PROBLEMS for Lecture 12

12.1 Five distinct collinear points A, B, C, D, E are given. Prove that

$$\langle A, B, C, D\rangle \cdot \langle A, B, D, E\rangle \cdot \langle A, B, E, C\rangle = 1.$$

12.2. How many different values does the cross-ratio of four points on a line take when the order of the points is changed?

12.3. Calculate the cross-ratio of four points $(x_i : y_i; 0)$, $i = 1, 2, 3, 4$ lying on the infinite line Λ_∞.

12.4. Prove Theorem 12.4.3.

12.5. Four planes pass through a common line l, while the line m intersects all four planes. Prove that the cross-ratio of the intersection points of m with the planes does not depend on the choice of m.

12.6. State and prove the theorem dual to the Pappus theorem. Draw the corresponding picture.

12.7. State and prove the theorem dual to Desargues' theorem. Draw the corresponding picture.

12.8*. Prove that under projective duality any point on a conic is taken to a line tangent to the dual conic.

12.9. Using Exercise 12.8, state and prove the theorem dual to Pascal's theorem (the dual theorem is known as *Brianchon's Theorem*). Draw the corresponding picture.

12.10. Three skew lines l, l_1, l_2 in $\mathbb{R}^3$ are given. To a point $A_1 \in l_1$ let us assign the point A_2 at which the line l_2 intersects the plane determined by A_1 and l. Prove that the assignment $A_1 \mapsto A_2$ is a projective map of l_1 onto l_2.

12.11. The lines $l_1, \dots l_{n-1}$ and l are given on the plane. The points $O_1, \dots, O_n$ are chosen on l. The lines containing the sides of a polygon $A_1, \dots, A_n$ pass through the points $O_1, \dots, O_n$ while its vertices $A_1, \dots, A_{n-1}$ move along the lines $l_1, \dots l_{n-1}$. Prove that the vertex A_n also moves along a straight line.

12.12. Compute the cross-ratios of the quadruple of points A, B, C, D in the figure below.

12.13. Prove the triangle inequality for the hyperbolic metric by using appropriate projective transformations.

12.14. Prove the Euclidean version of Pascal's theorem for the case of the circle.