Dynasty Fellowship Report 2010

Valentina Kiritchenko

1. Main results

In 2010, I mostly worked, together with Evgeny Smirnov and Vladlen Timorin, on a new approach to Schubert calculus on the varieties of complete flags in $\mathbb{C}^n$ using the volume polynomial associated with Gelfand–Zetlin polytopes. This approach allows us to compute the intersection products of Schubert cycles by intersecting faces of a polytope. An example for $n = 3$ is discussed in [K1], some of the results were announced in [K2], and a full version of all results with proofs appears in the preprint [KST]. The main results are described below. To make the exposition clearer, I first review some previously known results that we used.

Our approach uses polytope rings introduced by Khovanskii and Pukhlikov. With each convex polytope P, they associated a graded commutative ring R_P that lives in degrees up to $\dim(P)$ and satisfies Poincaré duality. That is, polytope ring behaves like the Chow ring (or cohomology ring) of a smooth variety. In fact, they proved that for an *integrally simple* polytope P (simple means that there are exactly $d = \dim(P)$ edges meeting at each vertex, and integrally simple means that primitive integer vectors parallel to the edges generate the lattice $\mathbb{Z}^d$) the ring R_P is isomorphic to the Chow ring of the corresponding smooth toric variety X_P. Single faces of P give rise to certain elements of R_P (representing cycles given by the closures of the torus orbits in X_P), which generate R_P as an additive group. If $[F]$ is the element of R_P corresponding to a face F, then $[F] \cdot [G] = [F \cap G]$ in R_P, provided that F and G are transverse. If $[F]$ and $[G]$ are not transverse one can always replace $[F]$ (using linear relations in R_P) by a linear combination of faces that are transverse to G (there is a well-known algorithm for this).

What happens for non-simple P? Kiumars Kaveh has related the polytope rings of some non-simple polytopes to the Chow rings of smooth non-toric spherical varieties. In particular, he observed that the ring R_P for the *Gelfand–Zetlin polytope* $P = P_\lambda$ (which is not simple) associated with a strictly dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ of the group $GL_n(\mathbb{C})$ is isomorphic to the Chow ring of the variety X of complete flags in $\mathbb{C}^n$.

This was the starting point of our investigation. First, we developed techniques for multiplying two elements in the ring of any non-simple polytope P by multiplying lifts of these elements to the ring of a simple polytope Q that resolves P [Section 2, KST]. This result is used substantially in our computations in the ring of a Gelfand-Zetlin polytope (which is not simple), and might be used for more general polytopes associated with other spherical varieties. In particular, this allows one to associate each element of R_P with a linear combination of faces of P (though not every face of P corresponds to an element of R_P). This is achieved by constructing an intermediate $\mathbb{Z}$-module $M_{P,Q}$ such that there are functorial injection $R_P \to M_{P,Q}$ and surjection $R_Q \to M_{P,Q}$ (unfortunately, there is no functorial homomorphism between R_P and R_Q for non-simple P).

Next, we answered the following natural question: how to express Schubert cycles as linear combinations of faces of the Gelfand–Zetlin polytope P_λ? Given two Schubert cycles X^w and $X^{w'}$, we can represent X^w and $X^{w'}$ as sums of explicitly described faces [Theorem 4.3, Corollary 4.5, KST] so that every face occurring in the decomposition of X^w is transverse to every face occurring in the decomposition of $X^{w'}$ [Corollary 4.7, KST]. This allows us to represent the intersection of any two Schubert cycles by linear combinations of faces with nonnegative coefficients which might lead to a transparent Littlewood–Richardson rule for the varieties of complete flags. One of our presentations for Schubert cycles is formally similar to the Fomin–Kirillov theorem on Schubert polynomials in terms of *pipe-dreams*, but can not be deduced from the latter because the elements $[F]$ will usually not belong to R_P (only to M_P) and hence cannot be identified with monomials in the corresponding Schubert polynomial. Pipe-dreams are simple combinatorial objects whose relation to faces of Gelfand-Zetlin polytopes was first noticed by Kogan.

We also obtained explicit formulas for the Demazure characters (corresponding to the weight λ) of Schubert varieties in terms of exponential sums over lattice points in unions of faces in P_λ [Theorem 4.8, KST], which imply, in particular, formulas for the degrees of Schubert varieties via sums of volumes of faces and formulas for Hilbert functions of Schubert varieties via Erkhart polynomials. All these formulas are in the spirit of the theory of Newton polytopes (Gelfand–Zetlin polytopes naturally correspond to projective embeddings of the flag variety). Formulas for the Demazure characters and for the degrees of Schubert varieties generalize recent results of Postnikov and Stanley from very special Schubert varieties, namely Kempf (or (1 3 2)-avoiding) varieties, to all Schubert varieties. Besides, as a byproduct of our proof we found a geometric realization of *mitosis* (a combinatorial procedure for computing Schubert polynomials in terms of pipe-dreams introduced by Knutson and Miller) and a minimal realization of a simplex as a cubic complex different from the previously known realizations.

In 2010, Jens Hornbostel and I revised our paper on Schubert calculus in the algebraic cobordism of complete flag varieties for arbitrary reductive groups over fields of zero characteristic [HK]. For varieties with algebraic cellular decomposition, we added the isomorphism theorem between their algebraic and complex cobordism [Appendix, HK].

2. Publications and preprints

[K1] *Gelfand–Zetlin polytopes and flag varieties*, International Mathematics Research Notices, 2010, no. 13, 2512–2531

[K2] *From moment poytopes to string bodies*, Oberwolfach Reports, 19/2010, 30–33

[HK] joint with Jens Hornbostel, *Schubert calculus for algebraic cobordism*, Journal für die reine und angewandte Mathematik (Crelle), 27 pages, in press

[KST] joint with Evgeny Smirnov and Vladlen Timorin, *Schubert calculus on Gelfand–Zetlin polytopes*, http://www.mccme.ru/ valya/Schubert.pdf

[K3] *Исчислительная геометрия: метод Шаля и Шуберта*, submitted to "Kvant"

3. Talks

Invited conference talks

August	ICM2010 Satellite Conference on Complex Geometry, Group actions and Moduli spaces, Hyderabad, India
April	Oberwolfach workshop "Algebraic groups", Oberwolfach, Germany

Seminar talks

November	V.I. Arnold seminar, Moscow State University
October	Moscow Mathematical Society meeting
August	Tata Institute of Fundamental Research, Mumbai, India

4. International collaboration

Freie Universität Berlin	project "Horospherical varieties and polyhedral divisors", joint with Klaus Altmann and Lars Petersen
Tata Institute, Mumbai	project "Equivariant cobordism of spherical varieties", joint with Amalendu Krishna

5. Teaching

In winter 2010, I taught a graduate topics mini-course "Spherical varieties" at the Institute for Mathematics, Freie Universität Berlin. Since September, I have been working at the Faculty of Mathematics, Higher School of Economics. I conduct problem solving sessions for the 2d year undergraduate course Calculus II and help with problem solving sessions for Algebra II, Topology II and Geometry I. I am also one of the organizers of the undegraduate learning seminar "Symmetric functions, Grassmannians and flags".

I help to coordinate our PhD program in Mathematics that started in November. In particular, I was responsible for the admission exams.

I supervise the following undergraduate course projects:

Pavel Gusev (3d year)	"f-vectors of Gelfand–Zetlin polytopes"
Leonid Yanushevich (2d year)	"Continued fraction of e"
Dmitry Zarifyan (2d year),	"Monodromy of Gauss hypergeometric function"

In 2010, I wrote a popularizing article for high school students on the Schubert method for solving enumerative geometry problems [K3]. I also revised the first part of the notes to my course "Geometry of spherical varieties".