Instructions.

- Only students who want a grade for the course should submit solutions to this exam. The exam due date and time is above.
- Tell me by December 12th if you plan to take this exam, so I know who I will be receiving solutions from later.
- Do not discuss your work on the exam with anyone else. You may write to me if you have questions. My email address is `kconrad@math.uconn.edu`.
- If you will write your work by hand, make sure to write *neatly* (I do not want to read terrible handwriting). Give the final version to Alexey Zykin at the end of his algebraic number theory lecture (IUM, room 310, 21:00). He will email a copy of your solutions to me and I will write to you after I receive them.
- If you will type your work, email the final version to me as a .pdf file. (Use the website `http://www.pdfonline.com` if you can't convert your file into .pdf format by yourself.) I will confirm by email that I received the file, so if you do not hear back from me then it means I did not get anything from you.
- Good luck!

1. (Computations)

   (a) Let $x \in \mathbf{Z}_5$ be the solution of the equation $x^2 = -1$ such that $x \equiv 2 \bmod 5\mathbf{Z}_5$. Compute $\{x/25\}_5$.

   (b) Let $\mathbf{a} = (3, -1/3, -1/3, 1, 1, \dots) \in \mathbf{A}_{\mathbf{Q}}$, where the unwritten terms are all 1. Compute $\Psi(\mathbf{a})$, where Ψ is the standard character on $\mathbf{A}_{\mathbf{Q}}$.

   (c) Let $\mathbf{b} = (10, 20, 30, 40, 1, 1, \dots) \in J_{\mathbf{Q}}$, where the unwritten terms are all 1. Using the isomorphism $J_{\mathbf{Q}} \cong \mathbf{Q}^\times \times \mathbf{R}_{>0} \times \prod_p \mathbf{Z}_p^\times$, write $\mathbf{b}$ in the form $qt\mathbf{u}$, where $q \in \mathbf{Q}^\times$, $t > 0$, and $\mathbf{u} \in \prod_p \mathbf{Z}_p^\times$.

2. Let G be a locally compact abelian group, H be a compact open subgroup, and let $f \in L^1(G)$. Fix a Haar measure μ on G to define the Fourier transform $\widehat{f}\colon \widehat{G} \to \mathbf{C}$.

   a) If $f(g) = 0$ for all $g \notin H$, show $\widehat{f}\colon \widehat{G} \to \mathbf{C}$ is constant on $H^\perp$-cosets in $\widehat{G}$: $\widehat{f}(\chi\psi) = \widehat{f}(\chi)$ if $\psi \in H^\perp$.

   b) If f is constant on H-cosets in G (that is, $f(gh) = f(g)$ for all $h \in H$), show $\widehat{f}(\chi) = 0$ for all $\chi \notin H^\perp$.

   (Hint for part b: Write $\widehat{f}(\chi)$ as an iterated integral over H and G/H using Weil's formula.)

3. Show $\mathbf{A}_{\mathbf{Q}}$ and $J_{\mathbf{Q}}$ are both σ-compact, *i.e.*, they can each be written as a countable union of compact subsets.

4. Let ψ be the standard character on $\mathbf{Q}_p$ and dx be the standard Haar measure on $\mathbf{Q}_p$.

   a) For $n \in \mathbf{Z}$, show

$$\int_{|x|_p = 1/p^n} \psi(xy)\, dx = \begin{cases} 1/p^n - 1/p^{n+1}, & \text{if } y \in (1/p^n)\mathbf{Z}_p, \\ -1/p^{n+1}, & \text{if } y \in (1/p^{n+1})\mathbf{Z}_p - (1/p^n)\mathbf{Z}_p, \\ 0, & \text{otherwise.} \end{cases}$$

   (Hint: Write the integral as a difference $\int_{p^n\mathbf{Z}_p} - \int_{p^{n+1}\mathbf{Z}_p}$.)

b) For $x \in \mathbf{Q}_p$, set $f(x) = |x|_p \xi_{\mathbf{Z}_p}(x)$. Show

$$\widehat{f}(y) = \begin{cases} \frac{1}{1+1/p}, & \text{if } y \in \mathbf{Z}_p, \\ -\frac{1}{|y|_p^2} \frac{p}{1+1/p}, & \text{if } y \notin \mathbf{Z}_p. \end{cases}$$

5. Let p be a prime.

   a) In the topological group $\mathbf{R} \times \mathbf{Q}_p$, show $\mathbf{Z}$ (embedded diagonally) is discrete but not co-compact and $\mathbf{Z}[1/p]$ (embedded diagonally) is both discrete and co-compact. Here $\mathbf{Z}[1/p] = \{a/p^n : a \in \mathbf{Z}, n \geq 0\}$ is the set of fractions with p-power denominator.

   b) Show counting measure on $\mathbf{Z}[1/p]$, the Haar measure $dx \times dx_p$ on $\mathbf{R} \times \mathbf{Q}_p$ where dx_p is the Haar measure on $\mathbf{Q}_p$ which assigns $\mathbf{Z}_p$ measure 1, and the normalized Haar measure on $(\mathbf{R} \times \mathbf{Q}_p)/\mathbf{Z}[1/p]$ are Weil compatible.

6. Let p be a prime. The group $\mathbf{R} \times \mathbf{Q}_p$ is self-dual. For $(x, y) \in \mathbf{R} \times \mathbf{Q}_p$, define $\chi_{(x,y)} \in \widehat{\mathbf{R} \times \mathbf{Q}_p}$ by $\chi_{(x,y)}(u, v) = e^{-2\pi i x u} e^{2\pi i \{yv\}_p}$. (Note the minus sign!) This is a self-duality on $\mathbf{R} \times \mathbf{Q}_p$.

   a) For every $t \in \mathbf{Z}[1/p]$, show $t = \{t\}_p$ in $\mathbf{Q}/\mathbf{Z}$.

   b) Relative to the self-duality of $\mathbf{R} \times \mathbf{Q}_p$ described above, show $\mathbf{Z}[1/p]^\perp = \mathbf{Z}[1/p]$, where $\mathbf{Z}[1/p]$ is viewed inside $\mathbf{R} \times \mathbf{Q}_p$ diagonally.

7. (Bonus) Using the ideas from the last two questions, for any finite subset S in $V_{\mathbf{Q}}$ such that $\infty \in S$, find an example of a lattice L (that is, a discrete and co-compact subgroup) inside $\prod_{v \in S} \mathbf{Q}_v$ such that $L^\perp = L$ relative to a suitable self-duality on the group $\prod_{v \in S} \mathbf{Q}_v$. Can you also find a lattice L such that $L^\perp = L$ in the group which is constructed like the adeles where one factor $\mathbf{Q}_p$ (p a prime) is not used?