The Beautiful Geometry of Discrete Painlevé Equations

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Outline

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Differential Painlevé Equations

Painlevé equations are second-order algebraic differential equations satisfying the Painlevé Property: the general solution of the equation is free of movable (i.e., dependent on the constants of integration) critical points where it loses local single-valuedness (e.g., branch points like $\sqrt{x - c}$) — i.e., uniformizability of a general solution.

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- R. Fuchs, L. Schlesinger, and R. Garnier (1907–12) — relationship to Isomonodromic Deformations of Fuchsian systems.

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Above equations are equations with constant coefficients. When coefficients of an ODE are polynomial (or, more generally, analytic) functions in the independent variable t we get such important special functions of mathematical physics as the Gauss Hypergeometric functions, Kummer functions, Hermite functions and Hermite polynomials, Bessel functions, Airy functions, and many others.

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In certain sense, the Painlevé property is an attempt to single out the equations that have a meaningful notion of a general solution and the associated Riemann surface — integrability.

Classification Scheme for Painlevé Equations

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n=1: L. Fuchs, H. Poincaré

- $\left(\frac{\mathrm{dy}}{\mathrm{dt}}\right)^2 = 4\mathrm{y}^3 - \mathrm{g}_2\mathrm{y} - \mathrm{g}_3, \qquad \mathrm{g}_2, \mathrm{g}_3 \in \mathbb{C}$ Weierstrass $\wp(\mathrm{t}|\mathrm{g}_2, \mathrm{g}_3)$
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- d-$\mathrm{P_V}$: $\begin{cases} \mathrm{f + \bar{f} = a_3 + \dfrac{a_1}{g+1} + \dfrac{a_0}{sg+1}} \\ \mathrm{g\bar{g} = \dfrac{(\bar{f} + a_2 - \delta)(\bar{f} + a_2 - \delta + a_4)}{s\bar{f}(\bar{f} - a_3)}} \end{cases}$, $\qquad \delta = \mathrm{a_0 + a_1 + 2a_2 + a_3 + a_4}$

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- d-$\mathrm{P_{II}}$: $\mathrm{x_{n+1} + x_{n-1} = \dfrac{z_n x_n + a}{1 - x_n^2}}$;
- d-$\mathrm{P_V}$: $$\begin{cases} \mathrm{f + \bar{f} = a_3 + \dfrac{a_1}{g+1} + \dfrac{a_0}{sg+1}} \\ \mathrm{g\bar{g} = \dfrac{(\bar{f} + a_2 - \delta)(\bar{f} + a_2 - \delta + a_4)}{s\bar{f}(\bar{f} - a_3)}} \end{cases}, \qquad \delta = \mathrm{a_0 + a_1 + 2a_2 + a_3 + a_4}$$
- q-$\mathrm{P_{VI}}$: $$\begin{cases} \mathrm{\bar{g} = \dfrac{b_3 b_4}{g} \dfrac{(f - b_5)(f - b_6)}{(f - b_7)(f - b_8)}} \\ \mathrm{\bar{f} = \dfrac{b_7 b_8}{f} \dfrac{(\bar{g} - qb_1)(\bar{g} - qb_2)}{(\bar{g} - b_3)(\bar{g} - b_4)}} \end{cases}, \qquad \mathrm{q = (b_3 b_4 b_5 b_6)/(b_1 b_2 b_7 b_8)}$$

What are Discrete Painlevé Equations

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As with the differential Painlevé equations, it is not obvious that a given recurrence relation is in the discrete Painlevé class. The naming convention, based on the continuous limit, is also not a very good one – ambiguous and does not cover all the cases. Correct approach is through the algebro-geometric theory due to H. Sakai.
Analogue of the Painlevé property — singularity confinement.

Classification Scheme for Discrete Painlevé Equations

In 2001 H. Sakai, developing the ideas of K. Okamoto in the differential case, proposed a classification scheme for Painlevé equations based on algebraic geometry. To each equation corresponds a pair of orthogonal sub-lattices $(\Pi(\mathrm{R}), \Pi(\mathrm{R}^{\perp}))$ — the surface and the symmetry sub-lattice in the $\mathrm{E}_8^{(1)}$ lattice, and a translation element in $\tilde{\mathrm{W}}(\mathrm{R}^{\perp})$.

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$$\begin{tikzcd}
\left(E_8^{(1)}\right)^e \arrow[ddd] & & & & & & & \text{q-}P_{\mathrm{I}} & \\
& & & & & & & \left(A_1^{(1)}\right)^q_{|\alpha|^2=8} \arrow[ddr] & \\
& & & \text{q-}P_{\mathrm{VI}} & \text{q-}P_{\mathrm{V}} & \text{q-}P_{\mathrm{IV}},\,\text{q-}P_{\mathrm{III}} & & & \\
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\left(E_8^{(1)}\right)^\delta \arrow[r] & \left(E_7^{(1)}\right)^\delta \arrow[r] & \left(E_6^{(1)}\right)^\delta \arrow[r] & \left(D_4^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_3^{(1)}\right)^{c,\delta} \arrow[r]\arrow[ddr] & \left((2A_1)^{(1)}\right)^{c,\delta} \arrow[r]\arrow[ddr] & \left(A_1^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_0^{(1)}\right)^c \arrow[dd] & \\
& & & P_{\mathrm{VI}},\,\text{d-}P_{\mathrm{V}} & \begin{array}{c} P_{\mathrm{V}},\,\text{d-}P_{\mathrm{IV}} \\ \text{d-}P_{\mathrm{III}} \end{array} & \begin{array}{c} P_{\mathrm{III}} \\ \text{alt. d-}P_{\mathrm{II}} \end{array} & & & \\
& & & & & \left(A_2^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_1^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_0^{(1)}\right)^c & \\
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Symmetry-type classification scheme for Painlevé equations

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Symmetry-type classification scheme for Painlevé equations

One of the objectives of my talk today is to explain the main ingredients of this scheme.

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Symmetry-type classification scheme for Painlevé equations

One of the objectives of my talk today is to explain the main ingredients of this scheme.

But first, an example of applications.

An Example: Statistics of the Longest Increasing Subsequences in Permutations

Let $\mathcal{S}_{\mathrm{N}}$ be the usual permutation group and let $\pi \in \mathcal{S}_{\mathrm{n}}$. Let $\mathrm{l}_{\mathrm{n}}(\pi)$ be the length of the maximal increasing subsequence in π.

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Example:

If $\pi = 7, 2, 8, 1, 3, 4, 10, 6, 9, 5$, then $\mathrm{l}_{10}(\pi) =$?.

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Let $\mathrm{L}_\mathrm{n} := \mathrm{l}_\mathrm{n}(\pi)$ be the corresponding random variable on $\mathcal{S}_\mathrm{n}$ equipped with the uniform probability distribution. Define

$$\mathrm{p}^\mathrm{n}_\mathrm{k} := \mathrm{P}(\mathrm{L}_\mathrm{n} \leq \mathrm{k}) = \frac{\mathsf{Card}(\pi \in \mathcal{S}_\mathrm{n} | \mathrm{l}_\mathrm{n}(\pi) \leq \mathrm{k})}{\mathrm{n}!}.$$

What is the behavior of $\mathrm{p}^\mathrm{n}_\mathrm{k}$ as $\mathrm{n} \to \infty$?
In particular, what is $\mathbb{E}(\mathrm{L}_\mathrm{n})$, $\sigma(\mathrm{L}_\mathrm{n})$ as $\mathrm{n} \to \infty$?

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Theorem (Vershik-Kerov; Pilpel, Logan-Shepp; Ulam)

$$\mathbb{E}(\mathrm{L}_\mathrm{n}) \sim 2\sqrt{\mathrm{n}},$$
$$\sigma(\mathrm{L}_\mathrm{n}) \sim \mathrm{o}(\sqrt{\mathrm{n}})$$

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where $\mu_\infty = 1.711...$, $\sigma_\infty = 0.902...$.

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$$P\left(\frac{L_n - 2n^{1/2}}{n^{1/6}} \leq t\right) \to F(t) \qquad \text{as } n \to \infty, \quad -\infty < t < \infty,$$

where

$$F(t) = \exp\left(-\int_t^\infty (x-t)u(x)^2\,dx\right),$$

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Solutions of Painlevé equations are called Painlevé transcendents. They are purely nonlinear special functions.

Painlevé transcendents appear in a wide range of important problems in pure and applied mathematics and mathematical physics (from WDVV equations and quantum cohomology to asymptotics of nonlinear waves and in a wide range of statistical and probabilistic models as above). In particular, Fredholm Determinants describing certain eigenvalue statistics of Random Matrix Models satisfy Painlevé equations (C. Tracy, H. Widom), this enables computation of asymptotics of such statistics.

Discretization and the Borodin-Okounkov-Olshanski Theorem

Theorem (Borodin; B-Okounkov-Olshanski)

Let us consider the poissonization of p_k^n:

$$p_k^{(\eta)} := e^{-\eta^2} \sum_{n=0}^{\infty} \frac{\eta^{2n}}{n!} p_k^n = e^{-\eta^2} \operatorname{det}[f_{i-j}]_{i,j=1}^{k}, \qquad \text{where} \sum_{m=-\infty}^{\infty} f_m \zeta_m = e^{\eta(\zeta+\zeta^{-1})}$$

$$= e^{-\eta^2} \sum_{\lambda_1 \leq k} \left(\frac{\operatorname{dim} \lambda}{|\lambda|!} \eta^{|\lambda|} \right)^2 = \operatorname{det} \left(1 - K\Big|_{\{k+1,k+2,\dots\}} \right).$$

Then

$$\frac{p_{k+1}^{(\eta)} p_{k-1}^{(\eta)}}{(p_k^{(\eta)})^2} = 1 - x_k^2,$$

where

$$x_{n+1} + x_{n-1} = \frac{n x_n}{\eta(x_n^2 - 1)}, n \geq 1, x_0 = -1, x_1 = \frac{f_1}{f_0}.$$

This last equation on x_n is known as the discrete Painlevé II.

The Second Painlevé Equation P_{II}

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Hamiltonian Form of $\mathrm{P_{II}}$

Hamiltonian system form: put $q = y$ and $p = y' + y^2 + t/2$:

$$\mathrm{H_{II}(b)}: \qquad \begin{cases} q' = p - q^2 - \dfrac{t}{2} = \dfrac{\partial \mathcal{H}}{\partial p} \\ p' = 2qp + b = -\dfrac{\partial \mathcal{H}}{\partial q} \end{cases}$$

where $\mathcal{H} = \mathcal{H}_{\mathrm{II}}(q, p, t; b) = \frac{1}{2}p(p - 2q^2 - t) - bq$ is the Painlevé-II Hamiltonian.

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Note that the Hamiltonian is time-dependent — Painlevé equations are non-autonomous.

Compactification from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{P}^1 \times \mathbb{P}^1$

Originally $\mathrm{H}_{\mathrm{II}}(\mathrm{b})$ is defined on $\mathbb{C} \times \mathbb{C}$. Following K. Okamoto, we extend it to $\mathbb{P}^1 \times \mathbb{P}^1$ using $\mathrm{Q} = 1/\mathrm{q}$ and $\mathrm{P} = 1/\mathrm{p}$ to get 4 charts (q, p), (Q, p), (q, P), and (Q, P):

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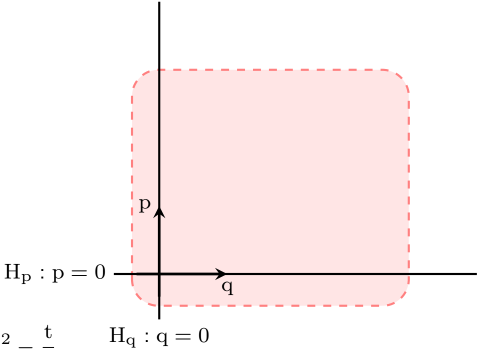

$$\begin{cases} \mathrm{q}' = \mathrm{p} - \mathrm{q}^2 - \dfrac{\mathrm{t}}{2} \\ \mathrm{p}' = 2\mathrm{qp} + \mathrm{b} \end{cases}$$

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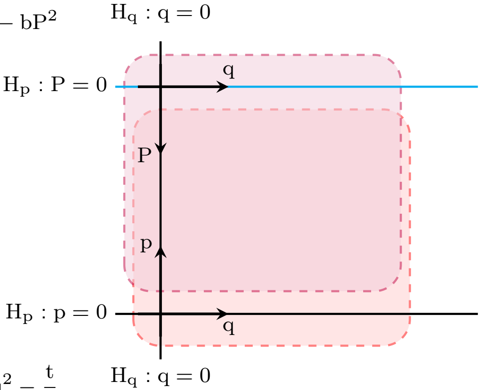

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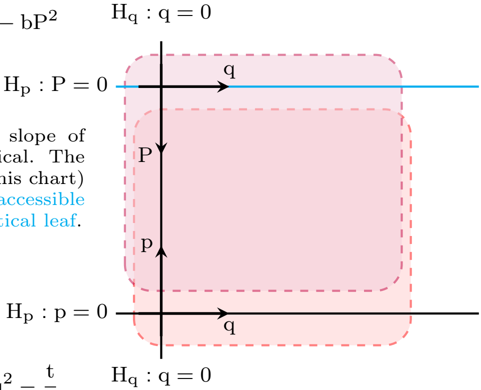

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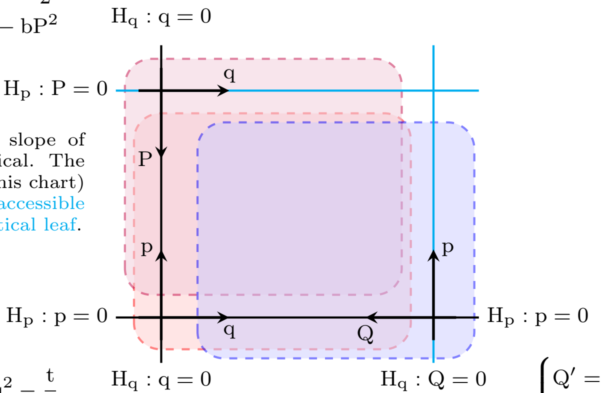

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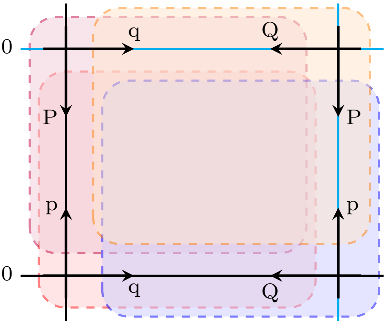

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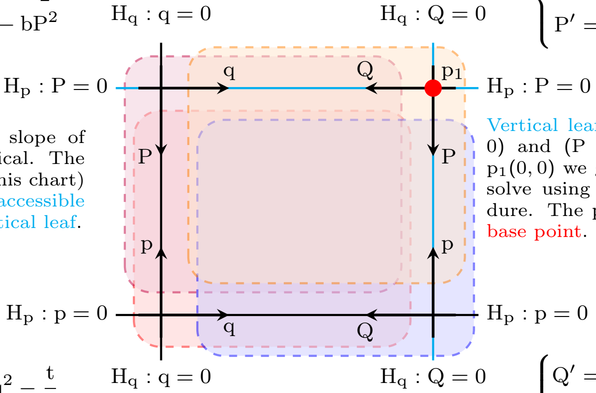

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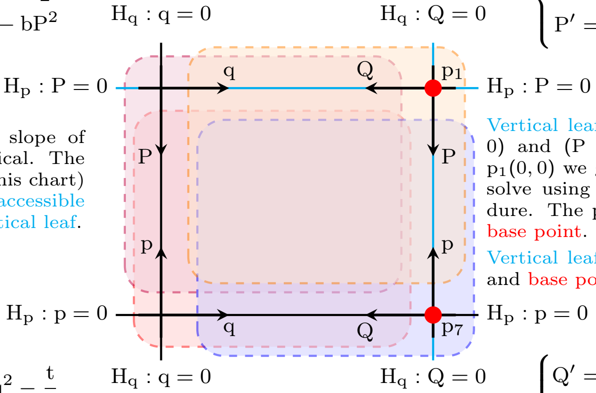

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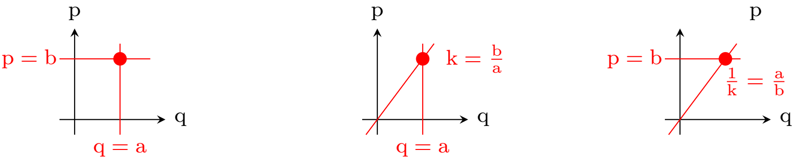

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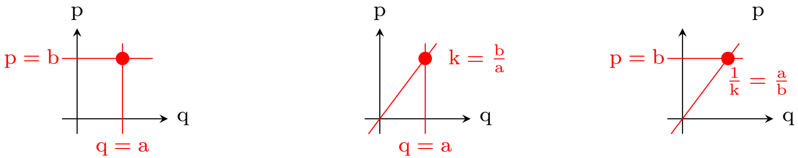

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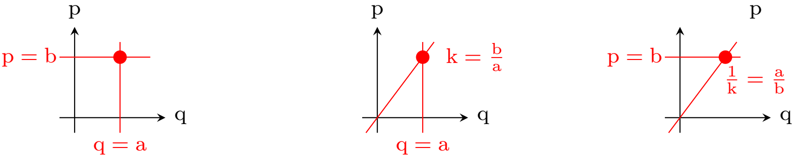

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- Then consider, in the space $\mathbb{C}^2 \times \mathbb{P}^1$ with coordinates $(q, p; [\xi_0 : \xi_1])$, the set $\mathcal{S}$ cut out by the equation $q\xi_0 = p\xi_1$.

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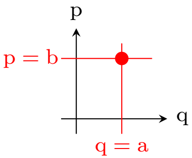

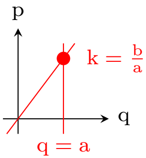

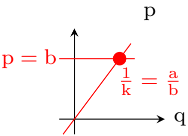

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- In view of the above, for $(q, p) \neq (0, 0)$, the restriction of the projection $\pi : \mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ on $\mathcal{S}$ is an isomorphism, but $\pi^{-1}(0, 0) \simeq \mathbb{P}^1$. It is called the exceptional divisor and is denoted by E.

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The set $\mathrm{S} = \mathrm{V}(\mathrm{q}\xi_0 - \mathrm{p}\xi_1)$ is covered by two charts (u, v) and (U, V). For a blowup with the center at $(\mathrm{q}_0, \mathrm{p}_0)$ these charts are $(\mathrm{q}, \mathrm{p}, [\xi_0 : \xi_1]) = (\mathrm{u} + \mathrm{q}_0, \mathrm{uv} + \mathrm{p}_0, [\mathrm{u} : 1])$ and $(\mathrm{q}, \mathrm{p}, [\xi_0 : \xi_1]) = (\mathrm{UV} + \mathrm{q}_0, \mathrm{V} + \mathrm{p}_0, [1 : \mathrm{V}])$.

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Algebraically blowup is a coordinate substitution $\mathrm{q} = \mathrm{u} + \mathrm{q}_0 = \mathrm{UV} + \mathrm{q}_0$ and $\mathrm{p} = \mathrm{uv} + \mathrm{p}_0 = \mathrm{V} + \mathrm{p}_0$.

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Algebraically blowup is a coordinate substitution $q = u + q_0 = UV + q_0$ and $p = uv + p_0 = V + p_0$.

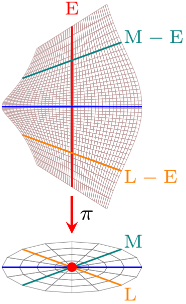
$$L \bullet M = 1$$

$$(L - E) \bullet (M - E) = 0$$

$$E \bullet E = -1$$

If $L^2 = L \bullet L = m$ then

$$(L - E)^2 = m - 1$$



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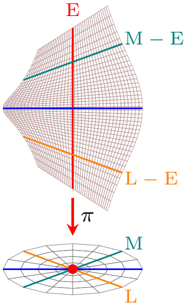
$$\mathrm{L} \bullet \mathrm{M} = 1$$

$$(\mathrm{L} - \mathrm{E}) \bullet (\mathrm{M} - \mathrm{E}) = 0$$

$$\mathrm{E} \bullet \mathrm{E} = -1$$

If $\mathrm{L}^2 = \mathrm{L} \bullet \mathrm{L} = \mathrm{m}$ then

$$(\mathrm{L} - \mathrm{E})^2 = \mathrm{m} - 1$$



Note that we need to distinguish the total transform $\pi^{-1}(\mathrm{L})$ and the proper transform $\overline{\pi^{-1}(\mathrm{L} - (0,0))}$ that we denote by L − E. Exceptional divisor has the self-intersection $\mathrm{E}^2 = -1$. Such curves are called −1-curves.

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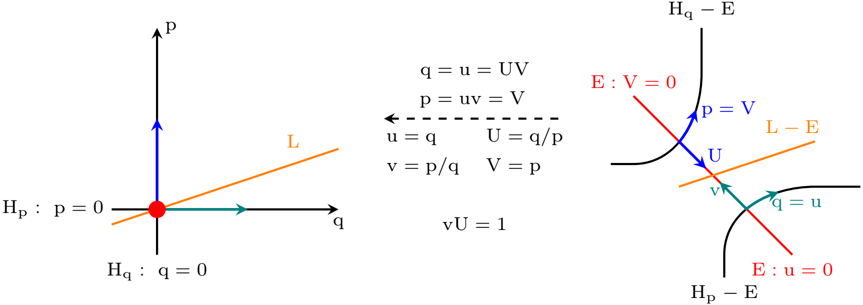

$$q = u = UV$$
$$p = uv = V$$

$u = q \qquad U = q/p$

$v = p/q \qquad V = p$

$$vU = 1$$

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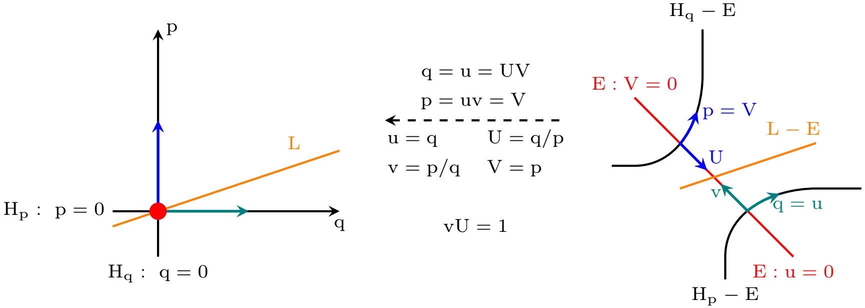

Note the proper transform notation and coordinates on E:

- E and $H_q - E$ intersect at $(U = 0, V = 0)$;
- E and $H_p - E$ intersect at $(u = 0, v = 0)$;
- if the line L had a slope 1/3, E and L − E intersect at $(u = 0, v = 1/3)$ or $(U = 3, V = 0)$.

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$$\begin{cases} Q' = 1 - Q^2p + \dfrac{t}{2}Q^2 = u_7' = 1 - (u_7)^3v_7 + \dfrac{t}{2}(u_7)^2 \\[2ex] p' = \dfrac{2p}{Q} + b = u_7'v_7 + u_7v_7' = \dfrac{2u_7v_7}{u_7} + b \end{cases} \Longrightarrow \begin{cases} u_7' = 1 - (u_7)^3v_7 + \dfrac{t}{2}(u_7)^2 \\[2ex] v_7' = \dfrac{v_7 + b}{u_7} + (u_7)^2(v_7)^2 - \dfrac{t}{2}(u_7v_7) \end{cases}$$

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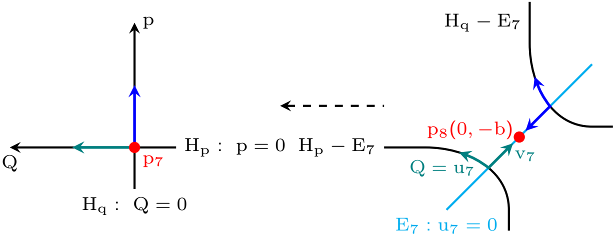

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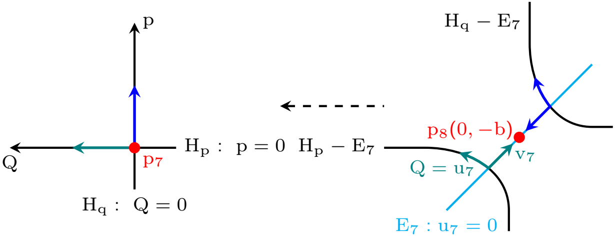
We are now ready to resolve the base points of P_{II} using the blowup procedure.
For example, recall that in coordinates (Q, p) we had a base point $p_7(0, 0)$. Blowing it up we get the charts (u_7, v_7) and (U_7, V_7) with $Q = u_7 = U_7V_7$ and $p = u_7v_7 = V_7$. Extending to the (u_7, v_7) chart (computations in chart (U_7, V_7) are similar) gives the following system.

$$\begin{cases} Q' = 1 - Q^2p + \frac{t}{2}Q^2 = u_7' = 1 - (u_7)^3v_7 + \frac{t}{2}(u_7)^2 \\ p' = \frac{2p}{Q} + b = u_7'v_7 + u_7v_7' = \frac{2u_7v_7}{u_7} + b \end{cases} \Longrightarrow \begin{cases} u_7' = 1 - (u_7)^3v_7 + \frac{t}{2}(u_7)^2 \\ v_7' = \frac{v_7 + b}{u_7} + (u_7)^2(v_7)^2 - \frac{t}{2}(u_7v_7) \end{cases}$$



For $u_7 = 0$ (i.e., on the exceptional curve E_7) we get a vertical leaf except when $v_7 = -b$ at which point v_7' is indeterminate. So we get a new base point $p_8(0, -b)$ in this chart. Blowing it up and taking the proper transform of E_7 gives us vertical leaf $\mathcal{D}_6 = E_7 - E_8$ of self-intersection -2 and the computation in (u_8, v_8) and (U_8, V_8) charts shows that there are no new base points.

The Space of Initial Conditions for P_{II}

Applying the blowup to the base points $\mathrm{p}_{\mathrm{i}} \in \mathbb{P}^1 \times \mathbb{P}^1$, extending to the new charts $(\mathrm{u}_{\mathrm{i}}, \mathrm{v}_{\mathrm{i}})$ and $(\mathrm{U}_{\mathrm{i}}, \mathrm{V}_{\mathrm{i}})$, checking new exceptional divisors E_{i} for base points and blowing them up until everything is resolved, and finally removing the vertical leaves, we get the surface X that is called the Okamoto space of Initial Conditions for P_{II}. For all Painlevé equations, X is a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 8 points (or $\mathbb{P}^2$ at 9 points), with the configuration of the removed vertical leaves $\mathcal{D}_{\mathrm{i}}$ essentially characterizing the equation. For P_{II} we get the following.

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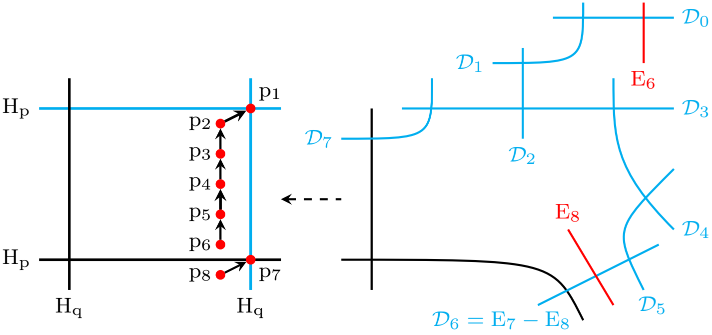

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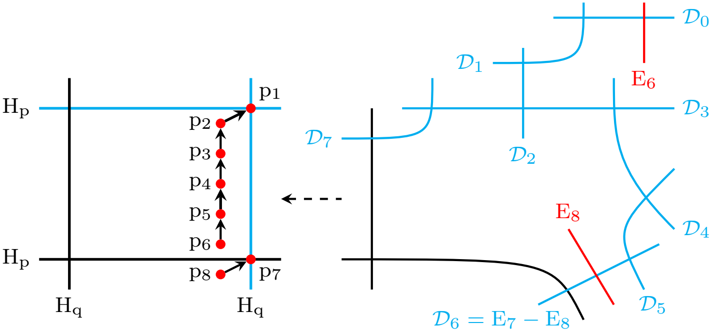

$$p_1\,(Q = 0, P = 0) \leftarrow p_2\left(u_1 = Q = 0, v_1 = \frac{P}{Q} = 0\right) \leftarrow p_3\left(u_2 = u_1 = 0, v_2 = \frac{u_1}{v_1} = \frac{1}{2}\right) \leftarrow p_4\,\Bigg(u_3 = u_2 = 0,$$

$$v_3 = \frac{v_2 - 1/2}{u_2} = 0\Bigg) \leftarrow p_5\left(u_4 = u_3 = 0, v_4 = \frac{v_3}{u_3} = -\frac{t}{4}\right) \leftarrow p_6\left(u_5 = u_4 = 0, v_5 = \frac{v_4 + t/4}{u_4} = 0\right).$$

The Space of Initial Conditions for P_{II}: the Dynkin diagram

Note that all vertical leaves are -2-curves. This configuration can be described by a Dynkin diagram where nodes are -2-curves and connected nodes correspond to intersecting curves.

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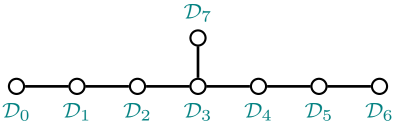

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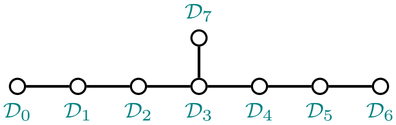

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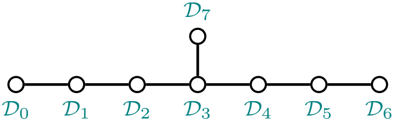

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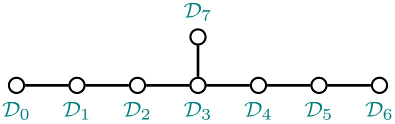

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For other Painlevé equations, we get:

$$\mathrm{P}_{\mathrm{III}} = \mathrm{P}(\mathrm{D}_6^{(1)}), \quad \mathrm{P}'_{\mathrm{III}} = \mathrm{P}(\mathrm{D}_7^{(1)}), \quad \mathrm{P}''_{\mathrm{III}} = \mathrm{P}(\mathrm{D}_8^{(1)}), \quad \mathrm{P}_{\mathrm{IV}} = \mathrm{P}(\mathrm{E}_6^{(1)}), \quad \mathrm{P}_{\mathrm{V}} = \mathrm{P}(\mathrm{D}_5^{(1)}), \quad \mathrm{P}_{\mathrm{VI}} =$$

Bäcklund Transformations of $\mathrm{P_{II}}$

Bäcklund Transformation is a symmetry of the family $\mathrm{P_{II}}(\mathrm{b})$. That is, it preserves the form of the equation, but can change the value of the parameter b.

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$$\mathrm{P_{II}(b)}: \quad y'' = 2y^3 + ty + (b - 1/2) \qquad \mathrm{H_{II}(b)}: \quad \begin{cases} q' = p - q^2 - \dfrac{t}{2} = \dfrac{\partial \mathcal{H}}{\partial p} \\ p' = 2qp + b = -\dfrac{\partial \mathcal{H}}{\partial q} \end{cases}$$

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$$s : (q, p, t; b) \to \left(\tilde{q} = q + b/p, \tilde{p} = p, \tilde{t} = t; \tilde{b} = -b\right)$$

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It is easy to verify that both s and r are Bäcklund transformations,

$$s : \mathrm{P_{II}(b)} \mapsto \mathrm{P_{II}(-b)} \qquad y \mapsto \tilde{y} = y + \frac{b}{y' + y^2 + t/2},$$

$$r : \mathrm{P_{II}(b)} \mapsto \mathrm{P_{II}(1 - b)} \qquad y \mapsto \tilde{y} = -y.$$

Bäcklund Transformations and Reflections

Note that each of the Bäcklund transformations r and s is an involution, $\mathrm{s}^2 = \mathrm{r}^2 = \mathrm{e}$. In fact, their actions on the parameter b is a reflection about $\mathrm{b} = 0$ for $\mathrm{s} : \mathrm{b} \mapsto -\mathrm{b}$ and a reflection about $\mathrm{b} = 1/2$ for $\mathrm{r} : \mathrm{b} \mapsto 1 - \mathrm{b}$:

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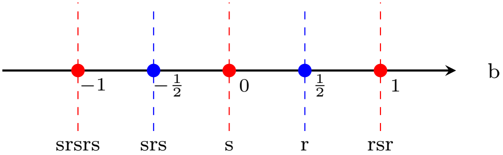

Action of Bäcklund transformations on the parameter space

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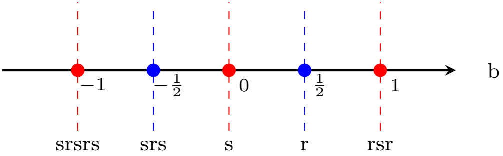

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The reflections s and r generate the affine Weyl group $W(A_1^{(1)})$ and adding an involution $\sigma = (\mathrm{sr})$ switching the mirrors generates an extended affine Weyl group $\widetilde{W}(A_1^{(1)})$:

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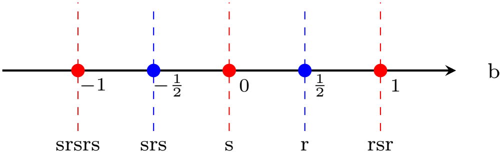

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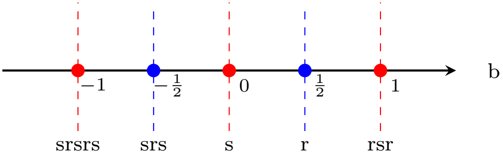

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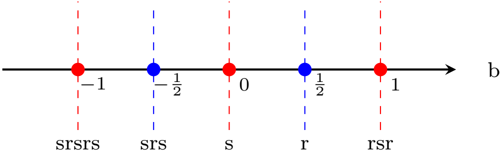

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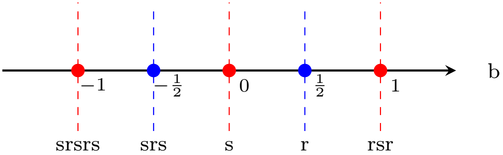

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- $H_{II}(0)$: $p' = 2qp$ has a solution $p = 0$, and then $q' = -q^2 - t/2$ is a Riccati equation. Setting $q = u'/u$ reduces it to the Airy equation $u'' + (t/2)u = 0$. If φ_0 and φ_1 are two fundamental solutions of the Airy equation, we get a one-parameter family of solutions $(q, p, b) = (\frac{c_0\varphi_0' + c_1\varphi_1'}{c_0\varphi_0 + c_1\varphi_1}, 0, 0)$.

Difference Painlevé Equations

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Composing basic Bäcklund transformations acting on the parameter space results in translations $\mathrm{T}_1 = \mathrm{r} \circ \mathrm{s} : \mathrm{b} \mapsto \mathrm{b} + 1$ and $\mathrm{T}_{-1} = \mathrm{s} \circ \mathrm{r} : \mathrm{b} \mapsto \mathrm{b} - 1$. This is a time step in the independent variable, whereas the resulting dynamic on the phase space of (q, p) variables is known as a discrete Painlevé equation alt. d-P_{I}.

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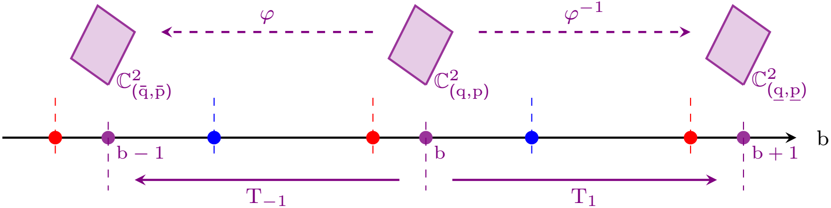

$$T_{-1} : \begin{cases} \bar{q} = -q + \dfrac{1-b}{2q^2 - p + t} \\ \bar{p} = 2q^2 - p + t \\ \bar{b} = b - 1 \end{cases} \qquad T_1 : \begin{cases} \underline{q} = -\dfrac{b}{p} - q \\ \underline{p} = 2\left(q + \dfrac{b}{p}\right)^2 - p + t \\ \underline{b} = b + 1 \end{cases}$$

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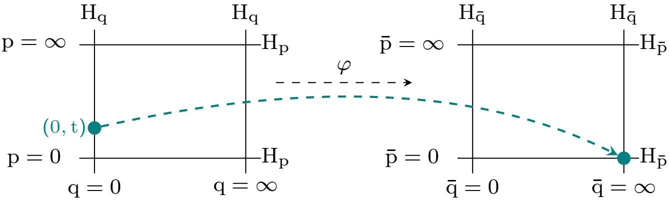

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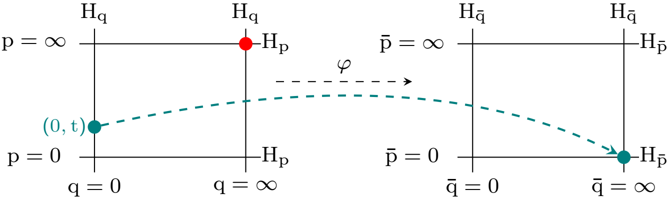

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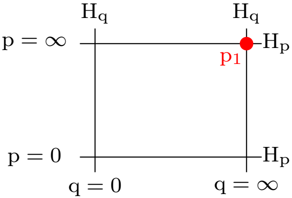

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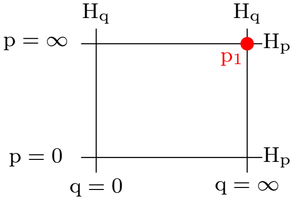

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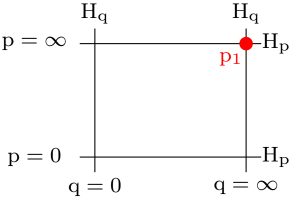

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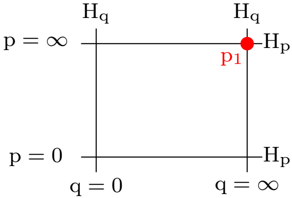

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Resolve it using the blowup procedure. In the blowup coordinates (u_1, v_1), $Q = u_1$, $P = u_1v_1$:

$$\begin{cases} \bar{q} = \dfrac{u_1^3v_1\left((1-b)u_1 - t\right) + u_1^2 - 2u_1v_1}{u_1(2u_1v_1 - u_1^2 + tu_1^3v_1)} = \dfrac{u_1^2v_1\left((1-b)u_1 - t\right) + u_1 - 2v_1}{u_1(2v_1 - u_1 + tu_1^2v_1)} = \dfrac{-2v_1}{0} = \infty, \\ \bar{p} = \dfrac{2u_1v_1 - u_1^2 + tu_1^3v_1}{u_1^3v_1} = \dfrac{2v_1 - u_1 + tu_1^2v_1}{u_1^2} = \dfrac{2v_1}{0} = \infty \qquad \text{on } E_1\text{: } u_1 = 0\,. \end{cases}$$

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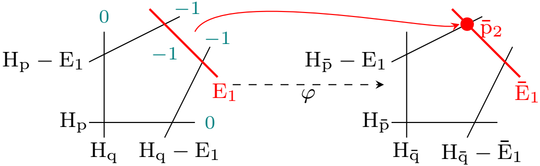

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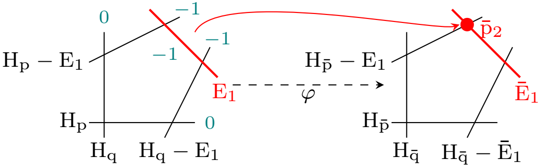

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Note that

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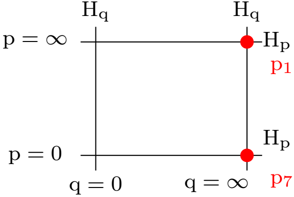

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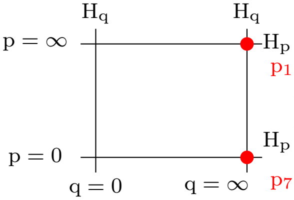

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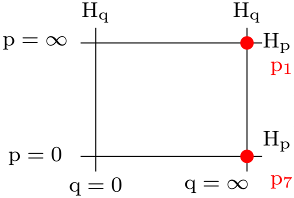

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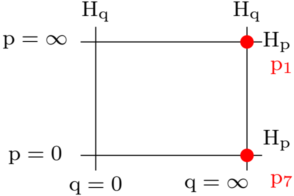

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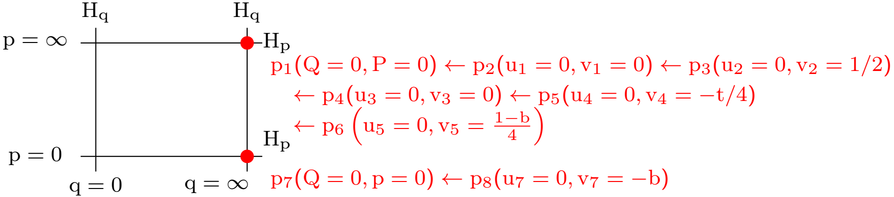

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$$-\mathcal{K}_{\mathcal{X}} = 2\mathcal{H}_f + 2\mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8 = \sum_i m_i \mathcal{D}_i.$$

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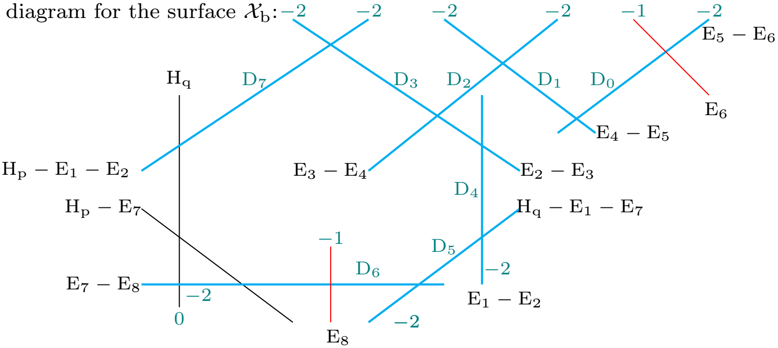

The Okamoto Space of initial conditions $\mathcal{X}_b$ for alt. d-P_I

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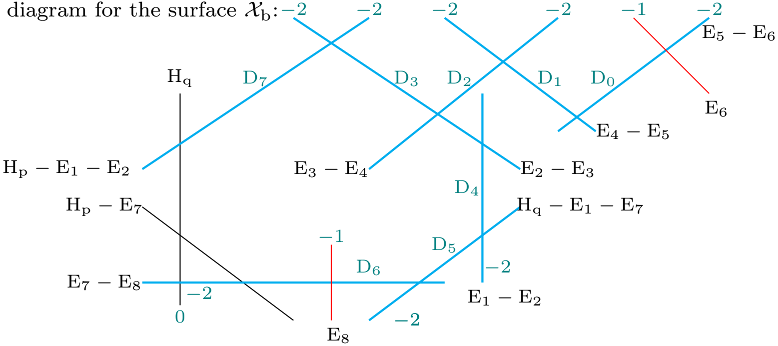
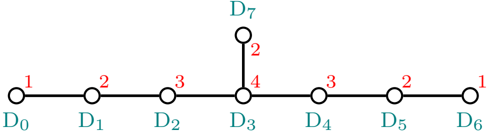


The Okamoto Space of initial conditions $\mathcal{X}_{\mathrm{b}}$ for alt. d-P_{I}

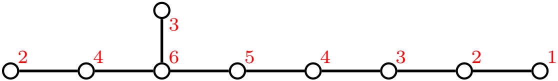
From here we see that the configuration of the irreducible components of $-\mathcal{K}_{\mathcal{X}}$ is given by the affine Dynkin diagram of type $\mathrm{E}_7^{(1)}$ (hence alt. d-P_{I} is also called d-P$(\mathrm{E}_7^{(1)})$):



$$
\begin{aligned}
-\mathrm{K}_{\mathcal{X}} &= 2\mathcal{H}_{\mathrm{q}} + 2\mathcal{H}_{\mathrm{p}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8 \\
&= \mathrm{D}_0 + 2\mathrm{D}_1 + 3\mathrm{D}_2 + 4\mathrm{D}_3 + 3\mathrm{D}_4 + 2\mathrm{D}_5 + \mathrm{D}_6 + 2\mathrm{D}_7.
\end{aligned}
$$

The Symmetry Sub-Lattice

In general, blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at eight points results in a surface $\mathcal{X}$. Its Picard lattice $\mathsf{Pic}(\mathcal{X})$ has rank 10, and the orthogonal complement in $\mathsf{Pic}(\mathcal{X})$ of the class of the anti-canonical divisor $-\mathrm{K}_{\mathcal{X}}$ has the affine type $\mathrm{E}_8^{(1)}$



with the anti-canonical divisor class

$$-\mathcal{K}_{\mathcal{X}} = 2\mathcal{H}_{\mathrm{f}} + 2\mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8$$

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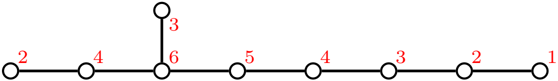

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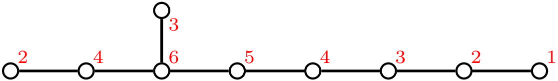

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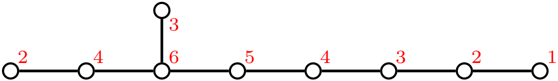

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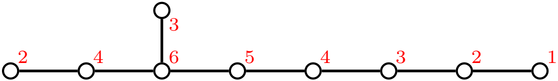

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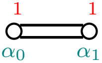

$$\alpha_0 = 2\mathcal{H}_{\mathrm{q}} + \mathcal{H}_{\mathrm{p}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6,$$
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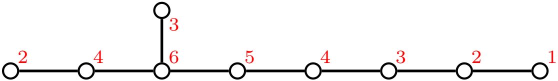

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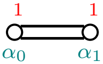

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- Note that $\Pi(\mathrm{R}) \cap \Pi(\mathrm{R}^{\perp}) = \mathsf{Span}_{\mathbb{Z}}(-\mathcal{K}_{\mathcal{X}})$.

$$-\mathrm{K}_{\mathcal{X}} = \mathrm{D}_0 + 2\mathrm{D}_1 + 3\mathrm{D}_2 + 4\mathrm{D}_3 + 3\mathrm{D}_4 + 2\mathrm{D}_5 + \mathrm{D}_6 + 2\mathrm{D}_7 = \alpha_0 + \alpha_1.$$

The Induced Dynamic on $\mathsf{Pic}(\mathcal{X})$

Step 4: Compute the induced map $\varphi_{*} : \mathsf{Pic}(\mathcal{X}_{\mathrm{b}}) \to \mathsf{Pic}(\mathcal{X}_{\bar{\mathrm{b}}})$

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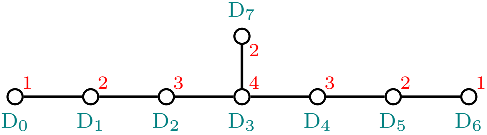

$$\begin{aligned}
\mathrm{D}_0 &= \mathrm{E}_5 - \mathrm{E}_6 & \mathrm{D}_4 &= \mathrm{E}_1 - \mathrm{E}_2\\
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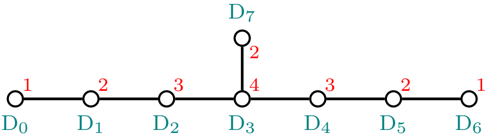

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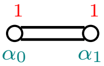

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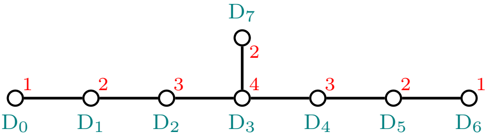

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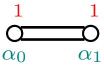

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\end{array}$$

Definition: A discrete Painlevé equation is a discrete dynamical system on the family $\mathcal{X}_{\mathrm{b}}$ induced by a translation in the $\mathsf{\Pi}(\mathrm{R}^{\perp})$ affine symmetry sub-lattice of the surface.

The Extended Affine Weyl Group $\widetilde{W}(A_1^{(1)})$

We now consider the reverse process: how, starting from the translation vector, to write down the corresponding discrete Painlevé equation.

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Let R and $\mathrm{R}^{\perp}$ be as above. Let us describe the extended affine Weyl symmetry group

$$\widetilde{\mathrm{W}}(\mathrm{A}_1^{(1)}) = \langle \mathrm{w}_0, \mathrm{w}_1, \sigma \mid \mathrm{w}_0^2 = \mathrm{w}_1^2 = \sigma^2 = \mathrm{e}, \sigma \mathrm{w}_0 = \mathrm{w}_1 \sigma \rangle.$$

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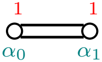
Recall:



$$\alpha_0 = 2\mathcal{H}_{\mathrm{q}} + \mathcal{H}_{\mathrm{p}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6$$
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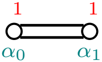

$$\begin{aligned} \alpha_0 &= 2\mathcal{H}_q + \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 \\ \alpha_1 &= \mathcal{H}_p - \mathcal{E}_7 - \mathcal{E}_8, \end{aligned}$$

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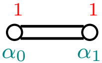

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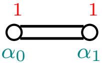

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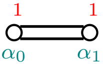

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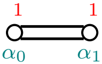

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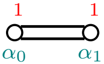

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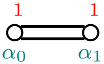

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What is the corresponding elementary bilinear transformation?

From Reflections to Elementary Birational Transformations

For each generator g of $\widetilde{\mathrm{W}}(\mathrm{A}_1^{(1)})$ we now want to construct a birational map $\psi_{\mathrm{g}} : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ such that, when extended to $\widetilde{\psi}_{\mathrm{g}} : \mathcal{X}_{\mathrm{b}} \to \mathcal{X}_{\tilde{\mathrm{b}}}$, the map $\widetilde{\psi}_{\mathrm{g}}$ is an isomorphism whose induces map $(\widetilde{\psi}_{\mathrm{g}})_*$ on $\mathsf{Pic}(\mathcal{X})$ coincides with g. We explain how to do it for w_1, since it is the simplest, and construct the underlying birational map ψ_1.

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- Similarly, from $|\mathcal{H}_{\bar{\mathrm{p}}}| = |\mathcal{H}_{\mathrm{p}}|$, we see that $\bar{\mathrm{p}} = \mathrm{p}$, also up to Möbius transformations.

From Reflections to Elementary Birational Transformations

- Thus, the mapping ψ_1 is given by

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- From $w_1(\mathcal{E}_1) = \bar{\mathcal{E}}_1$ we see that we want to have $\psi_1(p_1) = \bar{p}_1$, $p_1(\infty, \infty)$, i.e.

$$\bar{Q} = \frac{C(1 + bQP) + DQP}{A(1 + BQP) + bQP}\Big|_{Q=P=0} = \frac{C}{A} = 0, \quad \bar{P} = \frac{M + NP}{K + LP}\Big|_{Q=P=0} = \frac{M}{K} = 0,$$

  so $C = M = 0$. Without the loss of generality we can put $\bar{q} = \dfrac{A(qp + b) + Bp}{p}$, $\bar{p} = Kp + L$. Applying the same argument to the other points p_i we get $\bar{q} = q + b/p$, $\bar{p} = p$, which is exactly the elementary Bäcklund transformation s; $\psi_1 = s$.

From Reflections to Elementary Birational Transformations

- Thus, the mapping ψ_1 is given by

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- We can also see the action on parameters. From $\mathrm{w}_1(\mathcal{H}_\mathrm{p} - \mathcal{E}_7) = \bar{\mathcal{E}}_8$ we see that the line $\mathrm{p} = 0$ (whose proper transform is $\mathrm{H}_\mathrm{p} - \mathrm{E}_7$), when written in coordinates $\bar{\mathrm{u}}_7 = \bar{\mathrm{Q}}$ and $\bar{\mathrm{v}}_7 = \bar{\mathrm{p}}/\bar{\mathrm{Q}} = \overline{\mathrm{qp}}$, should collapse to the point $\bar{\mathrm{p}}_8(0, -\bar{\mathrm{b}})$. We get

$$(\bar{\mathrm{u}}_7, \bar{\mathrm{v}}_7)(\mathrm{p}=0) = \left(\frac{\mathrm{p}}{\mathrm{qp}+\mathrm{b}}, \mathrm{qp}+\mathrm{b}\right)\Big|_{\mathrm{p}=0} = (0, \mathrm{b}) = (0, -\bar{\mathrm{b}}),$$

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$$\bar{q} = \frac{A(qp+b)+Bp}{C(qp+b)+Dp}, \qquad \bar{p} = \frac{Kp+L}{Mp+N}$$

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and so $\bar{b} = -b$, as it should be.

- In the same way we can show that $\psi_\sigma = r$, and hence $\psi_0 = rsr$.

Application of the Geometric Approach

Are the following two equations: Same? Different? Equivalent? Related?

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$$\begin{cases} \bar{x} = \dfrac{(\alpha-\beta)(\alpha x(\theta_1^1-\theta_1^2)+(1+\theta_0^2)(x(y-\theta_1^2)+y(\theta_0^1-\theta_0^2)))}{(\alpha-\beta)(x(y-\theta_1^2)+(\theta_0^1-\theta_0^2)y)-\alpha(\theta_1^1+1)(\theta_0^1-\theta_0^2)} \\ \bar{y} = \dfrac{(\alpha-\beta)(y(x+\theta_0^1-\theta_0^2)-\theta_1^2 x)}{\alpha(\theta_0^1-\theta_0^2)} \end{cases}, \tag{1}$$

where θ_i^j and κ_i are some parameters and

$$\alpha(x,y) = \frac{\left(y r_1 + \frac{x(\theta_0^2 r_1 + r_2)}{x+\theta_0^1-\theta_0^2}\right)}{(x+y)(\theta_1^1-\theta_1^2)}, \qquad \beta(x,y) = \frac{((y+\theta_0^2)r_1 + r_2)}{(x+y)(\theta_1^1-\theta_1^2)},$$

$$r_1(x,y) = \kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_3\kappa_1-(y-\theta_1^2)(x-\theta_0^2)-\theta_0^1(y+\theta_0^2)-\theta_1^1(\theta_0^1+\theta_0^2+\theta_1^2),$$

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$$r_1(x,y) = \kappa_1\kappa_2 + \kappa_2\kappa_3 + \kappa_3\kappa_1 - (y-\theta_1^2)(x-\theta_0^2) - \theta_0^1(y+\theta_0^2) - \theta_1^1(\theta_0^1+\theta_0^2+\theta_1^2),$$

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$$\begin{cases} (f+g)(\bar{f}+g) = \dfrac{(g+b_1)(g+b_2)(g+b_3)(g+b_4)}{(g-b_5-\delta)(g-b_6-\delta)} \\[2ex] (\bar{f}+g)(\bar{f}+\bar{g}) = \dfrac{(\bar{f}-b_1)(\bar{f}-b_2)(\bar{f}-b_3)(\bar{f}-b_4)}{(\bar{f}+b_7-\delta)(\bar{f}-b_8-\delta)} \end{cases}, \tag{2}$$

where $b_1, \ldots, b_8$ are some parameters and $\delta = b_1 + \cdots + b_8$.

Both equations are in fact very natural expressions (in their respective settings, of course) of difference Painlevé equations of type d-P $\left(\mathrm{A}_2^{(1)*}\right)$ with symmetry $\widetilde{\mathrm{W}}\left(\mathrm{E}_6^{(1)}\right)$, and so a question about the relationship between the them is a very reasonable one.

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Main result: These two equations are equivalent through an explicit change of variables transforming one equation into the other:

$$f = \frac{x(y-\theta_1^1) + y(\theta_0^1+\kappa_1) + (\theta_0^2+\kappa_1)(\theta_0^1+\theta_0^2+\theta_1^1+2\kappa_1)}{y+\theta_0^2+\kappa_1}$$

$$g = \frac{x(y-\theta_0^2-\theta_1^1-\kappa_1) + y(\theta_0^1-\theta_0^2) + (\theta_0^2+\kappa_1)(\theta_0^1+\theta_0^2+2\kappa_1)}{x-\theta_0^2-\kappa_1}$$

Canonical Model of the Okamoto Surface of Type $A_2^{(1)*}$

Let us start by understanding the structure of a generalized Halphen surface of type $A_2^{(1)*}$.

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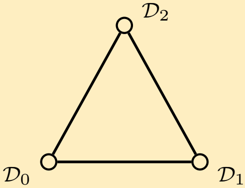
Dynkin diagram $A_2^{(1)}$ and the anti-canonical divisor decomposition



$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \qquad -\mathcal{K}_{\mathcal{X}} = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{D}_2$$

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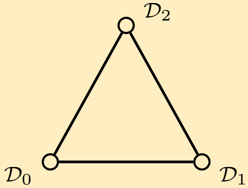

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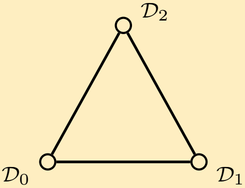

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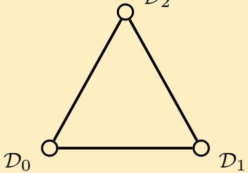

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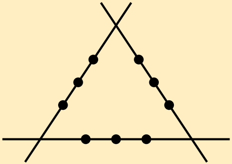
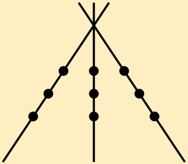


$A_2^{(1)}$ surface (multiplicative)



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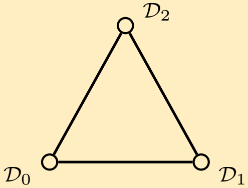

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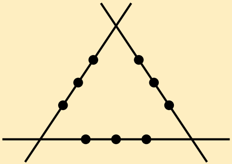

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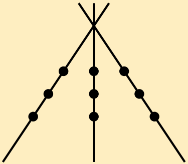

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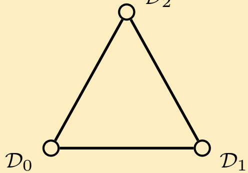

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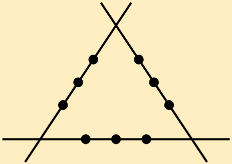

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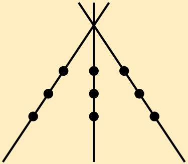

$\mathrm{A}_2^{(1)*}$ surface (additive)

We are interested in the additive dynamic given by $\mathrm{A}_2^{(1)*}$, so we want all of the irreducible components of the anti-canonical divisor to intersect at one point.

Again, without the loss of generality (i.e., acting by affine transformations on each of the two $\mathbb{P}^1$ factors) we can assume that the component $\mathrm{D}_1 = \mathrm{H_f} - \mathrm{E}_5 - \mathrm{E}_6$ under the blowing down map projects to the line $\mathrm{f} = \infty$ (and so there are two blowup points $\mathrm{p}_5(\infty, \mathrm{b}_5)$ and $\mathrm{p}_6(\infty, \mathrm{b}_6)$ on that line), the component $\mathrm{D}_2 = \mathrm{H_g} - \mathrm{E}_7 - \mathrm{E}_8$ projects to the line $\mathrm{g} = \infty$ with points $\mathrm{p}_7(-\mathrm{b}_6, \infty)$ and $\mathrm{p}_8(-\mathrm{b}_8, \infty)$, and the component $\mathrm{D}_0 = \mathrm{H_f} + \mathrm{H_g} - \mathrm{E}_1 - \mathrm{E}_2 - \mathrm{E}_3 - \mathrm{E}_4$ projects to the line $\mathrm{f} + \mathrm{g} = 0$.

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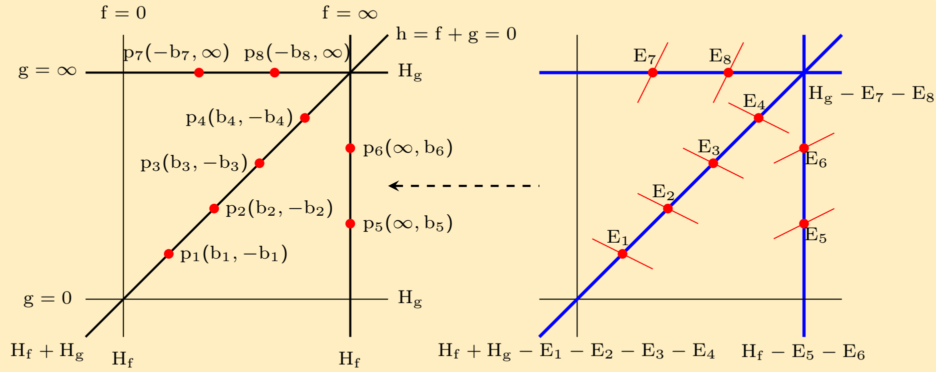

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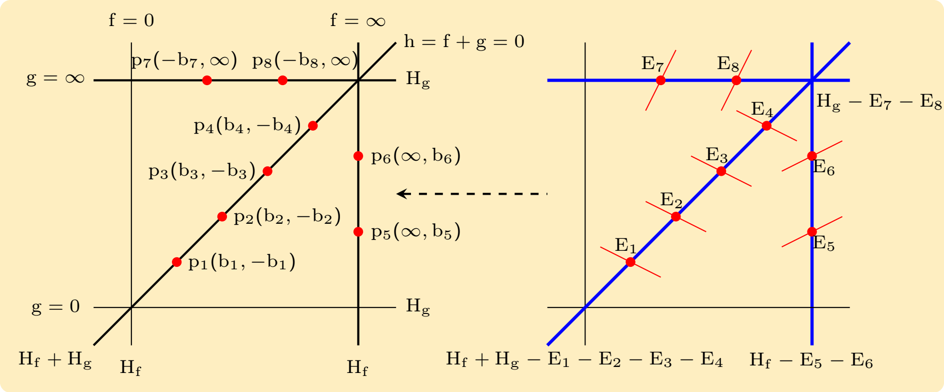
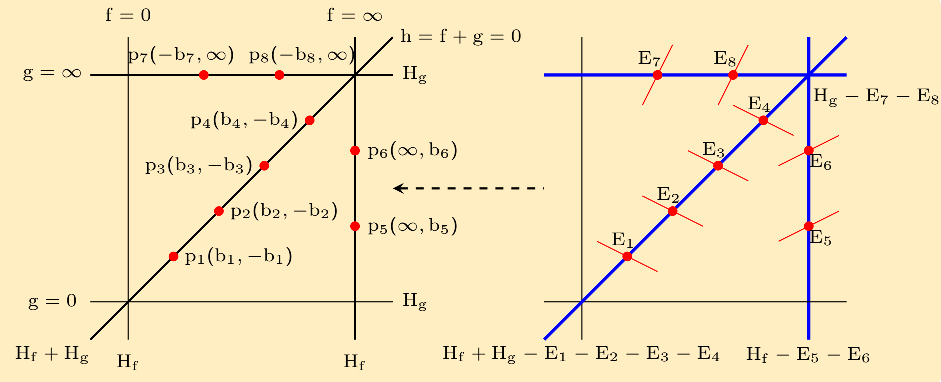


Note that the lines in the above configuration form a pole divisor of the symplectic form

$$\omega = \frac{df \wedge dg}{(f+g)} = -\frac{dF \wedge dg}{F(1+Fg)} = -\frac{df \wedge dG}{G(fG+1)} = \frac{dF \wedge dG}{(F+G)} = \frac{dh \wedge dg}{h} = \cdots$$

Canonical Model of the Okamoto Surface of Type $A_2^{(1)*}$

Thus, we get the following geometric realization of a (family of) surface(s) $\mathcal{X}_b$ of type $A_2^{(1)*}$:



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There is still a two-parameter family of transformations preserving this configuration:

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; f, g \biggr) \sim \biggl(\begin{matrix} \alpha b_1 + \beta & \alpha b_2 + \beta & \alpha b_3 + \beta & \alpha b_4 + \beta \\ \alpha b_5 - \beta & \alpha b_6 - \beta & \alpha b_7 - \beta & \alpha b_8 - \beta \end{matrix}; \alpha f + \beta, \alpha g - \beta \biggr), \ \alpha \neq 0.$$

The Symmetry Group and the Symmetry Sub-Lattice

A more invariant way to parameterize the surface is to use the so-called Period Map. For that we first need to define the symmetry sublattice.

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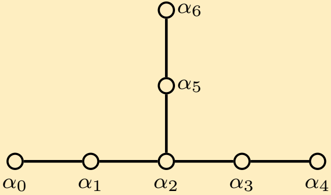
Symmetry sublattice $\mathrm{Q} \triangleleft \mathsf{Pic}(\mathcal{X})$

$$\mathrm{Q} = (\mathsf{Span}_{\mathbb{Z}}\{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2\})^{\perp} = \mathrm{Q}\left((\mathrm{A}_2^{(1)})^{\perp}\right) = \mathsf{Span}_{\mathbb{Z}}\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} = \mathrm{Q}\left(\mathrm{E}_6^{(1)}\right),$$

where the simple roots α_{i} are given by



$$\alpha_0 = \mathcal{E}_3 - \mathcal{E}_4, \qquad \alpha_4 = \mathcal{E}_7 - \mathcal{E}_8,$$
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$$\alpha_2 = \mathcal{E}_1 - \mathcal{E}_2, \qquad \alpha_6 = \mathcal{E}_5 - \mathcal{E}_6.$$
$$\alpha_3 = \mathcal{H}_{\mathrm{f}} - \mathcal{E}_1 - \mathcal{E}_7,$$

Note also that $\delta = -\mathcal{K}_{\mathcal{X}} = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$.

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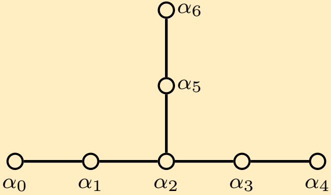


$$\begin{aligned} \alpha_0 &= \mathcal{E}_3 - \mathcal{E}_4, & \alpha_4 &= \mathcal{E}_7 - \mathcal{E}_8, \\ \alpha_1 &= \mathcal{E}_2 - \mathcal{E}_3, & \alpha_5 &= \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_5, \\ \alpha_2 &= \mathcal{E}_1 - \mathcal{E}_2, & \alpha_6 &= \mathcal{E}_5 - \mathcal{E}_6. \\ \alpha_3 &= \mathcal{H}_{\mathrm{f}} - \mathcal{E}_1 - \mathcal{E}_7, \end{aligned}$$

Note also that $\delta = -\mathcal{K}_{\mathcal{X}} = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$.

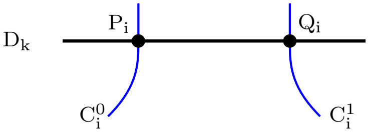
The period mapping is the map

$$\chi : \mathrm{Q} \to \mathbb{C}, \qquad \chi(\alpha_{\mathrm{i}}) = \mathrm{a}_{\mathrm{i}}$$

defined on the simple roots and then extended by the linearity.

The Period Map

$$\chi(\alpha_i) = \chi\left([C_i^1] - [C_1^0]\right) = \int_{P_i}^{Q_i} \frac{1}{2\pi i} \oint_{D_k} \omega$$
$$= \int_{P_i}^{Q_i} \mathsf{res}_{D_k}\, \omega, \quad \omega = \frac{df \wedge dg}{f+g}$$



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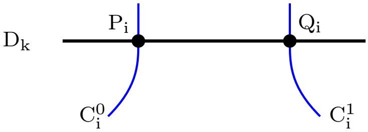
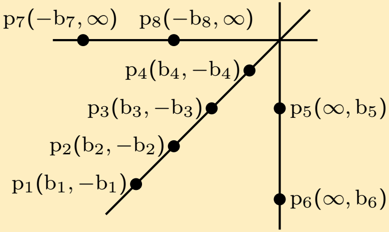


Examples of the Period Map computations



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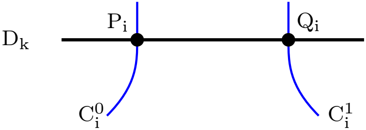

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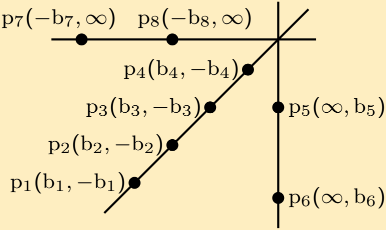
  $D_k = D_0 = \{h = f + g = 0\}$

  $$\omega = \frac{df \wedge dg}{f + g} = \frac{dh \wedge dg}{h}, \quad \mathsf{res}_{h=0}\,\omega = dg$$

$$\chi(\alpha_0) = \int_{-b_4}^{-b_3} dg = \boxed{b_4 - b_3 = a_0}$$



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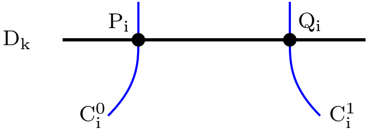

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  $D_k = D_2 = \{g = \infty\} = \{G = 0\}$

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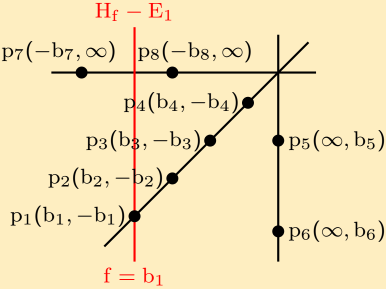
$$\chi(\alpha_3) = \int_{-b_7}^{b_1} df = \boxed{b_1 + b_7 = a_3}$$



The Period Map

The Period Map, $\mathrm{a_i} = \chi(\alpha_\mathrm{i})$ are the root variables

$$\begin{aligned} \mathrm{a}_0 &= \mathrm{b}_4 - \mathrm{b}_3, & \mathrm{a}_3 &= \mathrm{b}_1 + \mathrm{b}_7, & \mathrm{a}_6 &= \mathrm{b}_6 - \mathrm{b}_5, \\ \mathrm{a}_1 &= \mathrm{b}_3 - \mathrm{b}_2, & \mathrm{a}_4 &= \mathrm{b}_8 - \mathrm{b}_7, \\ \mathrm{a}_2 &= \mathrm{b}_2 - \mathrm{b}_1, & \mathrm{a}_5 &= \mathrm{b}_1 + \mathrm{b}_5. \end{aligned}$$

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\end{aligned}$$

Parameterization by the root variables a_i

$$\begin{pmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 & \mathrm{b}_4 \\ \mathrm{b}_5 & \mathrm{b}_6 & \mathrm{b}_7 & \mathrm{b}_8 \end{pmatrix} ; \mathrm{f}, \mathrm{g} \Big) = \begin{pmatrix} \mathrm{b}_1 & \mathrm{b}_1 + \mathrm{a}_2 & \mathrm{b}_1 + \mathrm{a}_1 + \mathrm{a}_2 & \mathrm{b}_1 + \mathrm{a}_0 + \mathrm{a}_1 + \mathrm{a}_2 \\ \mathrm{a}_5 - \mathrm{b}_1 & \mathrm{a}_5 + \mathrm{a}_6 - \mathrm{b}_1 & \mathrm{a}_3 - \mathrm{b}_1 & \mathrm{a}_3 + \mathrm{a}_4 - \mathrm{b}_1 \end{pmatrix} ; \mathrm{f}, \mathrm{g} \Big),$$

and so we see that b_1 is one free parameter (translation of the origin). To fix the global scaling parameter we usually normalize

$$\begin{aligned}
\chi(\delta) = \chi(-\mathcal{K}_{\mathcal{X}}) &= \chi(\mathrm{a}_0 + 2\mathrm{a}_1 + 3\mathrm{a}_2 + 2\mathrm{a}_3 + \mathrm{a}_4 + 2\mathrm{a}_5 + \mathrm{a}_6)\\
&= \mathrm{b}_1 + \mathrm{b}_2 + \mathrm{b}_3 + \mathrm{b}_4 + \mathrm{b}_5 + \mathrm{b}_6 + \mathrm{b}_7 + \mathrm{b}_8.
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Parameterization by the root variables $\mathrm{a_i}$

$$\begin{pmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 & \mathrm{b}_4 \\ \mathrm{b}_5 & \mathrm{b}_6 & \mathrm{b}_7 & \mathrm{b}_8 \end{pmatrix}; \mathrm{f}, \mathrm{g} \Big) = \begin{pmatrix} \mathrm{b}_1 & \mathrm{b}_1 + \mathrm{a}_2 & \mathrm{b}_1 + \mathrm{a}_1 + \mathrm{a}_2 & \mathrm{b}_1 + \mathrm{a}_0 + \mathrm{a}_1 + \mathrm{a}_2 \\ \mathrm{a}_5 - \mathrm{b}_1 & \mathrm{a}_5 + \mathrm{a}_6 - \mathrm{b}_1 & \mathrm{a}_3 - \mathrm{b}_1 & \mathrm{a}_3 + \mathrm{a}_4 - \mathrm{b}_1 \end{pmatrix}; \mathrm{f}, \mathrm{g} \Big),$$

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The usual normalization is to put $\chi(\delta) = 1$, and one can also ask the same for b_1. We will not do that, but we will require that, when resolving the normalization ambiguity, both $\chi(\delta)$ and b_1 are fixed — this ensures the group structure on the level of elementary birational maps.

The Extended Affine Weyl Symmetry Group $\widetilde{\mathrm{W}}\left(\mathrm{E}_6^{(1)}\right)$

The next step in understanding the structure of difference Painlevé equations of type d-P $\left(\mathrm{A}_2^{(1)*}\right)$ is to describe the realization of the symmetry group in terms of elementary bilinear maps.

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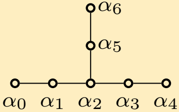
- The affine Weyl symmetry group of reflections $\mathrm{w}_\mathrm{i} = \mathrm{w}_{\alpha_\mathrm{i}}$

$$\mathrm{W}(\mathrm{E}_6^{(1)}) = \left\langle \mathrm{w}_0, \ldots, \mathrm{w}_6 \;\middle|\; \begin{array}{cc} \mathrm{w}_\mathrm{i}^2 = \mathrm{e} & \\ \mathrm{w}_\mathrm{i} \circ \mathrm{w}_\mathrm{j} = \mathrm{w}_\mathrm{j} \circ \mathrm{w}_\mathrm{i} & \text{when } \overset{\circ}{\alpha_\mathrm{i}} \quad \overset{\circ}{\alpha_\mathrm{j}} \\ \mathrm{w}_\mathrm{i} \circ \mathrm{w}_\mathrm{j} \circ \mathrm{w}_\mathrm{i} = \mathrm{w}_\mathrm{j} \circ \mathrm{w}_\mathrm{i} \circ \mathrm{w}_\mathrm{j} & \text{when } \overset{\circ\!\!-\!\!\!-\!\!\circ}{\alpha_\mathrm{i} \quad \alpha_\mathrm{j}} \end{array} \right\rangle$$



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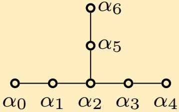
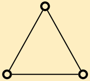
$$W(E_6^{(1)}) = \left\langle w_0,\dots,w_6 \;\middle|\; \begin{array}{cc} w_i^2 = e & \\ w_i\circ w_j = w_j\circ w_i & \text{when} \quad \overset{\circ}{\alpha_i}\quad\overset{\circ}{\alpha_j} \\ w_i\circ w_j\circ w_i = w_j\circ w_i\circ w_j & \text{when} \quad \overset{\circ\!-\!\!\!-\!\!\!-\!\circ}{\alpha_i\quad\alpha_j} \end{array}\right\rangle$$



- The finite group of Dynkin diagram automorphisms

$$\mathsf{Aut}\left(E_6^{(1)}\right) \simeq \mathsf{Aut}\left(A_2^{(1)}\right) \simeq \mathbb{D}_3,$$



where $\mathbb{D}_3 = \{e, m_0, m_1, m_2, r, r^2\} = \langle m_0, r \mid m_0^2 = r^3 = e, m_0 r = r^2 m_0\rangle$ is the usual dihedral group of the symmetries of a triangle.

The Affine Weyl Group $\mathrm{W}(\mathrm{E}_6^{(1)})$

Theorem

Reflections w_i are induced by the following elementary birational mappings (also denoted by w_i) on the family $\mathcal{X}_\mathrm{b}$ fixing b_1 and $\chi(\delta)$ (we put $\mathrm{b}_{\mathrm{i}\dots\mathrm{k}} = \mathrm{b}_\mathrm{i} + \cdots + \mathrm{b}_\mathrm{k}$, e.g., $\mathrm{b}_{12} = \mathrm{b}_1 + \mathrm{b}_2$ and so on)

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$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \begin{pmatrix} f \\ g \end{pmatrix} \xmapsto{w_0} \begin{pmatrix} b_1 & b_2 & b_4 & b_3 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix}; \begin{pmatrix} f \\ g \end{pmatrix},$$

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$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \,{}_{.}\, f \\ b_5 & b_6 & b_7 & b_8 \,{}^{,}\, g \end{pmatrix} \xmapsto{w_0} \begin{pmatrix} b_1 & b_2 & b_4 & b_3 \,{}_{.}\, f \\ b_5 & b_6 & b_7 & b_8 \,{}^{,}\, g \end{pmatrix},$$

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$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \,{}_{.}\, f \\ b_5 & b_6 & b_7 & b_8 \,{}^{,}\, g \end{pmatrix} \xmapsto{w_2} \begin{pmatrix} b_1 & b_{11} - b_2 & b_{13} - b_2 & b_{14} - b_2 \,{}_{.}\, f + b_1 - b_2 \\ b_{52} - b_1 & b_{62} - b_1 & b_{72} - b_1 & b_{82} - b_1 \,{}^{,}\, g - b_1 + b_2{}^{,} \end{pmatrix},$$

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$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xmapsto{w_0} \left(\begin{matrix} b_1 & b_2 & b_4 & b_3 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right),$$

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$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xmapsto{w_3} \left(\begin{matrix} b_1 & b_{217} & b_{317} & b_{417} \\ b_5 & b_6 & -b_{117} & b_8 - b_{17} \end{matrix} ; \begin{matrix} f + b_{17} \\ \frac{(g+b_1)(f+b_7)}{f-b_1} - b_1 \end{matrix}\right),$$

The Affine Weyl Group $W(E_6^{(1)})$

Theorem

Reflections w_i are induced by the following elementary birational mappings (also denoted by w_i) on the family $\mathcal{X}_b$ fixing b_1 and $\chi(\delta)$ (we put $b_{i\cdots k} = b_i + \cdots + b_k$, e.g., $b_{12} = b_1 + b_2$ and so on)

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xrightarrow{w_0} \left(\begin{matrix} b_1 & b_2 & b_4 & b_3 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right),$$

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$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xrightarrow{w_4} \left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_8 & b_7 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right),$$

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$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & . & f \\ b_5 & b_6 & b_7 & b_8 & , & g \end{pmatrix} \xmapsto{w_4} \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & . & f \\ b_5 & b_6 & b_8 & b_7 & , & g \end{pmatrix},$$

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & . & f \\ b_5 & b_6 & b_7 & b_8 & , & g \end{pmatrix} \xmapsto{w_5} \begin{pmatrix} b_1 & b_{215} & b_{315} & b_{415} & . & \frac{(f-b_1)(g-b_5)}{g+b_1} + b_1 \\ -b_{115} & b_6 - b_{15} & b_7 & b_8 & , & g - b_{15} \end{pmatrix},$$

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$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xrightarrow{w_0} \left(\begin{matrix} b_1 & b_2 & b_4 & b_3 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right),$$

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$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xrightarrow{w_5} \left(\begin{matrix} b_1 & b_{215} & b_{315} & b_{415} \\ -b_{115} & b_6 - b_{15} & b_7 & b_8 \end{matrix} ; \begin{matrix} \frac{(f-b_1)(g-b_5)}{g+b_1} + b_1 \\ g - b_{15} \end{matrix}\right),$$

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The Automorphism Group $\mathsf{Aut}(\mathrm{A}_2^{(1)}) \simeq \mathsf{Aut}(\mathrm{E}_6^{(1)}) \simeq \mathbb{D}_3$

Theorem

The acton of the automorphisms on the Picard lattice $\mathsf{Pic}(\mathcal{X})$, the symmetry sub-lattice $\mathsf{Span}_{\mathbb{Z}}\{\alpha_i\}$ and the surface sub-lattice $\mathsf{Span}_{\mathbb{Z}}\{\mathcal{D}_i\}$ is given by:

$$
\begin{aligned}
\mathrm{m}_0 &= (\mathcal{D}_1\mathcal{D}_2) = (\alpha_3\alpha_5)(\alpha_4\alpha_6),\\
&\qquad \mathcal{H}_f \to \mathcal{H}_g, \quad \mathcal{E}_1 \to \mathcal{E}_1, \quad \mathcal{E}_3 \to \mathcal{E}_3, \quad \mathcal{E}_5 \to \mathcal{E}_7, \quad \mathcal{E}_7 \to \mathcal{E}_5,\\
&\qquad \mathcal{H}_g \to \mathcal{H}_f, \quad \mathcal{E}_2 \to \mathcal{E}_2, \quad \mathcal{E}_4 \to \mathcal{E}_4, \quad \mathcal{E}_6 \to \mathcal{E}_8, \quad \mathcal{E}_8 \to \mathcal{E}_6;\\
\mathrm{m}_1 &= (\mathcal{D}_0\mathcal{D}_2) = (\alpha_0\alpha_4)(\alpha_1\alpha_3),\\
&\qquad \mathcal{H}_f \to \mathcal{H}_f, \quad \mathcal{E}_1 \to \mathcal{H}_f - \mathcal{E}_2, \quad \mathcal{E}_3 \to \mathcal{E}_7, \quad \mathcal{E}_5 \to \mathcal{E}_5, \quad \mathcal{E}_7 \to \mathcal{E}_3,\\
&\qquad \mathcal{H}_g \to \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{E}_2 \to \mathcal{H}_f - \mathcal{E}_1, \quad \mathcal{E}_4 \to \mathcal{E}_8, \quad \mathcal{E}_6 \to \mathcal{E}_6, \quad \mathcal{E}_8 \to \mathcal{E}_4;\\
\mathrm{m}_2 &= (\mathcal{D}_0\mathcal{D}_1) = (\alpha_0\alpha_6)(\alpha_1\alpha_5),\\
&\qquad \mathcal{H}_f \to \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{E}_1 \to \mathcal{H}_g - \mathcal{E}_2, \quad \mathcal{E}_3 \to \mathcal{E}_5, \quad \mathcal{E}_5 \to \mathcal{E}_3, \quad \mathcal{E}_7 \to \mathcal{E}_7,\\
&\qquad \mathcal{H}_g \to \mathcal{H}_g, \quad \mathcal{E}_2 \to \mathcal{H}_g - \mathcal{E}_1, \quad \mathcal{E}_4 \to \mathcal{E}_6, \quad \mathcal{E}_6 \to \mathcal{E}_4, \quad \mathcal{E}_8 \to \mathcal{E}_8;\\
\mathrm{r} &= (\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2) = (\alpha_0\alpha_6\alpha_4)(\alpha_1\alpha_5\alpha_3),\\
&\qquad \mathcal{H}_f \to \mathcal{H}_g, \quad \mathcal{E}_1 \to \mathcal{H}_g - \mathcal{E}_2, \quad \mathcal{E}_3 \to \mathcal{E}_5, \quad \mathcal{E}_5 \to \mathcal{E}_7, \quad \mathcal{E}_7 \to \mathcal{E}_3,\\
&\qquad \mathcal{H}_g \to \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{E}_2 \to \mathcal{H}_g - \mathcal{E}_1, \quad \mathcal{E}_4 \to \mathcal{E}_6, \quad \mathcal{E}_6 \to \mathcal{E}_8, \quad \mathcal{E}_8 \to \mathcal{E}_4;\\
\mathrm{r}^2 &= (\mathcal{D}_0\mathcal{D}_2\mathcal{D}_1) = (\alpha_0\alpha_4\alpha_6)(\alpha_1\alpha_3\alpha_5),\\
&\qquad \mathcal{H}_f \to \mathcal{H}_f + \mathcal{H}_g - \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{E}_1 \to \mathcal{H}_f - \mathcal{E}_2, \quad \mathcal{E}_3 \to \mathcal{E}_7, \quad \mathcal{E}_5 \to \mathcal{E}_3, \quad \mathcal{E}_7 \to \mathcal{E}_5,\\
&\qquad \mathcal{H}_g \to \mathcal{H}_f, \quad \mathcal{E}_2 \to \mathcal{H}_f - \mathcal{E}_1, \quad \mathcal{E}_4 \to \mathcal{E}_8, \quad \mathcal{E}_6 \to \mathcal{E}_4, \quad \mathcal{E}_8 \to \mathcal{E}_6.
\end{aligned}
$$

Sketch of the proof

This is almost obvious from looking at the diagrams. For example, for m_2 we have

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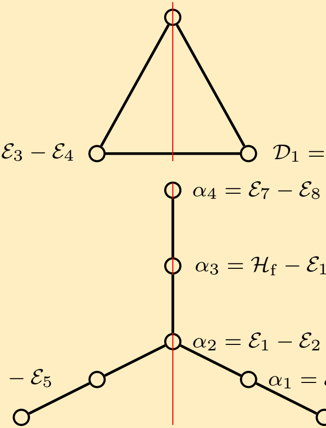

$\mathcal{D}_2 = \mathcal{H}_\mathrm{g} - \mathcal{E}_7 - \mathcal{E}_8$

$\mathcal{D}_0 = \mathcal{H}_\mathrm{f} + \mathcal{H}_\mathrm{g} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4$

$\mathcal{D}_1 = \mathcal{H}_\mathrm{f} - \mathcal{E}_5 - \mathcal{E}_6$

$\alpha_4 = \mathcal{E}_7 - \mathcal{E}_8$

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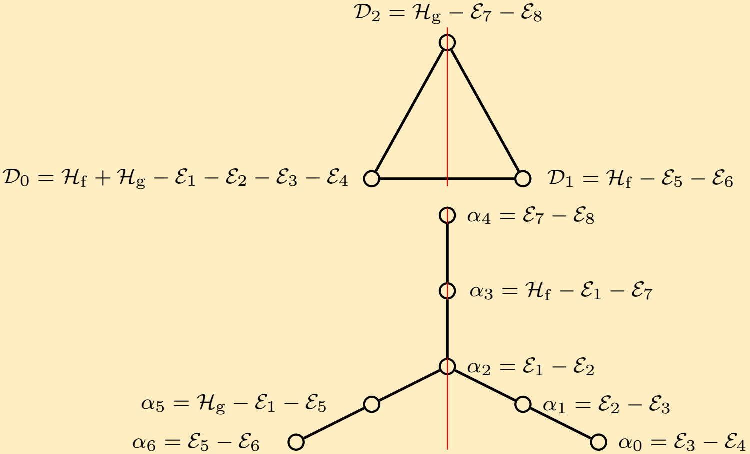

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Hence, m_2 is given by

$$\begin{array}{lllll} \mathcal{H}_{\mathrm{f}} \to \mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2, & \mathcal{E}_1 \to \mathcal{H}_{\mathrm{g}} - \mathcal{E}_2, & \mathcal{E}_3 \to \mathcal{E}_5, & \mathcal{E}_5 \to \mathcal{E}_3, & \mathcal{E}_7 \to \mathcal{E}_7, \\ \mathcal{H}_{\mathrm{g}} \to \mathcal{H}_{\mathrm{g}}, & \mathcal{E}_2 \to \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1, & \mathcal{E}_4 \to \mathcal{E}_6, & \mathcal{E}_6 \to \mathcal{E}_4, & \mathcal{E}_8 \to \mathcal{E}_8; \end{array}$$

The Automorphism Group $\mathsf{Aut}(A_2^{(1)}) \simeq \mathsf{Aut}(E_6^{(1)}) \simeq \mathbb{D}_3$

Theorem

The automorphisms are given by the following elementary birational maps on the family $\mathcal{X}_{b}$ fixing b_1 and $\chi(\delta)$

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xrightarrow{m_0} \left(\begin{matrix} b_1 & b_2 & b_4 & b_3 \\ b_7 & b_8 & b_5 & b_6 \end{matrix} ; \begin{matrix} -f \\ -g \end{matrix}\right),$$

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xrightarrow{m_1} \left(\begin{matrix} b_1 & b_2 & b_{127} & b_{128} \\ b_5 & b_6 & b_3 - b_{12} & b_4 - b_{12} \end{matrix} ; \begin{matrix} b_{12} - f \\ \frac{g(f-b_{12})-b_1 b_2}{f+g} \end{matrix}\right),$$

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xrightarrow{m_2} \left(\begin{matrix} b_1 & b_2 & b_{125} & b_{126} \\ b_3 - b_{12} & b_4 - b_{12} & b_7 & b_8 \end{matrix} ; \begin{matrix} \frac{f(g+b_{12})-b_1 b_2}{f+g} \\ -g - b_{12} \end{matrix}\right),$$

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xrightarrow{r} \left(\begin{matrix} b_1 & b_2 & b_{127} & b_{128} \\ b_3 - b_{12} & b_4 - b_{12} & b_5 & b_6 \end{matrix} ; \begin{matrix} -\frac{g(f-b_{12})-b_1 b_2}{f+g} \\ f - b_{12} \end{matrix}\right),$$

$$\left(\begin{matrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{matrix} ; \begin{matrix} f \\ g \end{matrix}\right) \xrightarrow{r^2} \left(\begin{matrix} b_1 & b_2 & b_{125} & b_{126} \\ b_7 & b_8 & b_3 - b_{12} & b_4 - b_{12} \end{matrix} ; \begin{matrix} g + b_{12} \\ -\frac{f(g+b_{12})-b_1 b_2}{f+g} \end{matrix}\right).$$

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Proof is similar to the previous theorem. Notice that the group structure is preserved on the level of the maps.

The Semi-Direct Product Structure

The extended affine Weyl group $\widetilde{\mathrm{W}}(\mathrm{E}_6^{(1)})$ is a semi-direct product of its normal subgroup $\mathrm{W}(\mathrm{E}_6^{(1)}) \triangleleft \widetilde{\mathrm{W}}(\mathrm{E}_6^{(1)})$ and the subgroup of the diagram automorphisms $\mathsf{Aut}(\mathrm{E}_6^{(1)})$,

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We have just described the group structure of $\mathrm{W}(\mathrm{E}_6^{(1)})$ and $\mathsf{Aut}(\mathrm{E}_6^{(1)})$ using generators and relations, so it remains to give the action of $\mathsf{Aut}(\mathrm{E}_6^{(1)})$ on $\mathrm{W}(\mathrm{E}_6^{(1)})$.

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We have just described the group structure of $\mathrm{W}(\mathrm{E}_6^{(1)})$ and $\mathsf{Aut}(\mathrm{E}_6^{(1)})$ using generators and relations, so it remains to give the action of $\mathsf{Aut}(\mathrm{E}_6^{(1)})$ on $\mathrm{W}(\mathrm{E}_6^{(1)})$.

But elements of $\mathsf{Aut}(\mathrm{E}_6^{(1)})$ act as permutations of the simple roots α_{i}, and so the action is just the corresponding permutation of the corresponding reflections, $\sigma_{\mathrm{t}} \mathrm{w}_{\alpha_{\mathrm{i}}} \sigma_{\mathrm{t}}^{-1} = \mathrm{w}_{\mathrm{t}(\alpha_{\mathrm{i}})}$, where t is the permutation of α_{i}'s corresponding to σ_{t}.

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Example: $\sigma_1 = \sigma_{\mathrm{m}_1} = (\alpha_0\alpha_4)(\alpha_1\alpha_3)$ acts as

$$\sigma_1\mathrm{w}_0\sigma_1 = \mathrm{w}_4, \quad \sigma_1\mathrm{w}_4\sigma_1 = \mathrm{w}_0, \quad \sigma_1\mathrm{w}_1\sigma_1 = \mathrm{w}_3, \quad \sigma_1\mathrm{w}_3\sigma_1 = \mathrm{w}_1, \qquad \sigma_1\mathrm{w}_{\mathrm{i}}\sigma_1 = \mathrm{w}_{\mathrm{i}} \quad \text{otherwise}.$$

Decomposition of Translation Elements

Finally, we need an algorithm for representing a translation element of $\widetilde{\mathrm{W}}(\mathrm{E}_6^1)$ as a composition of the generators of the group, then the corresponding discrete Painlevé equation can be written as a composition of elementary birational maps.

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For this, we use the following Lemma:

Reduction Lemma (V. Kac, Infinite dimensional Lie algebras, Lemma 3.11)

If $\mathrm{w}(\alpha_{\mathrm{i}}) < 0$, then

$$\mathrm{l}(\mathrm{w} \circ \mathrm{w}_{\mathrm{i}}) < \mathrm{l}(\mathrm{w}),$$

where $\mathrm{l}(\mathrm{w})$ is length of $\mathrm{w} \in \mathrm{W}$, and α_{i} is a simple root.

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As an example, consider the following translational mapping:

$$\varphi_* : (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$

where $\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$ as usual.

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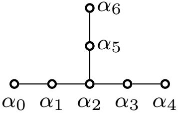
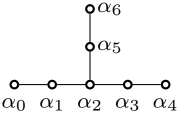
Put

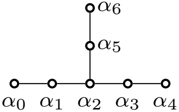
$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6).$$

Then the algorithm works as follows:


α_6
α_5
α_0
α_1
α_2
α_3
α_4

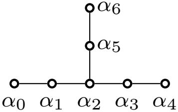


$$\varphi_*(\alpha) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$



$$\varphi_*(\alpha) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$

$$\left(\varphi_*^{(1)} = \varphi_* \circ \mathrm{w}_5\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \delta - \alpha_5, \alpha_{56} - \delta),$$



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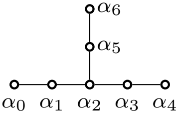

$$\varphi_*(\upalpha) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$

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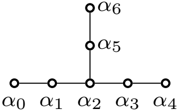
$$\left(\varphi_*^{(3)} = \varphi_*^{(2)} \circ \mathrm{w}_2\right)(\upalpha) = (\alpha_0, \alpha_{125} - \delta, \delta - \alpha_{25}, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$



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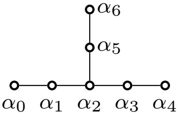
$$\left(\varphi_*^{(4)} = \varphi_*^{(3)} \circ \mathrm{w}_1\right)(\alpha) = (\alpha_{0125} - \delta, \delta - \alpha_{125}, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$



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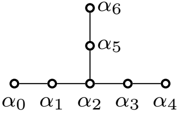
$$\left(\varphi_*^{(4)} = \varphi_*^{(3)} \circ \mathrm{w}_1\right)(\alpha) = (\alpha_{0125} - \delta, \delta - \alpha_{125}, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(5)} = \varphi_*^{(4)} \circ \mathrm{w}_0\right)(\alpha) = (\delta - \alpha_{0125}, \alpha_0, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$



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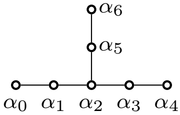
$$\left(\varphi_*^{(5)} = \varphi_*^{(4)} \circ \mathrm{w}_0\right)(\alpha) = (\delta - \alpha_{0125}, \alpha_0, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(6)} = \varphi_*^{(5)} \circ \mathrm{w}_5\right)(\alpha) = (\delta - \alpha_{0125}, \alpha_0, \alpha_{1256} - \delta, \alpha_{235}, \alpha_4, \delta - \alpha_{256}, \alpha_2),$$



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$$\left(\varphi_*^{(1)} = \varphi_* \circ \mathrm{w}_5\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \delta - \alpha_5, \alpha_{56} - \delta),$$

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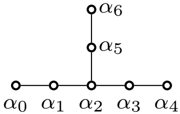
$$\left(\varphi_*^{(7)} = \varphi_*^{(6)} \circ \mathrm{w}_2\right)(\alpha) = (\alpha_{12233456}, -\alpha_{1223345}, \alpha_{01223345}, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),$$



$$
\begin{aligned}
\varphi_*(\alpha) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),\\
\left(\varphi_*^{(1)} = \varphi_* \circ \mathrm{w}_5\right)(\alpha) &= (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \delta - \alpha_5, \alpha_{56} - \delta),\\
\left(\varphi_*^{(2)} = \varphi_*^{(1)} \circ \mathrm{w}_6\right)(\alpha) &= (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \alpha_6, \delta - \alpha_{56}),\\
\left(\varphi_*^{(3)} = \varphi_*^{(2)} \circ \mathrm{w}_2\right)(\alpha) &= (\alpha_0, \alpha_{125} - \delta, \delta - \alpha_{25}, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),\\
\left(\varphi_*^{(4)} = \varphi_*^{(3)} \circ \mathrm{w}_1\right)(\alpha) &= (\alpha_{0125} - \delta, \delta - \alpha_{125}, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),\\
\left(\varphi_*^{(5)} = \varphi_*^{(4)} \circ \mathrm{w}_0\right)(\alpha) &= (\delta - \alpha_{0125}, \alpha_0, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),\\
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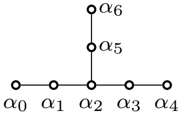
\left(\varphi_*^{(8)} = \varphi_*^{(7)} \circ \mathrm{w}_1\right)(\alpha) &= (\alpha_6, \alpha_{1223345}, \alpha_0, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),
\end{aligned}
$$



$$\varphi_*(\alpha) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$

$$\left(\varphi_*^{(1)} = \varphi_* \circ \mathrm{w}_5\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \delta - \alpha_5, \alpha_{56} - \delta),$$

$$\left(\varphi_*^{(2)} = \varphi_*^{(1)} \circ \mathrm{w}_6\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \alpha_6, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(3)} = \varphi_*^{(2)} \circ \mathrm{w}_2\right)(\alpha) = (\alpha_0, \alpha_{125} - \delta, \delta - \alpha_{25}, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

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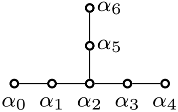
$$\left(\varphi_*^{(8)} = \varphi_*^{(7)} \circ \mathrm{w}_1\right)(\alpha) = (\alpha_6, \alpha_{1223345}, \alpha_0, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),$$

$$\left(\varphi_*^{(9)} = \varphi_*^{(8)} \circ \mathrm{w}_3\right)(\alpha) = (\alpha_6, \alpha_{1223345}, -\alpha_{1234}, \alpha_{01234}, -\alpha_{0123}, \alpha_1, \alpha_2),$$



$$\varphi_*(\alpha) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$

$$\left(\varphi_*^{(1)} = \varphi_* \circ \mathrm{w}_5\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \delta - \alpha_5, \alpha_{56} - \delta),$$

$$\left(\varphi_*^{(2)} = \varphi_*^{(1)} \circ \mathrm{w}_6\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \alpha_6, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(3)} = \varphi_*^{(2)} \circ \mathrm{w}_2\right)(\alpha) = (\alpha_0, \alpha_{125} - \delta, \delta - \alpha_{25}, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(4)} = \varphi_*^{(3)} \circ \mathrm{w}_1\right)(\alpha) = (\alpha_{0125} - \delta, \delta - \alpha_{125}, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

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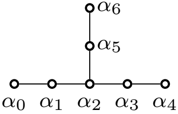
$$\left(\varphi_*^{(7)} = \varphi_*^{(6)} \circ \mathrm{w}_2\right)(\alpha) = (\alpha_{12233456}, -\alpha_{1223345}, \alpha_{01223345}, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),$$

$$\left(\varphi_*^{(8)} = \varphi_*^{(7)} \circ \mathrm{w}_1\right)(\alpha) = (\alpha_6, \alpha_{1223345}, \alpha_0, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),$$

$$\left(\varphi_*^{(9)} = \varphi_*^{(8)} \circ \mathrm{w}_3\right)(\alpha) = (\alpha_6, \alpha_{1223345}, -\alpha_{1234}, \alpha_{01234}, -\alpha_{0123}, \alpha_1, \alpha_2),$$

$$\left(\varphi_*^{(10)} = \varphi_*^{(9)} \circ \mathrm{w}_4\right)(\alpha) = (\alpha_6, \alpha_{1223345}, -\alpha_{1234}, \alpha_4, \alpha_{0123}, \alpha_1, \alpha_2),$$



$$\varphi_*(\alpha) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5 - \delta, \alpha_6),$$

$$\left(\varphi_*^{(1)} = \varphi_* \circ \mathrm{w}_5\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \delta - \alpha_5, \alpha_{56} - \delta),$$

$$\left(\varphi_*^{(2)} = \varphi_*^{(1)} \circ \mathrm{w}_6\right)(\alpha) = (\alpha_0, \alpha_1, \alpha_{25} - \delta, \alpha_3 + \delta, \alpha_4, \alpha_6, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(3)} = \varphi_*^{(2)} \circ \mathrm{w}_2\right)(\alpha) = (\alpha_0, \alpha_{125} - \delta, \delta - \alpha_{25}, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(4)} = \varphi_*^{(3)} \circ \mathrm{w}_1\right)(\alpha) = (\alpha_{0125} - \delta, \delta - \alpha_{125}, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

$$\left(\varphi_*^{(5)} = \varphi_*^{(4)} \circ \mathrm{w}_0\right)(\alpha) = (\delta - \alpha_{0125}, \alpha_0, \alpha_1, \alpha_{235}, \alpha_4, \alpha_{256} - \delta, \delta - \alpha_{56}),$$

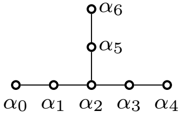
$$\left(\varphi_*^{(6)} = \varphi_*^{(5)} \circ \mathrm{w}_5\right)(\alpha) = (\delta - \alpha_{0125}, \alpha_0, \alpha_{1256} - \delta, \alpha_{235}, \alpha_4, \delta - \alpha_{256}, \alpha_2),$$

$$\left(\varphi_*^{(7)} = \varphi_*^{(6)} \circ \mathrm{w}_2\right)(\alpha) = (\alpha_{12233456}, -\alpha_{1223345}, \alpha_{01223345}, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),$$

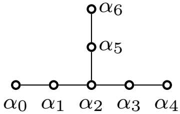
$$\left(\varphi_*^{(8)} = \varphi_*^{(7)} \circ \mathrm{w}_1\right)(\alpha) = (\alpha_6, \alpha_{1223345}, \alpha_0, -\alpha_{01234}, \alpha_4, \alpha_1, \alpha_2),$$

$$\left(\varphi_*^{(9)} = \varphi_*^{(8)} \circ \mathrm{w}_3\right)(\alpha) = (\alpha_6, \alpha_{1223345}, -\alpha_{1234}, \alpha_{01234}, -\alpha_{0123}, \alpha_1, \alpha_2),$$

$$\left(\varphi_*^{(10)} = \varphi_*^{(9)} \circ \mathrm{w}_4\right)(\alpha) = (\alpha_6, \alpha_{1223345}, -\alpha_{1234}, \alpha_4, \alpha_{0123}, \alpha_1, \alpha_2),$$



$$\left(\varphi_*^{(10)} = \varphi_*^{(9)} \circ \mathrm{w}_4\right)(\boldsymbol{\alpha}) = (\alpha_6, \alpha_{1223345}, -\alpha_{1234}, \alpha_4, \alpha_{0123}, \alpha_1, \alpha_2),$$



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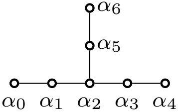

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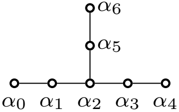

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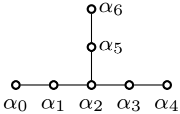

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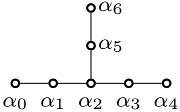

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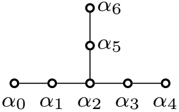

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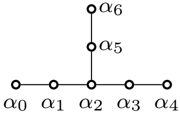

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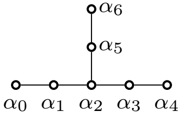

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Thus,

$$\varphi_* = \sigma_{\mathrm{r}} \circ \mathrm{w}_5 \circ \mathrm{w}_2 \circ \mathrm{w}_3 \circ \mathrm{w}_6 \circ \mathrm{w}_5 \circ \mathrm{w}_2 \circ \mathrm{w}_4 \circ \mathrm{w}_3 \circ \mathrm{w}_1 \circ \mathrm{w}_2 \circ \mathrm{w}_5 \circ \mathrm{w}_0 \circ \mathrm{w}_1 \circ \mathrm{w}_2 \circ \mathrm{w}_6 \circ \mathrm{w}_5$$

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First let us review these equations.

Difference Painlevé Equation of Type d-P($A_2^{(1)*}$): Deautonomization

The following example of a d-P($A_2^{(1)*}$) equation was first obtained by B. Grammaticos, A. Ramani, and Y. Ohta back around 1996 by applying the singularity confinement criterion to deautonomization of an integrable discrete autonomous mapping; due to the simplicity structure of the equation we will refer to it as a model example.

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The Model Example of d-P($A_2^{(1)*}$)

We consider a birational map $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ with parameters $b_1, \ldots, b_8$:

$$\varphi : \quad \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{pmatrix} \mapsto \begin{pmatrix} \bar{b}_1 & \bar{b}_2 & \bar{b}_3 & \bar{b}_4 \\ \bar{b}_5 & \bar{b}_6 & \bar{b}_7 & \bar{b}_8 \end{pmatrix},$$

$$\delta = b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8$$

$$\bar{b}_1 = b_1, \quad \bar{b}_3 = b_3, \quad \bar{b}_5 = b_5 + \delta, \quad \bar{b}_7 = b_7 - \delta$$

$$\bar{b}_2 = b_2, \quad \bar{b}_4 = b_4, \quad \bar{b}_6 = b_6 + \delta, \quad \bar{b}_8 = b_8 - \delta,$$

and $\bar{f}$ and $\bar{g}$ are given by the equation

$$\begin{cases} (f+g)(\bar{f}+g) = \dfrac{(g+b_1)(g+b_2)(g+b_3)(g+b_4)}{(g-b_5)(g-b_6)} \\ (\bar{f}+g)(\bar{f}+\bar{g}) = \dfrac{(\bar{f}-\bar{b}_1)(\bar{f}-\bar{b}_2)(\bar{f}-\bar{b}_3)(\bar{f}-\bar{b}_4)}{(\bar{f}+\bar{b}_7)(\bar{f}+\bar{b}_8)} \end{cases}.$$

Difference Painlevé Equation of Type d-P($A_2^{(1)*}$): Deautonomization

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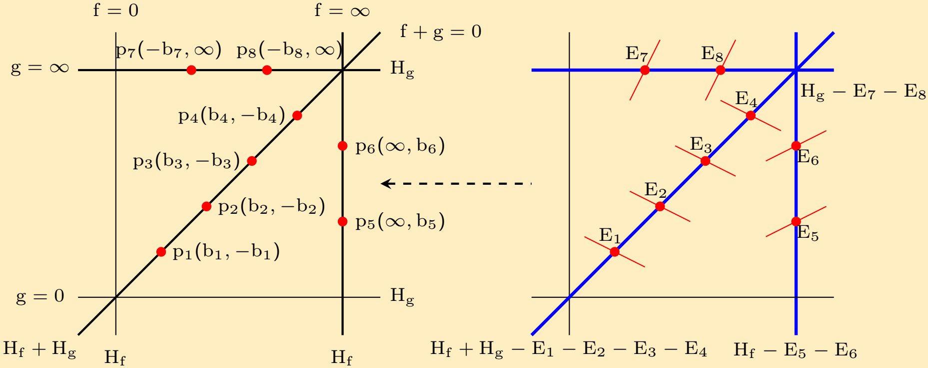

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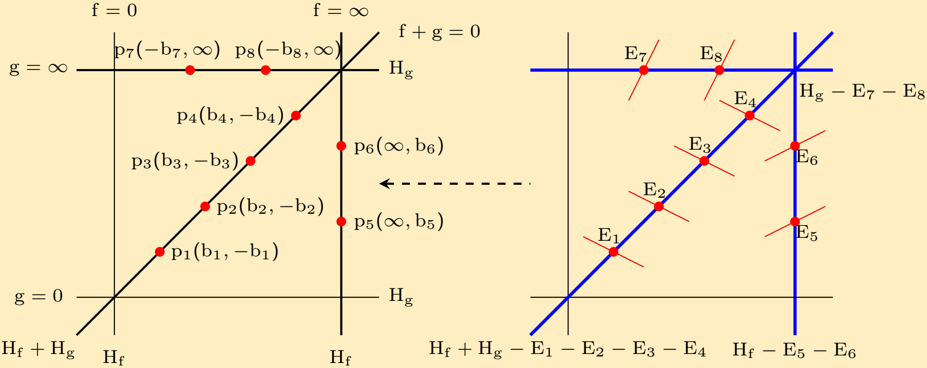

Now let us compute the action of this mapping on $\mathsf{Pic}(\mathcal{X})$

Difference Painlevé Equation of Type d-P($\mathrm{A}_2^{(1)*}$): Deautonomization

The action of φ_* on $\mathsf{Pic}(\mathcal{X})$

Finally, we compute the action of φ_* on $\mathsf{Pic}(\mathcal{X})$ to be

$$\begin{aligned}
\mathcal{H}_{\mathrm{f}} &\mapsto 6\mathcal{H}_{\mathrm{f}} + 3\mathcal{H}_{\mathrm{g}} - 2\mathcal{E}_1 - 2\mathcal{E}_2 - 2\mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - 3\mathcal{E}_7 - 3\mathcal{E}_8,\\
\mathcal{H}_{\mathrm{g}} &\mapsto 3\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_1 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_2 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_3 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_4 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_5 &\mapsto 3\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_6 &\mapsto 3\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_7 &\mapsto \mathcal{H}_{\mathrm{f}} - \mathcal{E}_8,\\
\mathcal{E}_8 &\mapsto \mathcal{H}_{\mathrm{f}} - \mathcal{E}_7,
\end{aligned}$$

Difference Painlevé Equation of Type d-P($\mathrm{A}_2^{(1)*}$): Deautonomization

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\mathcal{H}_{\mathrm{g}} &\mapsto 3\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_1 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_2 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_3 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
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\mathcal{E}_6 &\mapsto 3\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_7 &\mapsto \mathcal{H}_{\mathrm{f}} - \mathcal{E}_8,\\
\mathcal{E}_8 &\mapsto \mathcal{H}_{\mathrm{f}} - \mathcal{E}_7,
\end{aligned}$$

and so the induced action φ_* on the sub-lattice $\mathrm{R}^\perp$ is given by the following translation:

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + (0, 0, 0, 1, 0, -1, 0)\delta,$$

as well as the permutation $\sigma_{\mathrm{r}} = (\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2)$ of the irreducible components of $-\mathrm{K}_{\mathcal{X}}$.

Difference Painlevé Equation of Type d-P($A_2^{(1)*}$): Deautonomization

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$$\begin{aligned}
\mathcal{H}_{\mathrm{f}} &\mapsto 6\mathcal{H}_{\mathrm{f}} + 3\mathcal{H}_{\mathrm{g}} - 2\mathcal{E}_1 - 2\mathcal{E}_2 - 2\mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_6 - 3\mathcal{E}_7 - 3\mathcal{E}_8,\\
\mathcal{H}_{\mathrm{g}} &\mapsto 3\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_1 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_2 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_3 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_4 &\mapsto 2\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_5 &\mapsto 3\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_6 &\mapsto 3\mathcal{H}_{\mathrm{f}} + \mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_7 - \mathcal{E}_8,\\
\mathcal{E}_7 &\mapsto \mathcal{H}_{\mathrm{f}} - \mathcal{E}_8,\\
\mathcal{E}_8 &\mapsto \mathcal{H}_{\mathrm{f}} - \mathcal{E}_7,
\end{aligned}$$

and so the induced action φ_* on the sub-lattice $\mathrm{R}^{\perp}$ is given by the following translation:

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + (0, 0, 0, 1, 0, -1, 0)\delta,$$

as well as the permutation $\sigma_{\mathrm{r}} = (\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2)$ of the irreducible components of $-\mathrm{K}_{\mathcal{X}}$.

Hence $\varphi_* = \sigma_{\mathrm{r}} \circ \mathrm{w}_5 \circ \mathrm{w}_2 \circ \mathrm{w}_3 \circ \mathrm{w}_6 \circ \mathrm{w}_5 \circ \mathrm{w}_2 \circ \mathrm{w}_4 \circ \mathrm{w}_3 \circ \mathrm{w}_1 \circ \mathrm{w}_2 \circ \mathrm{w}_5 \circ \mathrm{w}_0 \circ \mathrm{w}_1 \circ \mathrm{w}_2 \circ \mathrm{w}_6 \circ \mathrm{w}_5$.

Sakai's Classification Scheme for Discrete Painlevé Equations.

In 2001 H. Sakai, developing the ideas of K. Okamoto in the differential case, proposed a classification scheme for Painlevé equations based on algebraic geometry. To each equation corresponds a pair of orthogonal sub-lattices $(\mathsf{\Pi}(\mathrm{R}), \mathsf{\Pi}(\mathrm{R}^{\perp}))$ — the surface and the symmetry sub-lattice in the $\mathrm{E}_8^{(1)}$ lattice, and a translation element in $\tilde{\mathrm{W}}(\mathrm{R}^{\perp})$.

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$$\begin{tikzcd}
\left(E_8^{(1)}\right)^e \arrow[d] & & & & & & & \overset{q\text{-}P_{\mathrm{I}}}{\left(A_1^{(1)}\right)^q_{|\alpha|^2=8}} \arrow[dr] & \\
\left(E_8^{(1)}\right)^q \arrow[r] \arrow[d] & \left(E_7^{(1)}\right)^q \arrow[r] \arrow[d] & \left(E_6^{(1)}\right)^q \arrow[r] \arrow[d] & \overset{q\text{-}P_{\mathrm{VI}}}{\left(D_5^{(1)}\right)^q} \arrow[r] \arrow[d] & \overset{q\text{-}P_{\mathrm{V}}}{\left(A_4^{(1)}\right)^q} \arrow[r] \arrow[d] & \overset{q\text{-}P_{\mathrm{IV}},\, q\text{-}P_{\mathrm{III}}}{\left((A_1{+}A_2)^{(1)}\right)^q} \arrow[r] \arrow[d] \arrow[dd, bend right] & \left((A_1{+}A_1)^{(1)}\right)^q_{|\alpha|^2=14} \arrow[r] \arrow[d] \arrow[ur] & \left(A_1^{(1)}\right)^q_{|\alpha|^2=4} \arrow[d] \arrow[ddl] & \left(A_0^{(1)}\right)^q \arrow[ddl] \\
\left(E_8^{(1)}\right)^\delta \arrow[r] & \left(E_7^{(1)}\right)^\delta \arrow[r] & \left(E_6^{(1)}\right)^\delta \arrow[r] & \underset{P_{\mathrm{VI}},\, d\text{-}P_{\mathrm{V}}}{\left(D_4^{(1)}\right)^{c,\delta}} \arrow[r] & \underset{\substack{P_{\mathrm{V}},\, d\text{-}P_{\mathrm{IV}} \\ d\text{-}P_{\mathrm{III}}}}{\left(A_3^{(1)}\right)^{c,\delta}} \arrow[r] \arrow[dr] & \underset{\substack{P_{\mathrm{III}} \\ \text{alt. } d\text{-}P_{\mathrm{II}}}}{\left((2A_1)^{(1)}\right)^{c,\delta}} \arrow[r] \arrow[dr] & \left(A_1^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_0^{(1)}\right)^c \arrow[d] & \\
 & & & & & \underset{P_{\mathrm{IV}},\, d\text{-}P_{\mathrm{II}}}{\left(A_2^{(1)}\right)^{c,\delta}} \arrow[r] & \underset{P_{\mathrm{II}},\, \text{alt.}d\text{-}P_{\mathrm{I}}}{\left(A_1^{(1)}\right)^{c,\delta}} \arrow[r] & \underset{P_{\mathrm{I}}}{\left(A_0^{(1)}\right)^c} &
\end{tikzcd}$$

Symmetry-type classification scheme for Painlevé equations

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$$\begin{tikzcd}
\left(E_8^{(1)}\right)^e \arrow[d] & & & & & & & \overset{\text{q-}P_{\mathrm{I}}}{\left(A_1^{(1)}\right)^q_{|\alpha|^2=8}} \arrow[dr] & \\
\left(E_8^{(1)}\right)^q \arrow[r] \arrow[d] & \left(E_7^{(1)}\right)^q \arrow[r] \arrow[d] & \left(E_6^{(1)}\right)^q \arrow[r] \arrow[d] & \overset{\text{q-}P_{\mathrm{VI}}}{\left(D_5^{(1)}\right)^q} \arrow[r] \arrow[d] & \overset{\text{q-}P_{\mathrm{V}}}{\left(A_4^{(1)}\right)^q} \arrow[r] \arrow[d] & \overset{\text{q-}P_{\mathrm{IV}},\,\text{q-}P_{\mathrm{III}}}{\left((A_1{+}A_2)^{(1)}\right)^q} \arrow[r] \arrow[d] \arrow[dd, bend right] & \left((A_1{+}A_1)^{(1)}\right)^q_{|\alpha|^2=14} \arrow[r] \arrow[d] \arrow[ur] & \left(A_1^{(1)}\right)^q_{|\alpha|^2=4} \arrow[d] \arrow[ddl] & \left(A_0^{(1)}\right)^q \arrow[ddl] \\
\left(E_8^{(1)}\right)^\delta \arrow[r] & \left(E_7^{(1)}\right)^\delta \arrow[r] & \left(E_6^{(1)}\right)^\delta \arrow[r] & \underset{P_{\mathrm{VI}},\,\text{d-}P_{\mathrm{V}}}{\left(D_4^{(1)}\right)^{c,\delta}} \arrow[r] & \underset{\substack{P_{\mathrm{V}},\,\text{d-}P_{\mathrm{IV}}\\ \text{d-}P_{\mathrm{III}}}}{\left(A_3^{(1)}\right)^{c,\delta}} \arrow[r] \arrow[dr] & \underset{\substack{P_{\mathrm{III}}\\ \text{alt. d-}P_{\mathrm{II}}}}{\left((2A_1)^{(1)}\right)^{c,\delta}} \arrow[r] \arrow[dr] & \left(A_1^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_0^{(1)}\right)^c \arrow[d] & \\
& & & & & \underset{P_{\mathrm{IV}},\,\text{d-}P_{\mathrm{II}}}{\left(A_2^{(1)}\right)^{c,\delta}} \arrow[r] & \underset{P_{\mathrm{II}},\,\text{alt.d-}P_{\mathrm{I}}}{\left(A_1^{(1)}\right)^{c,\delta}} \arrow[r] & \underset{P_{\mathrm{I}}}{\left(A_0^{(1)}\right)^c} &
\end{tikzcd}$$

The differential part of the classification scheme

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$$\begin{tikzcd}
 & & & & & & & \text{q-}P_{\mathrm{I}} & \\
\left(E_8^{(1)}\right)^e \arrow[dd] & & & & & & & \left(A_1^{(1)}\right)^q_{|\alpha|^2=8} \arrow[ddr] & \\
 & & & \text{q-}P_{\mathrm{VI}} & \text{q-}P_{\mathrm{V}} & \text{q-}P_{\mathrm{IV}}, \text{q-}P_{\mathrm{III}} & & & \\
\left(E_8^{(1)}\right)^q \arrow[r] \arrow[d] & \left(E_7^{(1)}\right)^q \arrow[r] \arrow[d] & \left(E_6^{(1)}\right)^q \arrow[r] \arrow[d] & \left(D_5^{(1)}\right)^q \arrow[r] \arrow[d] & \left(A_4^{(1)}\right)^q \arrow[r] \arrow[d] & \left((A_1{+}A_2)^{(1)}\right)^q \arrow[r] \arrow[d] \arrow[ddd, bend right] & \left((A_1{+}A_1)^{(1)}\right)^q_{|\alpha|^2=14} \arrow[r] \arrow[d] \arrow[uur] & \left(A_1^{(1)}\right)^q_{|\alpha|^2=4} \arrow[d] \arrow[dddl] & \left(A_0^{(1)}\right)^q \arrow[dddl] \\
\left(E_8^{(1)}\right)^\delta \arrow[r] & \left(E_7^{(1)}\right)^\delta \arrow[r] & \left(E_6^{(1)}\right)^\delta \arrow[r] & \left(D_4^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_3^{(1)}\right)^{c,\delta} \arrow[r] \arrow[ddr] & \left((2A_1)^{(1)}\right)^{c,\delta} \arrow[r] \arrow[ddr] & \left(A_1^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_0^{(1)}\right)^c \arrow[dd] & \\
 & & & P_{\mathrm{VI}}, \text{d-}P_{\mathrm{V}} & P_{\mathrm{V}}, \text{d-}P_{\mathrm{IV}} \\ \text{d-}P_{\mathrm{III}} & P_{\mathrm{III}} \\ \text{alt. d-}P_{\mathrm{II}} & & & \\
 & & & & & \left(A_2^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_1^{(1)}\right)^{c,\delta} \arrow[r] & \left(A_0^{(1)}\right)^c & \\
 & & & & & P_{\mathrm{IV}}, \text{d-}P_{\mathrm{II}} & P_{\mathrm{II}}, \text{alt.d-}P_{\mathrm{I}} & P_{\mathrm{I}} &
\end{tikzcd}$$

The purely discrete part of the classification scheme: why Painlevé?

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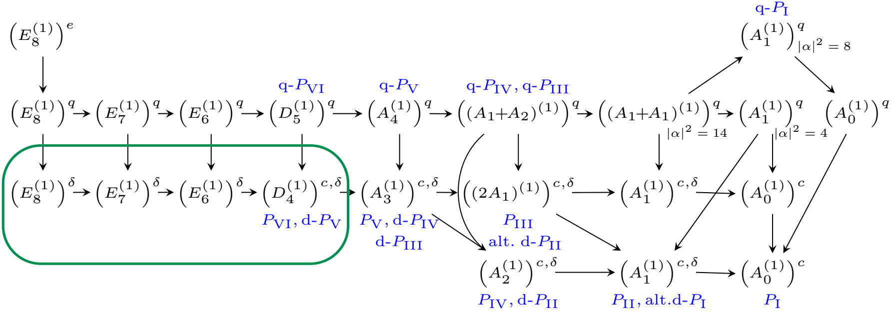

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Isomonodromic approach: difference Painlevé equations as reductions from Schlesinger transformations of Fuchsian systems (our project)

Difference Painlevé Equation of Type d-P($A_2^{(1)*}$): Schlesinger Transformations

Difference Painlevé Equation of Type d-P($\mathrm{A}_2^{(1)*}$): Schlesinger Transformations

So pure difference Painlevé equations in Sakai's scheme are (summetry and surface types):

$$\left(\mathrm{E}_8^{(1)}\right)^{\delta} \to \left(\mathrm{E}_7^{(1)}\right) \to \left(\mathrm{E}_6^{(1)}\right) \to \cdots \quad \text{or} \quad \left(\mathrm{A}_0^{(1)}\right)^{*} \to \left(\mathrm{A}_1^{(1)}\right)^{*} \to \left(\mathrm{A}_2^{(1)}\right)^{*} \to \cdots$$

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P.Boalch has identified the Fuchsian systems whose Schlesinger transformations have the required symmetry type (spectral type $1^3 1^3 1^3$ for d-P($\widetilde{A}_2^*$)).

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However, each discrete Painlevé equation is characterized not only by the symmetry or the surface type, but also by the actual translation direction in $\mathsf{Pic}(X)$ and to identify that explicit computations are needed.

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Take $n = 2$ finite poles $z_0 = 0$, $z_1 = 1$, matrix size $m = 3$, and $\mathsf{rank}(A_i) = 2$:

$$A(z) = \frac{A_0}{z} + \frac{A_1}{z-1}, \qquad A_i = B_i C_i^{\dagger} = \begin{bmatrix} b_{i,1} & b_{i,2} \end{bmatrix} \begin{bmatrix} c_i^{1\dagger} \\ c_i^{2\dagger} \end{bmatrix}.$$

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The corresponding Riemann scheme and the Fuchs relation are

$$\left\{ \begin{matrix} z=0 & z=1 & z=\infty \\ \theta_0^1 & \theta_1^1 & \kappa_1 \\ \theta_0^2 & \theta_1^2 & \kappa_2 \\ 0 & 0 & \kappa_3 \end{matrix} \right\}, \qquad \theta_0^1 + \theta_0^2 + \theta_1^1 + \theta_1^2 + \sum_{j=1}^{3} \kappa_j = 0.$$

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No continuous deformations but non-trivial Schlesinger transformations.

Using various gauge transformations we can normalize the b-vectors, and then use the condition $\mathrm{C}_{\mathrm{i}}^{\dagger}\mathrm{B}_{\mathrm{i}} = \Theta_{\mathrm{i}}$ to parameterize the $\mathrm{c}^{\dagger}$-vectors:

$$\mathrm{B}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathrm{C}_0^{\dagger} = \begin{bmatrix} \theta_0^1 & 0 & \alpha \\ 0 & \theta_0^2 & \beta \end{bmatrix}, \mathrm{B}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \mathrm{C}_1^{\dagger} = \begin{bmatrix} -\gamma - \theta_1^1 & \gamma & \theta_1^1 \\ \theta_1^2 - \delta & \delta & 0 \end{bmatrix}.$$

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Requiring that the eigenvalues of $A_\infty = -A_0 - A_1$ are κ_1, κ_2, and κ_3:

$$\begin{aligned} \mathsf{tr}(A_\infty) &= \kappa_1 + \kappa_2 + \kappa_3 \qquad \text{(the Fuchs relation)} \\ |A_\infty|_{11} + |A_\infty|_{22} + |A_\infty|_{33} &= \kappa_2\kappa_3 + \kappa_3\kappa_1 + \kappa_1\kappa_2 \\ \mathsf{det}(A_\infty) &= \kappa_1\kappa_2\kappa_3 \end{aligned}$$

imposes two linear constraints on four parameters α, β, γ, and δ.

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imposes two linear constraints on four parameters α, β, γ, and δ. We can write them as a linear system on α and β:

$$\begin{aligned} (\gamma + \delta + \theta_1^1 - \theta_1^2)\alpha - (\gamma + \delta)\beta &= \kappa_2\kappa_3 + \kappa_3\kappa_1 + \kappa_1\kappa_2 + (\theta_0^2 - \theta_0^1)\delta \\ &\quad - (\theta_0^2 + \theta_1^1)(\theta_0^1 + \theta_1^2) - \theta_0^2\theta_1^1), \\ -(\theta_0^2(\gamma + \delta + \theta_1^1 - \theta_1^2) + \theta_1^2\gamma + \theta_1^1\delta)\alpha + (\theta_0^1(\gamma + \delta) + \theta_1^2\gamma + \theta_1^1\delta)\beta &= \kappa_1\kappa_2\kappa_3 \\ &\quad + \theta_1^1((\theta_0^1 - \theta_0^2)\delta + \theta_0^2(\theta_0^1 + \theta_1^2)). \end{aligned}$$

Using various gauge transformations we can normalize the b-vectors, and then use the condition $\mathrm{C}_i^\dagger \mathrm{B}_i = \Theta_i$ to parameterize the $\mathrm{c}^\dagger$-vectors:

$$\mathrm{B}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathrm{C}_0^\dagger = \begin{bmatrix} \theta_0^1 & 0 & \alpha \\ 0 & \theta_0^2 & \beta \end{bmatrix}, \mathrm{B}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \mathrm{C}_1^\dagger = \begin{bmatrix} -\gamma - \theta_1^1 & \gamma & \theta_1^1 \\ \theta_1^2 - \delta & \delta & 0 \end{bmatrix}.$$

Requiring that the eigenvalues of $\mathrm{A}_\infty = -\mathrm{A}_0 - \mathrm{A}_1$ are κ_1, κ_2, and κ_3:

$$\begin{aligned} \mathsf{tr}(\mathrm{A}_\infty) &= \kappa_1 + \kappa_2 + \kappa_3 \qquad \text{(the Fuchs relation)} \\ |\mathrm{A}_\infty|_{11} + |\mathrm{A}_\infty|_{22} + |\mathrm{A}_\infty|_{33} &= \kappa_2\kappa_3 + \kappa_3\kappa_1 + \kappa_1\kappa_2 \\ \mathsf{det}(\mathrm{A}_\infty) &= \kappa_1\kappa_2\kappa_3 \end{aligned}$$

imposes two linear constraints on four parameters α, β, γ, and δ. We can write them as a linear system on α and β:

$$\begin{aligned} (\gamma + \delta + \theta_1^1 - \theta_1^2)\alpha - (\gamma + \delta)\beta &= \kappa_2\kappa_3 + \kappa_3\kappa_1 + \kappa_1\kappa_2 + (\theta_0^2 - \theta_0^1)\delta \\ &\qquad - (\theta_0^2 + \theta_1^1)(\theta_0^1 + \theta_1^2) - \theta_0^2\theta_1^1), \\ -(\theta_0^2(\gamma + \delta + \theta_1^1 - \theta_1^2) + \theta_1^2\gamma + \theta_1^1\delta)\alpha + (\theta_0^1(\gamma + \delta) + \theta_1^2\gamma + \theta_1^1\delta)\beta &= \kappa_1\kappa_2\kappa_3 \\ &\qquad + \theta_1^1((\theta_0^1 - \theta_0^2)\delta + \theta_0^2(\theta_0^1 + \theta_1^2)). \end{aligned}$$

Notice that the coefficients of the matrix of the above linear system are written in terms of the expressions $\gamma + \delta$, $\gamma + \delta + \theta_1^1 - \theta_1^2$, and $\theta_1^2\gamma + \theta_1^1\delta$.

Choose parameterization variables x and y to simplify the structure of the substitution rule (matrix entries and the determinant):

$$\mathrm{x} = \frac{(\gamma + \delta)(\theta_0^1 - \theta_0^2)}{\theta_1^1 - \theta_1^2}, \qquad \mathrm{y} = \frac{\theta_1^2\gamma + \theta_1^1\delta}{\gamma + \delta + \theta_1^1 - \theta_1^2}.$$

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This gives:

$$\alpha(\mathrm{x},\mathrm{y}) = \frac{\left(\mathrm{y}\mathrm{r}_1 + \frac{\mathrm{x}(\theta_0^2\mathrm{r}_1+\mathrm{r}_2)}{\mathrm{x}+\theta_0^1-\theta_0^2}\right)}{(\mathrm{x}+\mathrm{y})(\theta_1^1-\theta_1^2)}, \qquad \beta(\mathrm{x},\mathrm{y}) = \frac{((\mathrm{y}+\theta_0^2)\mathrm{r}_1+\mathrm{r}_2)}{(\mathrm{x}+\mathrm{y})(\theta_1^1-\theta_1^2)},$$

where r_1 and r_2 are the right-hand-sides of our linear system on α and β

$$\begin{aligned}\mathrm{r}_1 = \mathrm{r}_1(\mathrm{x},\mathrm{y}) &= \kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_3\kappa_1-(\mathrm{y}-\theta_1^2)(\mathrm{x}-\theta_0^2)-\theta_0^1(\mathrm{y}+\theta_0^2)\\ &\qquad -\theta_1^1(\theta_0^1+\theta_0^2+\theta_1^2),\\ \mathrm{r}_2 = \mathrm{r}_2(\mathrm{x},\mathrm{y}) &= \kappa_1\kappa_2\kappa_3+\theta_1^1((\mathrm{y}-\theta_1^2)(\mathrm{x}-\theta_0^2)+\theta_0^1(\mathrm{y}+\theta_0^2)).\end{aligned}$$

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Schlesinger evolution equations give us the map $\psi : (\mathrm{x},\mathrm{y}) \to (\bar{\mathrm{x}},\bar{\mathrm{y}})$:

$$\begin{cases} \bar{\mathrm{x}} = \dfrac{(\alpha-\beta)(\alpha\mathrm{x}(\theta_1^1-\theta_1^2)+(1+\theta_0^2)(\mathrm{x}(\mathrm{y}-\theta_1^2)+\mathrm{y}(\theta_0^1-\theta_0^2)))}{(\alpha-\beta)(\mathrm{x}(\mathrm{y}-\theta_1^2)+(\theta_0^1-\theta_0^2)\mathrm{y})-\alpha(\theta_1^1+1)(\theta_0^1-\theta_0^2)} \\ \bar{\mathrm{y}} = \dfrac{(\alpha-\beta)(\mathrm{y}(\mathrm{x}+\theta_0^1-\theta_0^2)-\theta_1^2\mathrm{x})}{\alpha(\theta_0^1-\theta_0^2)} \end{cases}.$$

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Very complicated! (Finding a simple form for this equation was one of the main motivations behind this project)

Difference Painlevé Equation of Type d-P($A_2^{(1)*}$): Schlesinger Transformations

The Okamoto surface for the map $\psi : (x, y) \to (\bar{x}, \bar{y})$ is given by the blow-up diagram:

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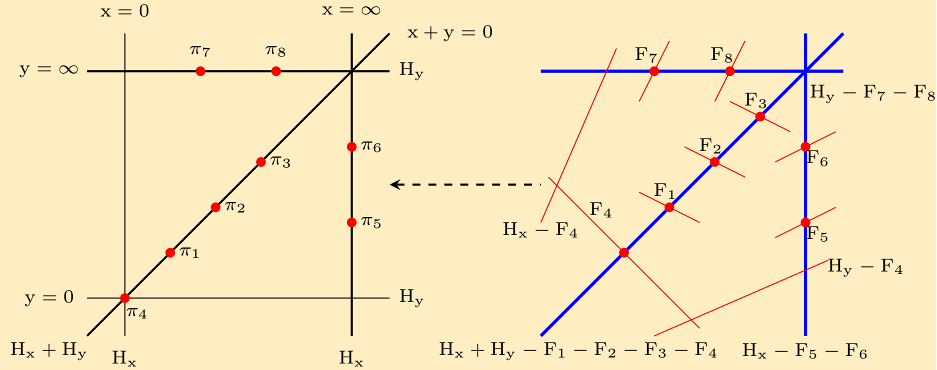

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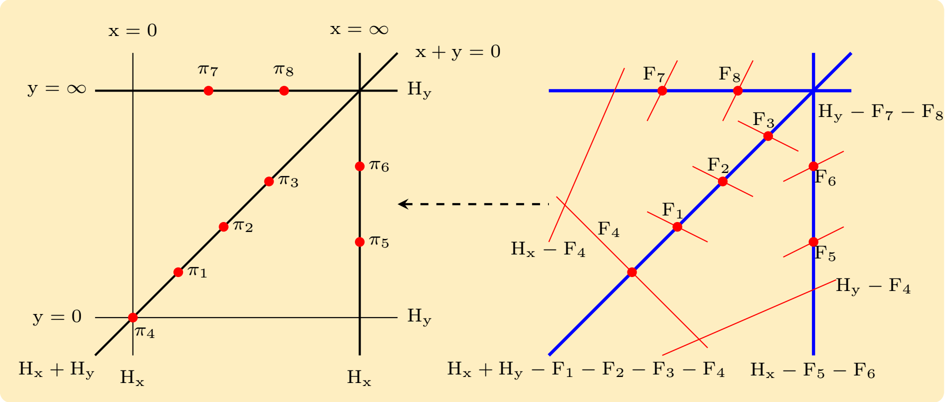

So we see that the configuration structure is the same, but the coordinates of the blowup points are now expressed in terms of the characteristic indices:

$$p_i(\theta_0^2+\kappa_i, -\theta_0^2-\kappa_i), \quad p_4(0,0), \quad p_5\left(\infty, \theta_1^1\right), \quad p_6\left(\infty, \theta_1^2\right), \quad p_7\left(\theta_0^2-\theta_0^1, \infty\right), \quad p_8\left(\theta_0^2+1, \infty\right).$$

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Difference Painlevé Equation of Type d-P($A_2^{(1)*}$): Schlesinger Transformations

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The action of ψ_* on $\mathsf{Pic}(\mathcal{X})$

$$\begin{aligned}
\mathcal{H}_{\mathrm{f}} &\mapsto 2\mathcal{H}_{\mathrm{f}} + 3\mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_5 - 2\mathcal{E}_8,\\
\mathcal{H}_{\mathrm{g}} &\mapsto 3\mathcal{H}_{\mathrm{f}} + 5\mathcal{H}_{\mathrm{g}} - 2\mathcal{E}_1 - 2\mathcal{E}_2 - 2\mathcal{E}_3 - 2\mathcal{E}_4 - 3\mathcal{E}_5 - \mathcal{E}_6 - 2\mathcal{E}_8,\\
\mathcal{E}_1 &\mapsto \mathcal{H}_{\mathrm{f}} + 2\mathcal{H}_{\mathrm{g}} - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8,\\
\mathcal{E}_2 &\mapsto \mathcal{H}_{\mathrm{f}} + 2\mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8,\\
\mathcal{E}_3 &\mapsto \mathcal{H}_{\mathrm{f}} + 2\mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_5 - \mathcal{E}_8,\\
\mathcal{E}_4 &\mapsto \mathcal{H}_{\mathrm{f}} + 2\mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_5 - \mathcal{E}_8,\\
\mathcal{E}_5 &\mapsto \mathcal{E}_7,\\
\mathcal{E}_6 &\mapsto 2\mathcal{H}_{\mathrm{f}} + 2\mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_5 - \mathcal{E}_8,\\
\mathcal{E}_7 &\mapsto 2\mathcal{H}_{\mathrm{f}} + 3\mathcal{H}_{\mathrm{g}} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - \mathcal{E}_4 - 2\mathcal{E}_5 - \mathcal{E}_6 - 2\mathcal{E}_8,\\
\mathcal{E}_8 &\mapsto \mathcal{H}_{\mathrm{g}} - \mathcal{E}_5,
\end{aligned}$$

and so the induced action φ_* on the sub-lattice $\mathrm{R}^{\perp}$ is given by the following translation:

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) + (0, 0, 0, -1, 1, 1, -1)\delta,$$

Comparison between different forms of d-P($\widetilde{A}_2^*$)

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- This can also be written as follows, with $\delta = \chi(-\mathcal{K}_\mathcal{X}) = \mathrm{b}_1 + \cdots + \mathrm{b}_8 (= -1)$:

$$\varphi : \begin{pmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 & \mathrm{b}_4 \\ \mathrm{b}_5 & \mathrm{b}_6 & \mathrm{b}_7 & \mathrm{b}_8 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 & \mathrm{b}_4 \\ \mathrm{b}_5 + \delta & \mathrm{b}_6 + \delta & \mathrm{b}_7 - \delta & \mathrm{b}_8 - \delta \end{pmatrix} \qquad \text{deautonomization}$$

$$\psi : \begin{pmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 & \mathrm{b}_4 \\ \mathrm{b}_5 & \mathrm{b}_6 & \mathrm{b}_7 & \mathrm{b}_8 \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \mathrm{b}_3 & \mathrm{b}_4 \\ \mathrm{b}_5 - \delta & \mathrm{b}_6 & \mathrm{b}_7 + \delta & \mathrm{b}_8 \end{pmatrix} \qquad \text{Schlesinger Transformations}$$

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- Riemann scheme $\left(\text{which gave d-P}(\mathrm{A}_2^{(1)*}) = \boldsymbol{\Sigma}_0(1,3) \circ \left\{ \begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix} \right\} \circ \boldsymbol{\Sigma}_0(1,3) \circ \left\{ \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right\}.\right)$:

$$\left\{ \begin{array}{ccc} z = 0 & z = 1 & z = \infty \\ \theta_0^1 & \theta_1^1 & \kappa_1 \\ \theta_0^2 & \theta_1^2 & \kappa_2 \\ 0 & 0 & \kappa_3 \end{array} \right\} \xrightarrow{\left\{ \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix} \right\}} \left\{ \begin{array}{ccc} z = 0 & z = 1 & z = \infty \\ \theta_0^1 - 1 & \theta_1^1 + 1 & \kappa_1 \\ \theta_0^2 & \theta_1^2 & \kappa_2 \\ 0 & 0 & \kappa_3 \end{array} \right\},$$

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Comparison between different forms of d-P($\widetilde{A}_2^*$)

- Translation directions:

$$\varphi_* : (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) \mapsto (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) + (0,0,0,1,0,-1,0)\delta$$
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- The best approach, however, is through the decomposition. In the same way as we did for φ_*, we can compute and compare the decomposition for ψ_*;

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- This gives us the equivalence!

$$\psi_* = \sigma_{\mathrm{r}} \circ \mathrm{w}_1 \circ \mathrm{w}_5 \circ \sigma_{\mathrm{r}^2} \circ \varphi_* \circ \mathrm{w}_5 \circ \mathrm{w}_3 = (\mathrm{w}_3 \circ \mathrm{w}_5) \circ \varphi_* \circ (\mathrm{w}_3 \circ \mathrm{w}_5)^{-1}$$

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$$\psi_* = \sigma_\mathrm{r} \circ \mathrm{w}_1 \circ \mathrm{w}_2 \circ \mathrm{w}_3 \circ \mathrm{w}_6 \circ \mathrm{w}_5 \circ \mathrm{w}_2 \circ \mathrm{w}_4 \circ \mathrm{w}_3 \circ \mathrm{w}_1 \circ \mathrm{w}_2 \circ \mathrm{w}_5 \circ \mathrm{w}_0 \circ \mathrm{w}_1 \circ \mathrm{w}_2 \circ \mathrm{w}_6 \circ \mathrm{w}_3$$

- This gives us the equivalence!

$$\psi_* = \sigma_\mathrm{r} \circ \mathrm{w}_1 \circ \mathrm{w}_5 \circ \sigma_{\mathrm{r}^2} \circ \varphi_* \circ \mathrm{w}_5 \circ \mathrm{w}_3 = (\mathrm{w}_3 \circ \mathrm{w}_5) \circ \varphi_* \circ (\mathrm{w}_3 \circ \mathrm{w}_5)^{-1}$$

- The mapping $\mathrm{w}_5 \circ \mathrm{w}_3$ gives us the change of variables between the two equations,

$$\mathrm{f} = \frac{\mathrm{x}(\mathrm{y} - \theta_1^1) + \mathrm{y}(\theta_0^1 + \kappa_1) + (\theta_0^2 + \kappa_1)(\theta_0^1 + \theta_0^2 + \theta_1^1 + 2\kappa_1)}{\mathrm{y} + \theta_0^2 + \kappa_1}$$
$$\mathrm{g} = \frac{\mathrm{x}(\mathrm{y} - \theta_0^2 - \theta_1^1 - \kappa_1) + \mathrm{y}(\theta_0^1 - \theta_0^2) + (\theta_0^2 + \kappa_1)(\theta_0^1 + \theta_0^2 + 2\kappa_1)}{\mathrm{x} - \theta_0^2 - \kappa_1}$$

References

Kazuo Okamoto, Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, Japan. J. Math. (N.S.) 5 (1979), no. 1, 1–79

Kazuo Okamoto, Studies on the Painlevé equations. I. Sixth Painlevé equation P_{VI} Ann. Mat. Pura Appl. (4), 1987, 337–381

Kazuo Okamoto, Studies on the Painlevé equations. III. Second and fourth Painlevé equations, P_{II} and P_{IV} Math. Ann., 275, 1986, 221–255

Hidetaka Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Comm. Math. Phys. 220 (2001), no. 1, 165–229.

Masatoshi Noumi, Painlevé equations through symmetry., Translations of Mathematical Monographs, 223. American Mathematical Society, (2004) 156 pp.

Philip Boalch, Quivers and difference Painlevé equations, Groups and symmetries, CRM Proc. Lecture Notes, vol. 47, Amer. Math. Soc., Providence, RI, 2009, pp. 25–51.

Anton Dzhamay, Hidetaka Sakai, Tomoyuki Takenawa, Discrete Hamiltonian Structure of Schlesinger Transformations, arXiv:1302.2972v2 [math-ph]

Anton Dzhamay and Tomoyuki Takenawa, Geometric Analysis of Reductions from Schlesinger Transformations to Difference Painlevé Equations, arXiv:1408.3778 [math-ph]

Kenji Kajiwara, Masatoshi Noumi, Yasuhiko Yamada, Geometric Aspects of Painlevé Equations, arXiv:1509.08186 [nlin.SI]