf bundle. (e) X is the set of unit vectors tangent to the standard S^2, $Y = S^2$, $f : X \to Y$ is the projection mapping a vector to its attachment point. (f) The same as 8(e) but S^2 is replaced with the sphere with g handles.