REPORT ON THE PIERRE DELIGNE FELLOWSHIP, 2011

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During 2011 I have been working on developing a representation theory approach to mirror symmetry and quantum cohomology of homogeneous spaces, associated with Lie groups of classical type.

According to Givental the $\mathfrak{gl}_N$-Whittaker functions, being solutions to the quantum cohomology D-module $QH^*(\mathrm{Fl}_N)$ of the complete flag variety $\mathrm{Fl}_N = GL_N(\mathbb{C})/B$, describe the corresponding equivariant Gromov-Witten invariants of Fl_N. However, the Givental's approach to representation theory description of quantum cohomology of homogeneous spaces is inapplicable to general incomplete flag varieties, since no relevant Whittaker model (Toda lattice) was known until very recently. In (5) this problem have been partially solved; namely, in (5) a generalization of the Whittaker model for principal series representations of $\mathfrak{gl}_N$ is constructed. This generalization produces parabolic $Gr_{m,\,N}$-Whittaker functions associated with the Grassmannian $\mathrm{Gr}_{m,\,N}$, which are expected to describe the $S^1 \times U_N$-eqiovariant Gromov-Witten invariants of $\mathrm{Gr}_{m,\,N}$.

In (8) we construct a Mellin-Barnes type integral representation of the specialized $\mathrm{Gr}_{m,N}$-Whittaker function, following an original generalization of Whittaker models to incomplete flag manifolds from (6):

$$\Psi^{(m,\,N)}_{\underline{\lambda}}(x,0,\ldots,0) \;=\; \int\limits_{\mathcal{C}} d\underline{\gamma}\; e^{-\frac{x}{\hbar}\sum\limits_{i=1}^{m}\gamma_i}\; \frac{\prod\limits_{i=1}^{m}\prod\limits_{j=1}^{N}\hbar^{\frac{\gamma_i-\lambda_j}{\hbar}}\;\Gamma\Big(\frac{\gamma_i-\lambda_j}{\hbar}\Big)}{\prod\limits_{\substack{i,k=1\\ k\neq i}}^{m}\hbar^{\frac{\gamma_i-\gamma_k}{\hbar}}\;\Gamma\Big(\frac{\gamma_i-\gamma_k}{\hbar}\Big)}\,.$$

with $\underline{\gamma} = (\gamma_1,\ldots,\gamma_m)$ and $\underline{\lambda} = (\lambda_1,\ldots,\lambda_N)$. Our derivation involves a generalization of the Gelfand-Zetlin realization to infinite-dimensional $U(\mathfrak{gl}_N)$-modules introduced Gerasimov, Kharchev, and Lebedev. Besides, our integral representation verifies the conjectural integral formula by Hori and Vafa, although our solution to the quantum cohomology D-module has a different asymptotic behavior.

In (2) we give a representation-theoretic derivation of the Givental-type stationary phase integral representation of the specialized $\mathrm{Gr}_{m,\,N}$-Whittaker function conjectured previously by Batyrev, Kim, Ciocan-Fontanine and van Straten. Namely, the following holds:

$$\Psi^{(m,N)}_{\underline{\lambda}}(x_{N,1},0,\ldots,0) \;=\; \int_{\mathcal{C}_{m,\,N}} \omega_{m,\,N}\quad e^{\mathcal{F}_{m,N}(\underline{x})}\,,$$

where

$$\mathcal{F}_{m,N}(\underline{x}) \;=\; \imath\Big(\sum_{k=1}^{m}\lambda_{N-m+k}\Big)x_{N,1} \;+\; \imath\sum_{n=1}^{N-m}(\lambda_n-\lambda_{n+1})\sum_{i=1}^{\min(m,n)}x_{n,i}$$
$$+\;\imath\sum_{n=1}^{m-1}(\lambda_{N-m+n}-\lambda_{N-m+n+1})\sum_{i=n+1}^{\min(N-m+n,\,m)}x_{N-m+n,\,i}$$

$$-\frac{1}{\hbar}\Big(e^{-x_{mm}}+e^{x_{N-m,\,1}-x_{N,1}}+\sum_{k=1}^{m}\sum_{i=1}^{N-1-m}e^{x_{i+k-1,\,k}-x_{i+k,\,k}}+\sum_{k=1}^{N-m}\sum_{i=1}^{m-1}e^{x_{k+i,\,i+1}-x_{k+i-1,\,i}}\Big).$$

and

$$\omega_{m,N}=\prod_{n=1}^{N-m}\prod_{k=1}^{\min(n,m)}dx_{n,k}\cdot\prod_{n=1}^{m-1}\prod_{i=n+1}^{\min(N-m+n,\,m)}dx_{N-m+n,\,i}\,.$$

This result uses the generalization of the Whittaker model for principal series $\mathcal{U}(\mathfrak{gl}_N)$-modules, and its Gauss-Givental realization in the space of functions of totally positive unipotent matrices. In the framework of (4,5,6) the above two integral representations are mirror symmetric one to the other; more precisely, there are two equivariant two-dimensional mirror dual topological field theories on a disc, such that certain correlation functions in these two theories reproduce the two integral representations of the $\mathrm{Gr}_{m,\,N}$-Whittaker functions.

The Givental's approach was extended by Givenal and Lee to the description of equivariant quantum K-theory of homogeneous spaces. The corresponding system of difference operators is given by the q-Toda lattice, and the corresponding solution is given by the q-deformed Whittaker function. The two integral representations, of the Mellin-Barnes type and of Givental type, for the q-deformed $\mathfrak{gl}_N$-Whittaker function were constructed previously in a series of papers (see (3) and references therein). In (1) we verify that under the limit $q\to 1$ the q-deformed Whittaker functions tend to the classical $\mathfrak{gl}_N$-Whittaker functions. In particular, this result illustrates an universality of the two types, the Mellin-Barnes one and Givental's one, of integral representations for the special functions on reductive groups and their deformations.

§1. Publications

(1) *On a classical limit of q-deformed Whittaker functions*, (with A. Gerasimov and D. Lebedev), to appear in Lett. Math. Phys., Preprint [math.AG/1101.4567] ;
(2) *On parabolic Whittaker functions II*, To appear in Cent. Eur. J. Math., Preprint [math.AG/1107.2998] ;
(3) *On q-deformed $\mathfrak{gl}_N$-Whittaker function III*, (with A. Gerasimov and D. Lebedev), Lett. Math. Phys. 97 (2011), 1-24 ;
(4) *Archimedean L-factors and topological field theories*, (with A. Gerasimov and D. Lebedev), Commun. Number Theory and Physics, 5 (2011) 57-101 ;
(5) *Archimedean L-factors and topological field theories II*, (with A. Gerasimov and D. Lebedev), Commun. Number Theory and Physics, 5 (2011) 101-134 ;
(6) *Parabolic Whittaker functions and Topological field theories I* (with A. Gerasimov and D. Lebedev), Commun. Number Theory and Physics, 5 (2011) 135-202 ;
(7) *New integral representations of Whittaker functions*, (with A. Gerasimov and D. Lebedev), To appear in Rus. Math. Surveys ;
(8) *On parabolic Whittaker functions*, To appear in Lett. Math. Phys., Preprint [math.AG/1011.4250] .

§2. Conferences

Talk "Parabolic Whittaker functions and quantum cohomology of homogeneous spaces" given at the conference "Classical and Quantum integrable systems", Protvino, January 2011.

§3. Teaching

In Spring 2011 I gave a course on orbit method and representation theory of noncompact groups to the 3-d and the 4-th year students of Moscow Institute of Physics and Technology (MIPT) at ITEP. This course continues a course on spinors, that I gave in 2010, and will be continued in Spring 2012.

Besides, together with A. Gerasimov and D. Lebedev, I organized a seminar on quantum field theory for the MIPT students at ITEP.