REPORT FOR THE "DYNASTY" FOUNDATION 2013

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Results

I. Discrete complex analysis

Various constructions of complex analysis on planar graphs were introduced by Isaacs, Ferrand, Duffin, Mercat, Dynnikov–Novikov,Bobenko–Mercat–Suris.Recently this subject is developed extensively due to applications to statistical physics, numerical analysis, computer graphics, and combinatorial geometry.

We develop linear discretization of complex analysis, originally introduced by R. Isaacs, J. Ferrand, R. Duffin, and C. Mercat. We prove convergence of discrete period matrices and discrete Abelian integrals to their continuous counterparts. We also prove a discrete counterpart of the Riemann–Roch theorem. The proofs use energy estimates inspired by electrical networks.

Let us give precise statement of the main result (we skip some standard definitions).

Let S be a *polyhedral surface*, i.e., an oriented 2-dimensional manifold without boundary equipped with a piecewise flat metric having isolated conical singularities. An example of a polyhedral surface is the surface of a polyhedron in 3-dimensional space. Let T be a geodesic triangulation of the polyhedral surface S such that all faces are flat triangles; in particular, all the singular points of the metric are vertices of T.

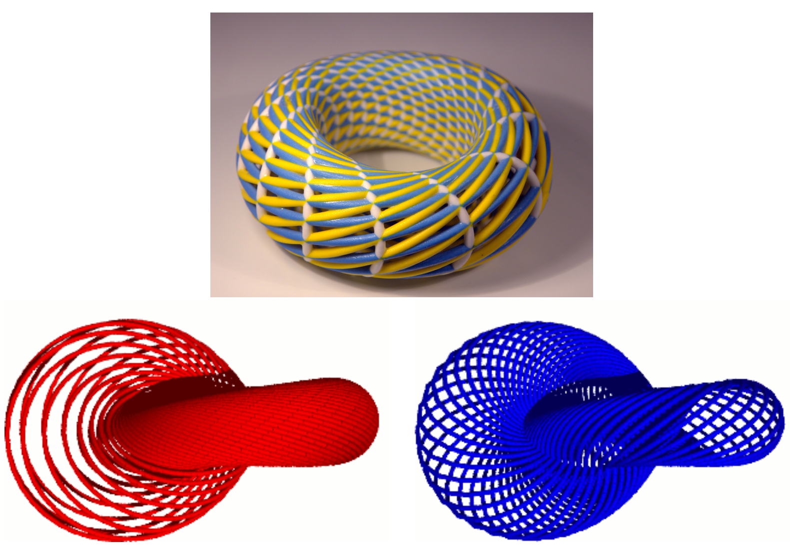

FIGURE 1. Examples of doubly circular surfaces; see Section III below

The polyhedral surface S has a natural complex structure. Indeed, identify each face $w \in \widetilde{T}^2$ with a triangle in the complex plane $\mathbb{C}$ by an orientation-preserving isometry. A function $f\colon \widetilde{S} \to \mathbb{C}$ is *analytic*, if it is continuous and its restriction to the interior of each face is analytic. Given the notion of an analytic function,the *period matrix* Π_S of the surface S is defined in a standard way.

We introduce the following discretization of analytic functions. Denote by T^0, T^1, $\vec{T}^1$, T^2 the sets of vertices, edges, oriented edges, faces, respectively. Introduce *cotan edge weights* by the formula

$$c(e) = \frac{1}{2}\cot\alpha_e + \frac{1}{2}\cot\beta_e,$$

where α_e and β_e are the angles opposite to an edge $e \in T^1$ in the 2 triangles sharing e; see Figure 2.

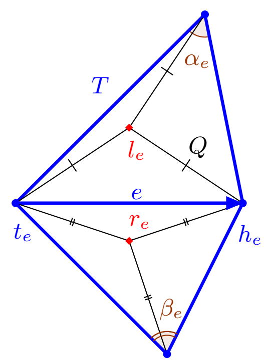

FIGURE 2. Notation associated with an edge $e \in T^1$.

For an oriented edge $e \in \vec{T}^1$ denote by $h_e \in T^0$, $t_e \in T^0$, $l_e \in T^2$, $r_e \in T^2$ the *head*, the *tail*, the *left shore*, the *right shore* of e, respectively; see Figure 2. Two functions $u\colon T^0 \to \mathbb{R}$ and $v\colon T^2 \to \mathbb{R}$ are *conjugate*, if for each oriented edge $e \in \vec{T}^1$ we have

$$v(l_e) - v(r_e) = c(e)(u(h_e) - u(t_e)). \tag{1}$$

The pair $f = (u\colon T^0 \to \mathbb{R}, v\colon T^2 \to \mathbb{R})$ of two conjugate functions is called a *discrete analytic function*. Given the notion of a discrete analytic function,the *discrete period matrix* Π_T of the triangulation T is defined in a standard way.

The *aperture* of a vertex $z \in T^0$ is the sum of all the face angles meeting at the vertex. Denote by γ_z the value 2π divided by the aperture. Denote $\gamma_S := \min\{1, \min_{z\in T^0}\gamma_z\}$. Clearly, the value γ_S depends only on the metric of S. For a $g \times g$ matrix Π denote $\|\Pi\| := \sqrt{\sum_{1\le k,l\le g}|\Pi_{kl}|^2}$.

Theorem (Bobenko-S., 2013). *For each number $\delta > 0$ there are two constants* $\mathrm{Const}_{\delta,S}$,$\mathrm{const}_{\delta,S} > 0$ *depending only on δ and the metric of the surface S such that for any triangulation T of S with the maximal edge length $h <$* $\mathrm{const}_{\delta,S}$ *and with the minimal face angle $> \delta$ we have*

$$\|\Pi_T - \Pi_S\| \le \mathrm{Const}_{\delta,S} \cdot \begin{cases} h, & \text{if } \gamma_S > 1/2; \\ h|\log h|, & \text{if } \gamma_S = 1/2; \\ h^{2\gamma_S}, & \text{if } \gamma_S < 1/2. \end{cases} \tag{2}$$

The approximation order in (2) agrees with numerical experiments; see Figure 3 and Table 1.

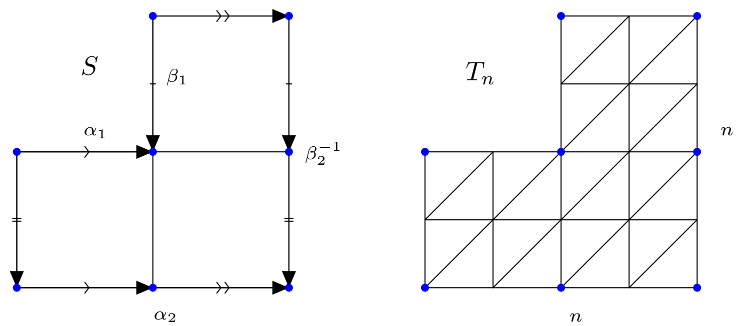

FIGURE 3. The model surface S for numeric experiments and its "triangulation" T_n for $n = 4$.

TABLE 1. Numerical experiments on approximation of period matrices by their discrete counterparts.

n	$\|\Pi_{T_n} - \Pi_S\|$	$\|\Pi_{T_n} - \Pi_S\| \cdot h^{-2\gamma_S}$
8	0.611	1.22
16	0.363	1.15
32	0.220	1.11
64	0.136	1.08
128	0.084	1.07
256	0.053	1.06

This result has been submitted for publication. It corrects a mistake in an unpublished result stated in the author's 2012 report for the "Dynasty" foundation, where the exceptional behavior in presence of vertices of large aperture was overlooked.

II. Rational classification of embeddings.

Given a manifold N and a number m, we study the following question: *is the set of isotopy classes of embeddings* $N \to S^m$ *finite*? In case when the manifold N is a sphere the answer was given by A. Haefliger in 1966. In case when the manifold N is a disjoint union of spheres the answer was given by D. Crowley, S. Ferry and the author in 2011.

We consider the next natural case when N is a product of two spheres. In the following theorem, $FCS(i, j) \subset \mathbb{Z}^2$ is a concrete set depending only on the parity of i and j which is defined in the paper.

Theorem (S., 2011). *Assume that* $m > 2p + q + 2$ *and* $m < p + 3q/2 + 2$. *Then the set of isotopy classes of smooth embeddings* $S^p \times S^q \to S^m$ *is infinite if and only if either* $q + 1$ *or* $p + q + 1$ *is divisible by* 4, *or there exists a point* (x, y) *in the set* $FCS(m - p - q, m - q)$ *such that* $(m - p - q - 2)x + (m - q - 2)y = m - 3$.

Our approach is based on a group structure on the set of embeddings and a new exact sequence, which in some sense reduces the classification of embeddings $S^p \times S^q \to S^m$ to the classification of embeddings $S^{p+q} \sqcup S^q \to S^m$ and $D^p \times S^q \to S^m$. The latter classification problems are reduced to homotopy ones, which are solved rationally.

In 2013 this result has been submitted for publication.

III. Classification of circular surfaces.

Motivated by potential applications in architecture, we study surfaces in Euclidean space containing at least 2 circles through each point; see Figure 1. We give new examples of such *doubly circular* surfaces in dimension 4 and partial classification results in dimension 3. Our approach is based on quaternionic parametrizations.

Example (Krasauskas–Pakharev–S., 2013) *The following surfaces are doubly circular:*

(1) $\Phi(s,t) = A_{11}(s,t)B_{11}(s,t)^{-1}$;
(2) $\Phi(s,t) = A_{10}(s)^{-1}B_{11}(s,t)C_{01}(t)^{-1}$;

where $A_{11}, B_{11} \in \mathbb{H}[s,t]$ *are bilinear,* $A_{10} \in \mathbb{H}[s]$ *and* $C_{01} \in \mathbb{H}[t]$ *are linear.*

Hereafter assume that the two circles through each point are not cospheric.

Theorem (S., 2011). *An analytic doubly circular surface in* $\mathbb{R}^3$ *is parametrized as*

$$x(s,s',t,t') : y(s,s',t,t') : z(s,s',t,t') : w(s,s',t,t'),$$

where x,y,z,w *are bihomogeneous biquadratic polynomials s.t.* $w \mid x^2+y^2+z^2$.

Theorem (Krasauskas–Pakharev–S., 2013) *An analytic doubly circular surface in* $R^3 = \mathrm{Im}\mathbb{H}$ *can be parametrized as* $Q_{13}(s,t)R_{13}(s,t)^{-1}$, *where the polynomials* $Q_{13}(s,t), R_{13}(s,t) \in \mathbb{H}[s,t]$ *have degree* 1 *in* t *and degree* ≤ 3 *in* s.

These results are prepared for publication.

Papers

[1] M. Skopenkov, The boundary value problem for discrete analytic functions, Adv. Math. 240 (2013) 61-87. http://arxiv.org/abs/1110.6737

[2] F. Nilov, M. Skopenkov, A surface containing a line and a circle through each point is a quadric, Geom. Dedicata 163:1 (2013), 301-310; http://arxiv.org/abs/1110.2338

[3] A. Bobenko, M. Skopenkov, Discrete Riemann surfaces: linear discretization and its convergence, submitted to J. für die reine und angewandte Mathematik (2013). http://arxiv.org/abs/1210.0561 (a new version submitted in 2013)

[4] M. Skopenkov, When the set of embeddings is finite?, submitted to Intern. J. Math (2013). http://arxiv.org/abs/1106.1878 (a new version submitted in 2013)

[5] A. Pakharev, M. Skopenkov, A. Ustinov, Through the net of resistors, submitted to Mat. Prosv. 3rd ser. (2013);

[6] A. Skopenkov, M. Skopenkov, Some short proofs of the unrealizability of hypergraphs, unpublished preprint.

In addition to [1]–[5], several talk abstracts have been published in 2013.

Scientific conferences and seminar talks

[1] International conference "Discrete curvature", Marseille, France, 18.11-22.11.

Talk "Discrete Riemann surfaces: convergence results"

[2] International conference "Multidimensional continued fractions", Graz, Austria, 22.06-26.06.

Talk "Tiling of a rectangle, alternating current, and continued fractions".

[3] I.M. Gelfand Centennial Conference, Moscow, 22.07-25.07.

Talk "Discrete complex analysis: convergence results"

[4] Christmas mathematical meetings of the "Dynasty" Foundation, Moscow, 8.01-11.01.

Talk "Discrete analytic functions: convergence results".

[5] Algebra and number theory, Saratov, 9-14.09.

Talk "Triangulations of surfaces by circular arcs"

[6] Visit to Technical University of Vienna, Austria, 2-5.07.

[7] Visit to Technical University of Linz, Austria, 4.07.

[8] Talks at several seminars in Moscow.

Teaching

[1] Complex analysis. Independent University of Moscow, II year students, February-May 2013, 4 hours per week.

Program (short version).

1. Survey of applications of complex analysis.
2. Simple properties of complex numbers and quaternions.
3. Analytic and holomorphic functions.
4. Integration of holomorphic functions.
5. Order of zeroes of holomorphic functions.
6. Conformal mappings.
7. Harmonic functions.

8.* Discrete harmonic functions.

9. Meromorphic functions.
10. Liuville' s theorem.

11.* Elliptic functions.

Course materials attached as a separate PDF file.

[2] Geometry. Independent University of Moscow, I year students, September-December 2013, 4 hours per week + distant excercise class http://dist-math.ru.

Program (short version).

1. Projective geometry.
2. Möbius geometry.
3. Spherical geometry.
4. Hyperbolic geometry.
5. Nine Caley-Klein planar geometries. Klein's concept of geometry.

Course materials attached as a separate PDF file.

[3] Informal supervision of 3 students in form of writing a joint research paper and consultations for their own research papers (3 such papers submitted or published in arxiv in 2013).

[4] Distant courses for mathematical olympiads winners (http://math.olymp.mioo.ru). Since 2013 working without any financial support.