# PROBLEMS for Lecture 7

**7.1.** Prove that inversion maps circles and straight lines to circles or straight lines.

**7.2.** Prove that inversion maps any circle orthogonal to the circle of inversion into itself.

**7.3.** Prove that inversion is conformal (i.e., it preserves the measure of angles).

**7.4.** Prove that if $P$ is point lying outside a circle $\gamma$ and $A$, $B$ are the intersection points with the circle of a line $l$ passing through $P$, then the product $|PA| \cdot |PB|$ (often called the *power of* $P$ *with respect to* $\gamma$) does not depend on the choice of $l$.

**7.5.** Prove that if $P$ is point lying inside a circle $\gamma$ and $A$, $B$ are the intersection points with the circle of a line $l$ passing through $P$, then the product $|PA| \cdot |PB|$ (often called the *power of* $P$ *with respect to* $\gamma$) does not depend on the choice of $l$.

**7.6.** Prove that inversion with respect to a circle orthogonal to a given circle $\mathcal{C}$ maps the disk bounded by $\mathcal{C}$ bijectively onto itself.

**7.7.** Prove that any Euclidean circle inside the disk model is also a hyperbolic circle. Does the ordinary (Euclidean) center coincide with its "hyperbolic center"?

**7.8.** Study Figure 7.11. Does it demonstrate any tilings of $\mathbb{H}^2$ by regular polygons? Of how many sides? Do you discern a Coxeter geometry in this picture with "hyperbolic Coxeter triangles" as fundamental domains? What are their angles?

**7.9.** Prove that any inversion of $\overline{\mathbb{C}}$ preserves the cross ratio of four points:

$$\langle z_1, z_2, z_3, z_4 \rangle := \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}.$$

**7.10*.** Using complex numbers, invent a formula for the distance between points on the Poincaré disk model and prove that "symmetry with respect to straight lines" (i.e., inversion) preserves this distance.

**7.11.** Prove that hyperbolic geometry is homogeneous in the sense that for any two flags (i.e., half planes with a marked point on the boundary) there exists an isometry taking one flag to the other.

**7.12.** Prove that the hyperbolic plane (as defined via the Poincaré disk model) can be tiled by regular pentagons.

**7.13.** Define inversion (together with the center and the sphere of inversion) in Euclidean space $\mathbb{R}^3$, state and prove its main properties: inversion takes planes and spheres to planes or spheres, any sphere orthogonal to the sphere of inversion to itself, any plane passing through the center of inversion to itself.

**7.14.** Using the previous exercise, prove that any inversion in $\mathbb{R}^3$ takes circles and straight lines to circles or straight lines.

**7.15.** Prove that any inversion in $\mathbb{R}^3$ is conformal (preserves the measure of angles).

**7.16.** Construct a model of hyperbolic space geometry on the open unit ball (use Exercise 7.13).

**7.17.** Prove that there is a unique common perpendicular joining any two nonintersecting lines.

**7.18.** Let $A_\infty P$ and $A_\infty P'$ be two parallel lines (with $A_\infty$ a point on the absolute). Given a point $M$ on $A_\infty P$, we say that $M' \in A_\infty P'$ is the *corresponding point* to $M$ if the angles $A_\infty MM'$ and $A_\infty M'M$ are equal. Prove that any point $M \in A_\infty P$ has a unique corresponding point on the line $A_\infty P'$.

**7.19.** The locus of all points corresponding to a point $M$ on $A_\infty P$ and lying on all the parallels to $A_\infty P$ is known as a *horocycle*. What do horocycles look like in the Poincaré disk model?