Report on the work done in 2015.

1. Results

What I did this year was a little bit of going off on a tangent, except that the tangent happens to be a rather old one -- in fact predating the current project by several years. Namely, it concerns a result from 2006 that was my first work on cyclic homology in positive characteristic -- a proof of Kontsevich-Soibelman Degeneration Conjecture. The conjecture asserts that for a smooth and proper DG algebra over a field of characteristic 0, the non-commutative Hodge-to-de Rham spectral sequence degenerates (if the DG algebra is Morita-equivalent to an algebraic variety, this is the standard Hodge-to-de Rham degeneration). Positive characteristic enters the picture via the method of the proof. It seems that classical analytic approach to Hodge-to-de Rham degeneration leads nowhere. What works is an algebraic approach suggested by Deligne and lllusie in 1987, based on reduction to positive characteristic and utilizing the properties of the Cartier map. What I did in 2006 was to generalize this whole story to the setting of general associative DG algebras.

Unfortunately, parts of the argument were quite technical, and somewhat ad hoc. Moreover, at some point the technicalities became overwhelming, so I had to impose a simplifying assumption -- namely, that the DG algebra in question is concentrated in non-negative cohomological degrees. This assumption was needed because I used simplicial methods and Dold-Kan equivalence. It is pretty unnatural, and although it does cover some interesting case such as that of the usual commutative algebraic varieties, it was pretty irksome to have to impose it.

For various reasons, I revisited the subject starting from the very end of last year, and by now I believe I'm almost finished with it.

I like my new approach much better. First of all, I no longer need to use simplicial methods, so the irksome unnatural assumption vanishes without a trace. Moreover, I now feel that I am much closer to understanding how things really work in the story. Namely, the new approach mimics that of Deligne and Illusie more closely, in that it relies on two completely different spectral sequence with the same first page -- the Hodge-to-de Rham spectral sequence, on one hand, and the so-called "conjugate spectral sequence", on the other hand. In the non-commutative case, the first page for both is a periodization of Hochschild homology. The first spectral sequence converges to periodic cyclic homology. It took a while to realize that the second sequence does not. What it converges to, in fact, is a completely new additive invariant of DG algebras and DG categories that is obtained by totalizing the periodic cyclic bicomplex in the "wrong" way. I call this new invariant "co-periodic cyclic homology".

The new invarint has been completely overlooked by the classics, probably because it is identically 0 over a field of characteristic 0. It has been suggested by M. Kontsevich back in 2005 that in positive characteristic, it is an interesting thing to consider, but the suggestion has not been taken seriously at the time. A new perspective came from recent work of A. Beilinson and B. Bhatt, where a very similar phenomenon appears in the study of derived de Rham cohomology. Starting from there, it was a relatively straightforward matter to figure out the whole story.

In the end, I proved that co-periodic cyclic homology is Morita-invariant, and moreover, it is an additive invariant in the sense of Keller (that is, it has a natural long exact sequence attached to a localization sequence of DG categories). It always has a conjugate spectral sequence converging to it. Then on one hand, there are many practically important examples when

periodic and co-periodic cyclic homology really are different. But on the other hand, I proved that for a smooth and cohomologically bounded DG algebra over a perfect field of positive characteristic, they are the same. Deligne-Illusie-type argument for degeneration follows more-or-less immediately from this.

These results take up four papers in total, three of them already posted as preprints, and the last one in preparation (should be posted early next year). The first is the short preprint arXiv:1509.08049. This is concerned purely with algebras, not DG algebras. It constructs a non-commutative version of the Cartier isomorphism, and contains a proof that in the commutative case it coincides with the standard one (this is also needed for further projected work on the de Rham-Witt complex). The second is arXiv:1509.08784. This is on the longish side; it lays the foundations of the theory of co-periodic cyclic homology, and in particular, it contains the proofs of the two results mentioned above. The third one is arXiv:1510.06258, and this serves a twofold purpose. Firstly, it is an illustration and a continuation of my earlier joint work with W. Lowen on deformations and square-zero extensions of abelian and exact categories. But secondly, it also treats the special case of the second Witt vectors extension $W_2(k)$ of a perfect field k, and it shows that this extension is intimately related to the cyclic power endofunctor of the category of k-vector spaces. This is needed for a DG algebra counterpart of the Deligne-Illusie condition of liftability to $W_2(k)$ that is used to insure the degeneration of the conjugate spectral sequence (and then deduce the degeneration of the Hodge-to-de Rham one). The remaining preprint is concerned with degeneration directly. It will not be too long (30-something pages), it is half-written by now, and as soon as I finish typing this report, I will proceed to finish it.

Somewhat unexpectedly, there was also a new discovery made while working on all this that has direct relevance to the project sketched in my original research proposal. Namely, the starting part of that story was a certain generalization of Witt vectors to a polynomial functor of two arguments, an algebra and a bimodule over it. Since then, I gave talks about this generalization numeruous times, but I still haven't finished writing the paper. What stops me is too much detail: I have a very general construction of polynomial Witt vectors, but it is quite involved technically, and without a striking application, it would be a hard sell. However, as a biproduct of the work done on co-periodic cyclic homology, I have now a much more direct and immediate alternative construction. It works in restricted generality, but it leads directly to a p-typical Hochschild-Witt complex, the non-commutative generalization of the de Rham-Witt complex of Deligne and Illusie. With this new insight, I'm pretty confident that I can produce a short and direct paper. The old approach can then be used to generalize and explain the story, and to study its relations to algeraic K-theory as sketched in my last report.

1a. Summing up.

Since this is the third and last report on my Dynasty foundation award, I'd better add a few words on the general outcome of the project. Let me state right away that formally speaking, none of the goals sketched in the original research proposal have been accomplished (including some fairly modest ones such as writing up the polynomial Witt vectors story). Somewhat paradoxically, I nevertheless consider the project a success. Indeed, originally, there was one big goal for it that was shaky, and a lot of specific intermediate goals that were sure to produce at least something. Well, as I explained in my last report, the big goal did not work out (nor did I have a lot of confidence in it to begin with). All the other ones worked out wonderfully -- in fact, a bit too wonderfully. I now have a very clear understanding of the things involved, and know much more that I did when I started the whole thing. None of the original planned papers are finished; however, they all grew into something more. Parts of this "more" are the preprints

posted this year, including the new discoveries made (for example, three years ago I didn't even suspect the existence of co-periodic cyclic homology). The remaining parts ought to be enough for several years of active work. I am very grateful to the Dynasty foundation for making it all possible.

2. Publications

Preprints:

D. Kaledin, {\em Bokstein homomorphism as a universal object}, arXiv:1510.06258.

D. Kaledin, *Co-periodic cyclic homology*, arXiv:1509.08784.

D. Kaledin, *Cartier isomorphism for unital associative algebras*, arXiv:1509.08049.

Papers:

These preprints from the last report are accepted, some are already out:

D. Kaledin, *K -theory as an Eilenberg-Maclane spectrum*, arXiv:1412.2537, to appear in Documenta Math.

D. Kaledin, A. Kuznetsov, *Refined blowups*, arXiv:1410.7121, to appear in Math. Res. Lett.

V. Baranovsky, V. Ginzburg, D. Kaledin, J. Pecharich, *Quantization of line bundles on Lagrangian subvarieties*, arXiv:1403.3493, published in Selecta Math.

Also, these two earlier preprints appeared this year:

D. Kaledin, *Trace theories and localization*, arXiv:1308.3743, published in Contemporary Math.

D. Kaledin, W. Lowen, *Cohomology of exact categories and (non-)additive sheaves*, arXiv:1308.3743, published in Adv. Math.

3. Conferences

Homological Mirror Symmetry, Miami, USA, January

Derived categories, algebra and representation theory, Warwick, UK, March

Derived categories and birational geometry, Warwick, UK, June.

Holomorphically Symplectic Manifolds, Levicto Terme, Italy, November.

Algebraic Geometry in Mexico, Puerto Escondido, Mexico, December.

Also, gave a series of three "Adem lectures" at CINVESTAV, Mexico City, Mexico, November.

4. Work in "scientific centers and international teams"

I suppose I should mention here a visiting position at the IBS Center for Geometry and Physics, POSTECH, Pohang, Rep. of Korea, September-October 2015

5. Teaching

I'm still supervising one graduate student, E. Balzin, jointly with Carlos Simpson. He ought to graduate in Spring 2016.

D. Kaledin
Steklov Math. Institute, Algebraic Geometry section.