Groups, ends and trees: exercises I

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Exercise 1. Show that $\mathrm{SL}(n, \mathbf{Z})$ is generated by the transvections: these are the matrices with 1's on the diagonal and one other entry equal to 1 and all the others 0.

Exercise 2. The elements of the *lamplighter group* $\mathbf{Z}_2 \wr \mathbf{Z} = \bigoplus_{\mathbf{Z}} \mathbf{Z}_2 \rtimes \mathbf{Z}$ can be represented by all the possible configurations of an infinite line of lamps, which can be ON or OFF, with only finitely many lamps ON, with a lamplighter sitting on some marked site. Show that the group is generated by two elements which can be described by the possible actions of the lamplighter: he can switch the light at the site where he seats ON/OFF (which corresponds to an element of order 2), and he can move one step right/left. Is this group finitely presented?

Exercise 3. Show that $(\mathbb{Q}, +)$ is not finitely generated.

Exercise 4. Show that the fundamental group of a bouquet of n circles is the free group over an alphabet S of size n: $\#S = n$. More generally, what is the fundamental group of a finite graph?

Exercise 5. Show that the free group $F(S)$ satisfies the following *universal property*: for any function $\varphi : S \longrightarrow G$ from S to a group G, there exists a unique group homomorphism $\Phi : F(S) \longrightarrow G$ such that $\Phi(s) = \varphi(s)$ for any $s \in S$.

Exercise 6. Show that two free groups $F(S)$ and $F(S')$ are isomorphic if and only if S and S' have the same cardinality.

Exercise 7. Prove that the quotient of a finitely generated group is finitely generated.

Exercise 8. Let N and K be two finitely generated groups, with N subgroup of a group G, such that $G/N = K$. Then G is finitely generated. If N and K are moreover finitely presented, so is G.

Exercise 9. Let $G = \langle s_1, s_2 \mid r \rangle$ be a group with two generators and one relation. Write $r = s_{i_1}^{\epsilon_1} \cdots s_{i_n}^{\epsilon_n}$, with $i_j = 1, 2$ and $\epsilon_j = \pm 1$. Let X_0 be the bouquet of two circles σ_1 and σ_2, and choose an orientation for both of them. In $\pi_1(X_0)$, the homotopy classes $s_1 = [\sigma_1]$ and $s_2 = [\sigma_2]$ generate the free group $F(s_1, s_2)$.

Then we take a disc D of dimension 2 and cut the boundary ∂D into n labelled and oriented intervals I_j, according to the relation $r = s_{i_1}^{\epsilon_1} \cdots s_{i_n}^{\epsilon_n}$: going cyclically around the boundary, we read the label s_{i_j} on I_j, and the orientation is positive if $\epsilon_j = 1$, negative otherwise.

Consider a continuous map $\phi : \partial D \longrightarrow X_0$, satisfying the condition that $\phi(I_j) = \sigma_{i_j}$ and the restriction of ϕ to the interior of I_j is injective and is orientation preserving if $\epsilon_j = 1$, and orientation reversing otherwise.

We define the space X_1 to be the topological space obtained from the disjoint union $X_0 \sqcup D$, and identifying every point $p \in \partial D$ with the image $\phi(p) \in X_0$.

Show that the fundamental group of X_1 is isomorphic to the group G.